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PRO MATHEMATICA

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CONTENIDO

Notes on flat pseudo-Riemannian manifolds

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(Traslated into English by Elizabeth Gasparim[2]***)***

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Abstract

In these notes we survey basic concepts of affine geometry and their interaction with Riemannian geometry. We give a characterization of affine manifolds which has as counterpart those pseudo-Riemannian manifolds whose Levi-Civita connection is flat. We show that no connected semisimple Lie group admits a left invariant flat affine connection. We characterize flat pseudo-Riemannian Lie groups. For a flat left-invariant pseudo-metric on a Lie group, we show the equivalence between the completeness of the Levi-Civita connection and unimodularity of the group. We emphasize the case of flat left invariant hyperbolic metrics on the cotangent bundle of a simply connected flat affine Lie group. We also discuss Lie groups with bi-invariant pseudo-metrics and the construction of orthogonal Lie algebras.

MSC(2010): **53C20, 22E60, 53A15**.

Keywords: Flat affine structures, flat pseudo-metrics, flat pseudo-Riemannian Lie groups, orthogonal Lie groups, orthogonal Lie algebras.

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1 Introduction

A real smooth manifold M of dimension n is called an **affine manifold** if it admits a maximal atlas whose change of coordinates are restrictions of affine transformations of $\mathbb{R}^n$. Having an affine structure over M is equivalent to having a flat and torsion free linear connection ∇ on TM (see Theorem 2.5). A pair (M, ∇), where ∇ is a flat affine connection (i.e. ∇ is a flat and torsion free linear connection) on M, is called a **flat affine manifold**. When $M = G$ is a Lie group and ∇ is a left invariant flat affine connection, the pair (G, ∇) is called a **flat affine Lie group**. If g is a pseudo-metric on M (respectively μ is a left invariant pseudo-metric on G) such that the Levi-Civita connection associated to g has vanishing curvature tensor, the pair (M, g) (respectively (G, μ)) is called a flat pseudo-Riemannian manifold (respectively flat pseudo-Riemannian Lie group).

These notes are organized as follows. The first two sections are devoted to the study of flat affine manifolds. Theorem 3.2 is essential because it gives a characterization of flat affine Lie groups that we will use throughout these notes. We show that no connected semisimple real Lie group admits a left invariant flat affine connection (Theorem 3.6). In Section 3 we introduce some basic concepts of Riemannian geometry and exhibit examples of flat affine structures compatible with pseudo-metrics. Section 4 is dedicated to the study of flat pseudo-Riemannian Lie groups. We give a characterization of such Lie groups and we show that the left-invariant affine structure defined by the Levi-Civita connection is geodesically complete if and only if the group is unimodular (Theorem 5.3). We also show that the cotangent bundle of a simply connected flat affine Lie group is endowed with an affine Lie group structure and a left invariant flat hyperbolic metric (Proposition 6.1). In the Section 7 we study **orthogonal Lie groups**, that is, Lie groups endowed with bi-invariant metrics. To study properties of orthogonal Lie groups we introduce the notion of **orthogonal Lie algebra**, which will be used in the method of **double orthogonal extension**. As an application, we

describe how to construct the oscillator Lie algebra of the oscillator Lie group which appear in several branches of Physics and Mathematical-Physics and give rise to particular solutions of the Einstein-Yang-Mills equations. Finally, we present another characterization of flat Riemannian Lie groups using some consequences of the presence of an orthogonal structure in a Lie algebra (Theorem 7.9).

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2 Flat affine manifolds

In what follows M will denote a connected paracompact real smooth manifold of dimension n. We will denote by $\mathfrak{X}(M)$ the Lie algebra of smooth vector fields over M and by $C^\infty(M)$ the associative algebra of functions on M with values in $\mathbb{R}$.

The objects of study of these notes are flat affine paracompact manifolds. In particular, we study flat affine structures that are compatible with pseudo-Riemannian metrics. A good understanding of the category of Lagrangian submanifolds requires a good knowledge of the category of flat affine manifolds (see [26, Thm 7.8]). Also, flat affine manifolds

with holonomy reduced to $\mathrm{Gl}(n,\mathbb{Z})$ appear naturally in integrable systems and Mirror symmetry (see [12]). Further applications of flat affine manifolds appear in the study of Hessian structures and Information Geometry (see [23, c. 6]).

Let V be a real finite dimensional vector space. The space of **affine transformations of** V is the Lie group $\mathrm{Aff}(V) = V \rtimes_{Id} \mathrm{GL}(V)$ determined by the semi-direct product of the Abelian Lie group $(V,+)$ with the Lie group $\mathrm{GL}(V)$ via the identity representation. Its Lie algebra is the product vector space $\mathfrak{aff}(V) = V \rtimes_{id} \mathfrak{gl}(V)$ with Lie bracket given by

$$[(x,t),(y,s)] = (t(y) - s(x), [t,s]_{\mathfrak{gl}(V)}),$$

for all $x, y \in V$ and $t, s \in \mathfrak{gl}(V)$.

We say that M **admits an affine structure** if there exists an maximal atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in J}$ of M having change of coordinates that are restrictions of affine transformations of $\mathbb{R}^n$, that is, for each $\alpha, \beta \in J$ with $U_\alpha \cap U_\beta \neq \emptyset$, there exists $\sigma_{\alpha\beta} \in \mathrm{Aff}(\mathbb{R}^n)$ such that

$$\varphi_\beta \circ \varphi_\alpha^{-1}\big|_{\varphi_\alpha(U_\alpha\cap U_\beta)} = \sigma_{\alpha\beta}\big|_{\varphi_\alpha(U_\alpha\cap U_\beta)}.$$

If G is a discrete Lie subgroup of $\mathrm{Aff}(\mathbb{R}^n)$ that acts freely and properly discontinuously over $\mathbb{R}^n$, then the quotient manifold $\mathbb{R}^n/G$ admits an affine structure such that the coordinate changes are restrictions of elements of G (see [22, p. 349]).

Example 2.1. Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, $U_1 = S^1 - \{(1,0)\}$ and $U_2 = S^1 - \{(0,1)\}$. If $\varphi_1 : U_1 \to (0, 2\pi)$ and $\varphi_2 : U_2 \to \left(-\frac{\pi}{2}, 3\frac{\pi}{2}\right)$ are defined respectively by

$$z \mapsto \arg(z) \quad \text{and} \quad z \mapsto \begin{cases} \arg(z) - \dfrac{\pi}{2}, & \text{if} \quad \arg(z) \in \left(\dfrac{\pi}{2}, 2\pi\right) \\ \arg(z) + 3\dfrac{\pi}{2}, & \text{if} \quad \arg(z) \in \left(0, \dfrac{\pi}{2}\right) \end{cases},$$

then the atlas $\{(U_1,\varphi_1),(U_2,\varphi_2)\}$ determines an affine structure for S^1.

Example 2.2. *Hopf manifolds.* Let $\lambda > 1$ be a fixed real number. Denote by G the group of transformations of $\mathbb{R}^n\backslash\{0\}$ defined by

$$\begin{aligned} T_n : \mathbb{R}^n\backslash\{0\} &\to \mathbb{R}^n\backslash\{0\} \\ x &\mapsto T_n(x) = \lambda^n \cdot x, \end{aligned}$$

for all $n \in \mathbb{Z}$. The set G is a discrete subgroup of $\mathrm{Aff}(\mathbb{R}^n)$ that acts freely and properly discontinuously over $\mathbb{R}^n\backslash\{0\}$. Therefore $\mathbb{R}^n\backslash\{0\}/G$ is an affine manifold called a **Hopf manifold** which we will denote by $\mathrm{Hopf}(\lambda, n)$. Topologically these manifolds are either the disjoint union of two Hopf circles $\mathbb{R}^+/G$, when $n = 1$, or diffeomorphic to $S^{n-1} \times S^1$ when $n > 1$.

Recall that a **linear connection** on a smooth manifold M is an $\mathbb{R}$-bilinear map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ that is $C^\infty(M)$-linear on the first component and satisfies

$$\nabla_X fY = X(f)Y + f\nabla_X Y,$$

for all $X, Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$. The **torsion tensor** T_∇ and **curvature tensor** R_∇ associated to a linear connection ∇ are defined respectively by

$$T_\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \tag{2.1}$$

and

$$R_\nabla(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{2.2}$$

for all $X, Y, Z \in \mathfrak{X}(M)$. When $T_\nabla = 0$ and $R_\nabla = 0$ we say that ∇ is a **flat affine connection** and the pair (M, ∇) is called a **flat affine manifold**.

Remark 2.3. The pair (M, ∇) is a flat affine manifold if and only if there exists an atlas for M such that the Christoffel symbols associated to ∇ vanish identically on all charts (see [5, p. 108]).

Example 2.4. If $(x^1, \cdots, x^n)$ are the usual coordinates in $\mathbb{R}^n$, the **usual linear connection** ∇^0 on $\mathbb{R}^n$ is defined as

$$\nabla^0_X Y = \sum_{j=1}^{n} X(f^j)\frac{\partial}{\partial x^j}, \quad \text{where} \quad Y = \sum_{j=1} f^j \frac{\partial}{\partial x^j}.$$

It is simple to verify that ∇^0 is a flat affine connection on $\mathbb{R}^n$ and that we have $\Gamma^k_{ij} = 0$ for all $i, j, k = 1, \cdots, n$.

From now on, by smooth manifolds we mean real manifolds that are C^∞ differentiable. The following characterization of affine manifolds appears in [2].

Theorem 2.5 (Auslander-Markus)**.** *A real smooth manifold M has an affine structure if and only if there exists a flat affine connection on M.*

Proof. Suppose that $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in J}$ is an affine structure for M. For each $\alpha \in J$, we endow the open set $\varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$ with the usual linear connection ∇^0. The pullback of $\nabla^0\big|_{\varphi_\alpha(U_\alpha)}$ by the diffeomorphism φ_α defines a flat affine connection ∇_α over U_α. We choose ∇ over M as the linear connection subjected to $\nabla|_{U_\alpha} = \nabla_\alpha$ for all $\alpha \in J$. To verify that ∇ is well defined observe that, for $\alpha, \beta \in J$ with $U_\alpha \cap U_\beta \neq \emptyset$, setting $\varphi_\alpha = (x^1, \cdots, x^n)$ and $\varphi_\beta = (y^1, \cdots, y^n)$ on $U_\alpha \cap U_\beta$, we have

$$(\Gamma^i_{jk})_\alpha = \sum_{l=1}^{n} \frac{\partial^2 y^l}{\partial x^j \partial x^k}\frac{\partial x^i}{\partial y^l} + \sum_{l,m,q=1}^{n} (\Gamma^l_{mq})_\beta \frac{\partial y^m}{\partial x^j}\frac{\partial y^q}{\partial x^k}\frac{\partial x^i}{\partial y^l}. \tag{2.3}$$

Given that the Christoffel symbols of ∇^0 vanish, we obtain $(\Gamma^l_{mq})_\beta = 0$ on U_β. Moreover we have $\dfrac{\partial^2 y^l}{\partial x^j \partial x^k} = 0$, since $\varphi_\beta \circ \varphi_\alpha^{-1}\big|_{\varphi_\alpha(U_\alpha\cap U_\beta)}$ is the restriction of an element of $\mathrm{Aff}(\mathbb{R}^n)$, and hence we obtain $(\Gamma^i_{jk})_\alpha = 0$, showing that ∇ is well defined. Furthermore, using equation (2.3) we can verify that ∇ is the unique flat affine connection that can be obtained in this fashion.

Reciprocally, suppose that ∇ is a flat affine connection on M. For each $p \in M$ there exists a neighborhood V_p of 0 in T_pM and a neighborhood U_p of p in M such that the exponential map associated to ∇,

denoted by $\exp_p \colon V_p \to U_p$, is a diffeomorphism (see [11, p. 148]). Given a basis $\{X_1, \cdots, X_n\}$ for the tangent space T_pM, we define local charts on U_p by

$$x^i \left(\exp_p \left(\sum_{j=1}^{n} a^j X_j \right) \right) = a^i,$$

if $\sum_{j=1}^{n} a^j X_j \in V_p$ for all $i = 1, \cdots, n$. Since ∇ is flat affine, there exists an atlas over M with respect to which we have $\Gamma_{ij}^k = 0$ for every chart. The computation of geodesic curves γ in a chart $(U, (y^1, \cdots, y^n))$ of such an atlas amounts to solving the system of ordinary differential equations $\dfrac{d^2 y^i(\gamma(t))}{dt^2} = 0$ for $i = 1, \cdots, n$, whose solution, for a fixed initial condition $\left(p, \sum_{j=1}^{n} a^j X_j\right) \in TM$, is unique. Therefore, setting $\dfrac{\partial^2 x*^l}{\partial x^j \partial x^k} = 0$ on each intersection, these normal coordinates $(U_p, (x^1, \cdots, x^n))$ generate a unique atlas over M for which the changes of coordinates are restrictions of elements of $\mathrm{Aff}(\mathbb{R}^n)$. □

In general, determining whether a smooth manifold admits a flat affine structure or not is a difficult question, and there are obstructions for the existence of said structures.

Example 2.6. We list some manifolds that do not admit flat affine structures:

- Compact simply connected manifolds (see [7]).
- Compact manifolds with finite fundamental group (see [2]).
- In particular for $n > 1$ the real n-sphere S^n, the real projective space $\mathbb{RP}^n$ and the group of rotations $O(n)^+$ do not admit flat affine structures.

Further topological obstructions for the existence of a flat affine structures are listed in [24].

Remark 2.7. There is no direct relation between the notion of affine variety as given in algebraic geometry (namely a set cut out by polynomial equations) and the definition of affine manifold in the way that we present it (when a manifold admits an affine structure). For example, for $n > 1$ the n-dimensional real sphere S^n is an affine algebraic variety but is not a flat affine manifold in the sense of our definition.

3 Flat affine Lie groups

In what follows G denotes a connected real Lie group. For each $\sigma \in G$, we denote by $L_\sigma : G \to G$ the map left multiplication by σ in G, that is, the map defined by $\tau \mapsto L_\sigma(\tau) = \sigma\tau$. The tangent space $T_\epsilon G$ of G at the identity and the Lie algebra of left invariant vector fields $\mathfrak{X}_l(G)$ on G are isomorphic vector spaces as follows. For each $x \in T_\epsilon G$, we associate the left invariant vector field x^+ defined by

$$x^+_\sigma = (L_\sigma)_{*,\epsilon}(x) = \left.\frac{d}{dt}\right|_{t=0} (\sigma \cdot \exp_G(tx)),$$

for all $\sigma \in G$. Under this isomorphism we give a structure of Lie algebra to $\mathfrak{g} = T_\epsilon G$ and call it the Lie algebra of G.

A linear connection ∇ on G is called **left invariant** if L_σ is an affine transformation of (G, ∇) for all $\sigma \in G$. More precisely, we must have

$$(L_{\sigma^{-1}})_* \left(\nabla_{(L_\sigma)_* X} (L_\sigma)_* Y\right) = \nabla_X Y,$$

for all $X, Y \in \mathfrak{X}(G)$ and $\sigma \in G$. From this definition it follows immediately that a connection ∇ on G is left invariant if and only if for all $x^+, y^+ \in \mathfrak{X}_l(G)$ we have $\nabla_{x^+} y^+ \in \mathfrak{X}_l(G)$.

Lemma 3.1. *There exists a bijective correspondence between left invariant linear connections on G and bilinear maps on $\mathfrak{g}$.*

Proof. If ∇ is a left invariant linear connection on G, then the assignment $\cdot : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, given by $(x, y) \mapsto x \cdot y = (\nabla_{x^+} y^+)(\epsilon)$ for all $x, y \in \mathfrak{g}$,

defines a bilinear map on $\mathfrak{g}$. Conversely, suppose that $\cdot : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a bilinear map on $\mathfrak{g}$. Define ∇ on G as the linear connection such that $\nabla_{x^+} y^+ = (x \cdot y)^+$ verifies

$$\nabla_{fx^+} y^+ = f.(x \cdot y)^+ \qquad \text{and} \qquad \nabla_{x^+} f y^+ = x^+(f) y^+ + f.(x \cdot y)^+, \tag{3.1}$$

for all $x, y \in \mathfrak{g}$ and $f \in C^\infty(G)$. Since the left invariant vector fields determine an absolute parallelism over G, we have that $\mathfrak{X}_l(G)$ generates $\mathfrak{X}(G)$ as a $C^\infty(G)$-module. Hence, each smooth vector field over G can be written as a $C^\infty(G)$-linear combination of left invariant vector fields. Using this fact together with the identities exhibited in (3.1) we can easily conclude that ∇ is a left invariant linear connection on G. □

When there exists a left invariant flat affine connection ∇ on G, the pair (G, ∇) is called a **flat affine Lie group**. To characterize flat affine Lie groups and to study their structure is an open problem which was proposed by J. Milnor in [20]. The following characterization of flat affine Lie groups was given in [10] and [15].

Theorem 3.2 (Koszul and Medina)**.** *Let G be a connected n-dimensional real Lie group, $\mathfrak{g}$ its Lie algebra and $\widetilde{G}$ its universal covering Lie group. Then, the following are equivalent.*

1. *There exists a left invariant flat affine connection on G.*
2. *There exists a bilinear map $\cdot : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ on $\mathfrak{g}$ such that*

$$[x, y] = x \cdot y - y \cdot x \tag{3.2}$$

   and

$$L_{[x,y]} = [L_x, L_y]_{\mathfrak{gl}(\mathfrak{g})}, \tag{3.3}$$

   for all $x, y \in \mathfrak{g}$, here $L_x : \mathfrak{g} \to \mathfrak{g}$ is the map defined by $y \mapsto L_x(y) = x \cdot y$.

3. *There exists a real n-dimensional vector space V and a Lie group homomorphism $\rho\colon \widetilde{G} \to Aff(V)$ such that the left action of $\widetilde{G}$ over V defined by $\sigma \cdot v = \rho(\sigma)(v)$ for all $(\sigma, v) \in \widetilde{G} \times V$, allows a point having open orbit and discrete isotropy.*

Proof. We first show that 1 implies 2. Let ∇ be a left invariant flat affine connection on G. By Lemma 3.1 we have that $L_x(y) = x \cdot y = (\nabla_{x^+} y^+)(\epsilon)$ defines a bilinear map on $\mathfrak{g}$. Substituting this equality into the formulas of torsion and curvature (2.1)-(2.2) for ∇, we obtain identities (3.2) and (3.3), respectively.

To get 2 implies 3, suppose that there exists a bilinear map $\cdot : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ on $\mathfrak{g}$ satisfying (3.2) and (3.3), where $L_x : \mathfrak{g} \to \mathfrak{g}$ is the linear map defined by $y \mapsto L_x(y) = x \cdot y$, for all $x, y \in \mathfrak{g}$. Then the map $\theta : \mathfrak{g} \to \mathfrak{aff}(\mathfrak{g})$, defined by $x \mapsto (x, L_x)$, is a well defined Lie algebra homomorphism. This follows from the fact that (3.2) and (3.3) imply that the map $L : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, defined by $x \mapsto L_x$, is a well defined Lie algebra homomorphism which satisfies $[x, y] = L_x(y) - L_y(x)$ for all $x, y \in \mathfrak{g}$. On the other hand, for $0 \in \mathfrak{g}$ the map $\psi_0 : \mathfrak{g} \to \mathfrak{g}$ given by $x \mapsto x + L_x(0) = x$ is a linear isomorphism. Thus, by means of the exponential map of G, we obtain a homomorphism of Lie groups $\rho : \widetilde{G} \to \mathrm{Aff}(\mathfrak{g})$ given by $\sigma \to (Q(\sigma), F_\sigma)$, where for $\sigma = \exp_G(x)$ we have

$$Q(\sigma) = \sum_{k=1}^{\infty} \frac{1}{k!}(L_x)^{k-1}(x) \quad \text{and} \quad F_\sigma = \mathrm{Exp}(L_x) = \sum_{k=0}^{\infty} \frac{1}{k!}(L_x)^k.$$

Since ψ_0 is surjective, the orbit of $0 \in \mathfrak{g}$ by the left action of $\widetilde{G}$ over $\mathfrak{g}$, defined by $\sigma \cdot 0 = Q(\sigma) + F_\sigma(0) = Q(\sigma)$ for all $\sigma \in \widetilde{G}$, is open. Moreover, by the injectivity of ψ_0 it follows that the isotropy of $0 \in \mathfrak{g}$ by the given action is dicrete. The latter implies that the orbital map $\pi : \widetilde{G} \to \mathrm{Orb}(0)$, given by $\sigma \mapsto Q(\sigma)$, is a local diffeomorphism and hence a covering map (see [15]).

Finally let us show that 3 implies 1. Let V be a real vector space of dimension n and assume that there exists a Lie group homomorphism $\rho : \widetilde{G} \to \mathrm{Aff}(V)$, defined by $\sigma \to (Q(\sigma), F_\sigma)$, which admits a point $v \in V$ with open orbit and discrete isotropy for the action of $\widetilde{G}$ on the left over V induced by ρ. The latter implies that the map $F : \widetilde{G} \to \mathrm{GL}(V)$ defined by $\sigma \mapsto F_\sigma$, is a Lie group homomorphism and $Q : \widetilde{G} \to V$, given by $\sigma \mapsto Q(\sigma)$, is a smooth map that satisfies

$$Q(\sigma\tau) = Q(\sigma) + F_\sigma(Q(\tau)),$$

for all $\sigma, \tau \in \widetilde{G}$. Moreover, the orbital map $\pi : \widetilde{G} \to \mathrm{Orb}(v)$ given by $\sigma \mapsto Q(\sigma) + F_\sigma(v)$ is a local diffeomorphism. Differentiating at the identity of $\widetilde{G}$, we obtain the Lie algebra homomorphism $\theta : \mathfrak{g} \to \mathfrak{aff}(V)$ given by $x \mapsto (q(x), f_x)$ for all $x \in \mathfrak{g}$, where the map $f : \mathfrak{g} \to \mathfrak{gl}(V)$ defined by $x \mapsto f_x$, is a Lie algebra homomorphism and $q : \mathfrak{g} \to V$, given by $x \mapsto q(x)$, is the linear map

$$q([x, y]) = f_x(q(y)) - f_y(q(x)), \tag{3.4}$$

for all $x, y \in \mathfrak{g}$. Moreover, the map $\psi_v : \mathfrak{g} \to V$ defined by $x \mapsto q(x) + f_x(v)$ is a linear isomorphism. Now, for each $x \in \mathfrak{g}$ we define

$$L_x = \psi_v^{-1} \circ f_x \circ \psi_v.$$

Since $f : \mathfrak{g} \to \mathfrak{gl}(V)$ is a Lie algebra homomorphism, we have $L_{[x,y]} = [L_x, L_y]_{\mathfrak{gl}(\mathfrak{g})}$ for all $x, y \in \mathfrak{g}$. On the other hand, since $q : \mathfrak{g} \to V$ satisfies (3.4), we conclude $[x, y] = L_x(y) - L_y(x)$ for all $x, y \in \mathfrak{g}$. Therefore, using Lemma 3.1, we obtain that the linear connection ∇ defined by

$$\nabla_{x^+} y^+ = (x \cdot y)^+ = (L_x(y))^+,$$

for all $x, y \in \mathfrak{g}$, is a left invariant flat affine connection on G. Using the linear isomorphism ψ_v, it can be easily drawn that the Lie algebra homomorphisms in $\mathfrak{g} \to \mathfrak{aff}(V)$ defined by $x \mapsto (x, L_x)$ and $x \to (q(x), f_x)$ are isomorphic. □

Example 3.3. *Dimension 2.* Recall that the Lie group of affine transformations of the real line is given by the product manifold $\mathrm{Aff}(\mathbb{R}) = \mathbb{R}^* \times \mathbb{R}$, with product $(a, b) \cdot (c, d) = (ac, ad + b)$. Its Lie algebra is identified with the vector space $\mathfrak{aff}(\mathbb{R}) = \mathrm{Vect}_{\mathbb{R}}\{e_1, e_2\}$ with Lie bracket $[e_1, e_2] = e_2$. Next we introduce is a family of left invariant flat affine connections on $\mathrm{Aff}(\mathbb{R})$ which are not isomorphic. For α real, set

$$\nabla_{e_1^+} e_1^+ = \alpha e_1^+, \quad \nabla_{e_1^+} e_2^+ = e_2^+, \quad \nabla_{e_2^+} e_1^+ = \nabla_{e_2^+} e_2^+ = 0.$$

Here $e_1^+ = x\dfrac{\partial}{\partial x}$ and $e_2^+ = x\dfrac{\partial}{\partial y}$ are the left invariant vector fields associated to e_1 and e_2, respectively. A description of left invariant flat affine structures over $\mathrm{Aff}(\mathbb{R})$ can be found in [18].

Example 3.4. *Dimension 3.* The **Heisenberg** Lie group of dimension 3 is given by the set of matrices

$$H_3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : \ x, y, z \in \mathbb{R} \right\}.$$

The Lie algebra of H_3 is identified with $\mathfrak{h}_3 = \mathrm{Vect}_{\mathbb{R}}\{e_1, e_2, e_3\}$ with Lie bracket $[e_1, e_2] = e_3$. The following is a left invariant flat affine connection on H_3:

$$\nabla_{e_1^+} = \nabla_{e_3^+} = 0, \quad \nabla_{e_2^+} e_1^+ = -e_3, \quad \nabla_{e_2^+} e_2^+ = e_1^+, \quad \nabla_{e_2^+} e_3^+ = 0.$$

The vector fields $e_1^+ = \dfrac{\partial}{\partial x}$, $e_2^+ = \dfrac{\partial}{\partial y} + x\dfrac{\partial}{\partial z}$, and $e_3^+ = \dfrac{\partial}{\partial z}$, denote the left invariant vector fields associated to e_1, e_2 and e_3, respectively.

Example 3.5. *Dimension 4.* The product manifold $\mathbb{R} \ltimes_\rho \mathbb{R}^3$ has the structure of a Lie group given by the semidirect product of the Abelian Lie group $(\mathbb{R}^3, +)$ with $(\mathbb{R}, +)$ via the Lie group homomorphism

$$\begin{aligned} \rho : \mathbb{R} &\longrightarrow \mathrm{GL}(\mathbb{R}^3) \\ t &\longmapsto \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Next we introduce a family of flat left invariant affine connections on $\mathbb{R} \ltimes_\rho \mathbb{R}^3$:

$$\nabla_{e_2^+} = \nabla_{e_3^+} = \nabla_{e_4^+} = 0, \quad \nabla_{e_1^+} e_1^+ = \alpha e_1^+,$$
$$\nabla_{e_2^+} e_2^+ = e_2^+, \quad \nabla_{e_1^+} e_3^+ = -e_3^+, \quad \nabla_{e_1^+} e_4^+ = 0,$$

for all $\alpha \in \mathbb{R}$. The vector fields

$$e_1^+ = \frac{\partial}{\partial t}, \quad e_2^+ = e^t \frac{\partial}{\partial x}, \quad e_3^+ = e^{-t} \frac{\partial}{\partial y}, \quad e_4^+ = \frac{\partial}{\partial z},$$

determine a basis for $\mathfrak{X}_l(\mathbb{R} \ltimes_\rho \mathbb{R}^3)$.

Recall that a Lie group G is called **semisimple** if its Lie algebra decomposes into a direct sum of simple Lie algebras. An interesting result, due to C. Chevalley and S. Eilenberg (see [6]) states that a Lie algebra $\mathfrak{g}$ is semisimple if and only if we have $H^1(\mathfrak{g}, \theta) = 0$ for every real representation θ of $\mathfrak{g}$ over a finite dimensional vector space. Accordingly, we have the following beautiful result of A. Bon-Yau Chu in [4].

Theorem 3.6 (Bon-Yau Chu)**.** *Let G be a real semisimple Lie group. Then G does not admit a left invariant flat affine connection.*

Proof. Let G be a semisimple real Lie group of dimension n and $\mathfrak{g}$ its Lie algebra. Since $\mathfrak{g}$ is semisimple, its derived ideal satisfies $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. This implies that every linear representation θ of $\mathfrak{g}$ on a finite dimensional vector space has trace $\mathrm{tr}(\theta(x)) = 0$ for all $x \in \mathfrak{g}$. Suppose that there exists a left invariant flat affine connection ∇ on G. Then, by Theorem 3.2, the map $L : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ defined by $x \mapsto L_x$, where $L_x : \mathfrak{g} \to \mathfrak{g}$ is the linear map given by $y \mapsto L_x(y) = x \cdot y = (\nabla_{x^+} y^+)(\epsilon)$, is a linear representation of $\mathfrak{g}$ on the vector space $\mathfrak{g}$. We denote by $C^p(\mathfrak{g}, L)$ and $H^p(\mathfrak{g}, L)$ the spaces of p-cochains and the p-th cohomology group of $\mathfrak{g}$ associated to the linear representation L, respectively. We define $\gamma \in C^1(\mathfrak{g}, L)$ by $\gamma(x) = x$ for all $x \in \mathfrak{g}$. Then, since ∇ is torsion free and left invariant, we have

$$\mathrm{d}\gamma(x, y) = L_x(\gamma(y)) - L_y(\gamma(x)) - \gamma([x, y]) = x \cdot y - y \cdot x - [x, y] = 0,$$

for all $x, y \in \mathfrak{g}$. Therefore, we have $\mathrm{d}\gamma = 0$. Since $\mathfrak{g}$ is semisimple, we obtain $H^1(\mathfrak{g}, L) = 0$. Hence, there exists $z \in C^0(\mathfrak{g}, L) = \mathfrak{g}$ such that $x = \gamma(x) = \mathrm{d}z(x) = L_x(z)$ for all $x \in \mathfrak{g}$. Once again, since the torsion tensor of ∇ is null we reach

$$x = L_x(z) = x \cdot z = z \cdot x - [z, x] = (L_z - \mathrm{ad}_z)(x),$$

which implies the relation $L_z = I + \mathrm{ad}_z$, where I and ad are the identity map and the adjoint representation of $\mathfrak{g}$, respectively. Since L and ad are linear representations of $\mathfrak{g}$, we obtain $0 = \mathrm{tr}(L_z) = \mathrm{tr}(I) + \mathrm{tr}(\mathrm{ad}_z) = \dim(\mathfrak{g}) = n$, which is a contradiction. □

Example 3.7. The special linear group $\mathrm{SL}(n,\mathbb{R})$, the special orthogonal group $\mathrm{SO}(n,\mathbb{R})$ and the symplectic linear group $\mathrm{Sp}(n,\mathbb{R})$ do not allow a structure of flat affine Lie group, given that they are semisimple.

4 Flat pseudo-Riemannian manifolds

Our next objective is to study left invariant flat affine structures over Lie groups in the case when these structures are compatible with a pseudo-Riemannian metric. To do so, we introduce the following structures from Riemannian geometry. Let M be a smooth connected paracompact manifold of real dimension n. For each $p \in M$, we denote by $L^2(T_pM,\mathbb{R})$ the set of all bilinear maps $\beta : T_pM \times T_pM \to \mathbb{R}$. Recall that the **index** ν of a symmetric bilinear form β on a real finite-dimensional vector space V is the largest integer that is the dimension of a subspace $W \subset V$ on which $\beta|_W$ is negative definite. Equivalently, if β is also non-degenerate, the index ν of V is the number of -1 in the diagonal of the matrix representation of β with respect to any orthonormal basis of V.

A **pseudo-metric** g on M is an assignment $p \mapsto g_p \in L^2(T_pM,\mathbb{R})$ such that the following conditions are met:

1. $g_p(X_p,Y_p) = g_p(Y_p,X_p)$ for all $X_p, Y_p \in T_pM$,
2. g_p is non-degenerate for all $p \in M$,
3. if $(U,(x^1,\cdots,x^n))$ is a chart of M, the coefficients g_{ij} of the local representation

$$g_p = \sum_{i,j=1}^{n} g_{ij}(p) \cdot dx^i|_p \otimes dx^j|_p,$$

   are smooth functions,
4. the index of g_p is the same for all $p \in M$.

In other words, a pseudo-metric is a field of tensors of type $(0,2)$ that is symmetric, non-degenerate and of constant index. The pair (M,g),

where g is a pseudo-metric on M, is called a **pseudo-Riemannian manifold**.

The common index ν of g_p in a pseudo-Riemannian manifold (M, g) is the **index** of M. When $\nu = 0$ we say that (M, g) is a **Riemannian manifold**. In such case g_p determines an inner product over T_pM for all $p \in M$. On the other hand, when $\nu = 1$ and $n \geq 2$ the pair (M, g) is called a **Lorentzian manifold**. In the first case, the signature of g is $(0, n)$ while in the second case $(1, n)$. A bilinear form over a finite dimensional real vector space that satisfies the first two conditions of our definition is called a **scalar product**. An inner product is a scalar product that is positive definite.

A linear connection ∇ on a pseudo-Riemannian manifold (M, g) is said to be **compatible with the pseudo-metric structure** of M if it satisfies $\nabla g = 0$, that is, if

$$X \cdot g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \tag{4.1}$$

for all $X, Y, Z \in \mathfrak{X}(M)$. The following result is usually called the **Fundamental theorem of pseudo-Riemannian Geometry**.

Theorem 4.1 (Levi-Civita)**.** *Given (M, g) a pseudo-Riemannian manifold, there exists a unique linear connection ∇ on M that is compatible with the pseudo-metric structure of M and has vanishing torsion tensor. Such a linear connection is characterized by the* **Koszul formula**

$$\begin{aligned} 2g(\nabla_X Y, Z) \quad &= X \cdot g(Y, Z) + Y \cdot g(Z, X) - Z \cdot g(X, Y) + \\ &\quad - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]), \end{aligned}$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

The linear connection of Theorem 4.1 is called the **Levi-Civita connection**. It is important to observe that the Koszul formula implies that the Christoffel symbols associated to the Levi-Civita connection satisfy the relation

$$\sum_{l=1}^{n} g_{lk} \Gamma^l_{ji} = \frac{1}{2} \left(\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ji}}{\partial x^k} \right), \tag{4.2}$$

for all $i, j, k = 1, \cdots, n$. When the curvature tensor of the Levi-Civita connection ∇ associated to a pseudo-Riemannian manifold (M, g) vanishes, the pseudo-metric g is called **flat**, and the pair (M, g) is a **flat pseudo-Riemannian manifold**.

The basic model of flat pseudo-Riemannian manifolds is the space $(\mathbb{R}^n_\nu, g_0^\nu, \nabla^0)$ where $\mathbb{R}^n_\nu$ equals $\mathbb{R}^n$ with pseudo-metric g_0^ν of index ν with $0 \leq \nu \leq n$, defined by

$$g_0^\nu = -\sum_{j=1}^{\nu} dx^j \otimes dx^j + \sum_{j=\nu+1}^{n} dx^j \otimes dx^j.$$

A simple computation shows that the usual linear connection ∇^0 of $\mathbb{R}^n$ is the Levi-Civita connection associated to g_0^ν. When $\nu = 0$, the pseudo-Riemannian manifold $\mathbb{R}^n_\nu$ reduces to $\mathbb{R}^n$. On the other hand, for $\nu = 1$ and $n \geq 2$, the manifold $\mathbb{R}^n_1$ is known as the **n-dimensional Minkowski space**. The Lorentzian manifold $(\mathbb{R}^4_1, g_0^1)$ is the basic model for relativistic space-time.

An **isometry** between two pseudo-Riemannian manifolds (M_1, g_1) and (M_2, g_2) is a diffeomorphism $f : M_1 \to M_2$ satisfying $f^* g_2 = g_1$, that is,

$$(g_2)_{F(p)}(F_{*,p}(X_p), F_{*,p}(Y_p)) = (g_1)_p(X_p, Y_p),$$

for all $X_p, Y_p \in T_p M_1$ with $p \in M_1$.

Remark 4.2. If (M, g) is a pseudo-Riemannian manifold and $f : M \to M$ is an isometry, the uniqueness of the Levi-Civita connection ∇ associated to g implies that f is an affine transformation of (M, ∇). More precisely, we have

$$f_*^{-1}\left(\nabla_{f_* X} f_* Y\right) = \nabla_X Y, \tag{4.3}$$

for all $X, Y \in \mathfrak{X}(M)$ (see [11, p. 161]).

If $O(n, \mathbb{R})$ denotes the linear orthogonal group, the **group of isometries** of $(\mathbb{R}^n, g_0)$ is the Lie group $\mathrm{OAff}(\mathbb{R}^n) = \mathbb{R}^n \rtimes_{Id} O(n, \mathbb{R})$ determined by the semi-direct product of the Abelian Lie group $(\mathbb{R}^n, +)$ and the orthogonal group $O(n, \mathbb{R})$ via the identity representation. An important

consequence of Theorem 2.5 to the case of Riemannian manifolds, which can be proven in a similar way, is the following result (see for instance [20]).

Proposition 4.3. *A real smooth manifold M of dimension n admits a flat Riemannian metric if and only if there exists an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in J}$ of M for which the changes of coordinates are restrictions of the elements of the group of isometries of $(\mathbb{R}^n, g_0)$; that is, for each $\alpha, \beta \in J$ with $U_\alpha \cap U_\beta \neq \emptyset$, there exists $\sigma_{\alpha\beta} \in OAff(\mathbb{R}^n)$ such that*

$$\varphi_\beta \circ \varphi_\alpha^{-1}\big|_{\varphi_\alpha(U_\alpha\cap U_\beta)} = \sigma_{\alpha\beta}|_{\varphi_\alpha(U_\alpha\cap U_\beta)}\,.$$

□

For the next examples, we denote by G_j the discrete subgroup of $\mathrm{OAff}(\mathbb{R}^2)$ which acts freely and properly discontinuously over $\mathbb{R}^2$, for $j = 1, 2, 3, 4$. Recall that in such a case the quotient manifold $\mathbb{R}^2/G_j$ admits an affine structure whose changes of coordinates are restrictions of elements of G_j (see [22, p. 349]). There exists four types of flat complete 2-dimensional Riemannian manifolds other than $(\mathbb{R}^2, g_0)$ they are given in the following example. See for instance [11, p. 209-224] for further details.

Example 4.4. *Ordinary cylinder.* Let G_1 be the set of transformations of $\mathbb{R}^2$ defined by

$$C_n(x, y) = (x + n, y), \quad \text{for all} \quad n \in \mathbb{Z}.$$

The quotient manifold $\mathbb{R}^2/G_1$ determined by the action of G_1 over $\mathbb{R}^2$ is diffeomorphic to the ordinary cylinder $S^1 \times \mathbb{R}$.

Example 4.5. *Ordinary torus.* Consider the set G_2 of transformations of $\mathbb{R}^2$ given by

$$T^{a,b}_{n,m}(x, y) = (x + ma + n, y + mb),$$

for all $n, m \in \mathbb{Z}$ and $a, b \in \mathbb{R}$, $b \neq 0$. The quotient manifold $\mathbb{R}^2/G_2$ determined by the action of G_2 over $\mathbb{R}^2$ is diffeomorphic to the ordinary torus.

Example 4.6. *Infinite Möbius band.* We denote by G_3 the set of transformations of $\mathbb{R}^2$ defined by

$$M_n(x, y) = (x + n, (-1)^n y),$$

for all $n \in \mathbb{Z}$. The quotient manifold $\mathbb{R}^2/G_3$ determined by the action of G_3 over $\mathbb{R}^2$ is diffeomorphic to the infinite Möbius band.

Example 4.7. *Klein bottle.* Let G_4 be the set of transformations of $\mathbb{R}^2$ given by

$$K^b_{n,m}(x, y) = (x + n, (-1)^n y + bm),$$

for all $n, m \in \mathbb{Z}$ and $b \in \mathbb{R}\backslash\{0\}$. The quotient manifold $\mathbb{R}^2/G_4$ determined by the action of G_4 over $\mathbb{R}^2$ is diffeomorphic to the Klein bottle.

The existence of partitions of unity for M helps us guaranty the existence of Riemannian metrics on M. Nevertheless, partitions of unity do not allow us to prove the existence of pseudo-metrics on M with index at least 1. In fact, there are topological obstructions to the existence of such pseudo-metrics. For example, a compact manifold M admits a Lorentzian metric if and only if its Euler characteristic $\chi(M)$ is equal to zero. This because in such cases we can guaranty the existence of a nowhere vanishing vector field on M (see [14] or [25, p. 207]). The only compact two dimensional surfaces satisfying this condition are the torus and the Klein bottle.

5 Flat pseudo-Riemannian Lie groups

Let G be a real connected Lie group of dimension n and $\mathfrak{g}$ its Lie algebra. The goal of this section is to discuss the open problem proposed by J. Milnor in [20] of describing left invariant flat affine structures in the case when G admits left invariant flat pseudo-metrics.

A pseudo-metric μ on G is called **left invariant** if $L^*_\sigma \mu = \mu$ for all $\sigma \in G$. In other words, μ is left invariant if L_σ is an isometry of (G, μ) for all $\sigma \in G$. The pair (G, μ), where μ is a left invariant pseudo-metric on G, is called a **pseudo-Riemannian Lie group**.

There is a faithful correspondence between left invariant pseudo-metrics on G and scalar products on $\mathfrak{g}$, depicted as follows. If μ is a left invariant pseudo-metric on G, then $\mu_\epsilon : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ defines a scalar product on $\mathfrak{g}$. On the other hand, given a scalar product $\mu_0 : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ for $\mathfrak{g}$, as a consequence of the chain rule, we can define a left invariant pseudo-metric μ on G by the formula

$$\mu_\sigma(X_\sigma, Y_\sigma) = \mu_0((L_{\sigma^{-1}})_{*,\sigma}(X_\sigma), (L_{\sigma^{-1}})_{*,\sigma}(Y_\sigma)), \tag{5.1}$$

for all $X_\sigma, Y_\sigma \in T_\sigma G$ with $\sigma \in G$. If (G, μ) is a pseudo-Riemannian Lie group, identity (4.3) implies that the Levi-Civita connection ∇ associated to μ is a left invariant linear connection. On the other hand, since μ is a left invariant pseudo-metric we get

$$\mu_\sigma(x^+_\sigma, y^+_\sigma) = \mu_\sigma((L_\sigma)_{*,\epsilon}(x), (L_\sigma)_{*,\epsilon}(y)) = \mu_\epsilon(x, y),$$

for all $x, y \in \mathfrak{g}$. This implies that the map $\mu(x^+, y^+) : G \to \mathbb{R}$ defined by $\sigma \mapsto \mu_\sigma(x^+_\sigma, y^+_\sigma)$, is constant for all $x^+, y^+ \in \mathfrak{X}_l(G)$. Therefore, putting $x \cdot y = L_x(y) = (\nabla_{x^+} y^+)(\epsilon)$ for $x, y \in \mathfrak{g}$, we have

$$[x, y] = x \cdot y - y \cdot x \quad \text{and} \tag{5.2}$$

$$\mu_\epsilon(L_x(y), z) + \mu_\epsilon(y, L_x(z)) = 0, \tag{5.3}$$

for $x, y, z \in \mathfrak{g}$. The bilinear map $\cdot : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ defined by $x \cdot y = L_x(y) = (\nabla_{x^+} y^+)(\epsilon)$ is called the **Levi-Civita product**. The Koszul formula implies that the Levi-Civita product is characterized by the expression

$$\mu_\epsilon(L_x(y), z) = \frac{1}{2}(\mu_\epsilon([x, y], z) - \mu_\epsilon([y, z], x) + \mu_\epsilon([z, x], y)), \tag{5.4}$$

where $x, y, z \in \mathfrak{g}$.

A pseudo-Riemannian Lie group (G, μ) is called **flat** if the curvature tensor of the Levi-Civita connection associated to μ is identically zero.

Let (V, μ_0) be a real finite-dimensional vector space with a scalar product μ_0. The group of **orthogonal transformations** of (V, μ_0), denoted by $O(V, \mu_0)$, is defined as the set of transformations $T : V \to V$

which satisfy $\mu_0(T(x), T(y)) = \mu_0(x, y)$ for all $x, y \in V$. It is a Lie group whose Lie algebra is the set $\mathfrak{o}(V, \mu_0)$ of endomorphisms $t : V \to V$ verifying the identity $\mu_0(t(x), y) + \mu_0(x, t(y)) = 0$ for all $x, y \in V$. The **group of isometries** of (V, μ_0), denoted by $\mathrm{OAff}(V)$, is defined as the semi-direct product $V \rtimes_{Id} O(V, \mu_0)$ of the Abelian Lie group $(V, +)$ and $O(V, \mu_0)$ via the identity representation.

Remark 5.1. If (G, μ) is a flat pseudo-Riemannian Lie group and ∇ is the Levi-Civita connection associated to μ, the map $L\colon \mathfrak{g} \to \mathfrak{o}(\mathfrak{g}, \mu_\epsilon)$ defined by $x \mapsto L_x$, where $L_x\colon \mathfrak{g} \to \mathfrak{g}$ is the linear map defined by $L_x(y) = (\nabla_{x^+} y^+)(\epsilon)$ for all $x, y \in \mathfrak{g}$, is a well defined Lie algebra homomorphism.

We can now have a first characterization of flat pseudo-Riemannian Lie groups as given by A. Aubert and A. Medina in [1].

Proposition 5.2 (Aubert-Medina)**.** *Let G be a real connected Lie group of dimension n, $\mathfrak{g}$ its Lie algebra, and $\widetilde{G}$ its universal covering Lie group. Then, the following are equivalent.*

1. *There exists a left invariant flat pseudo-metric on G.*
2. *There exist a scalar product $\mu_0 : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ and a bilinear map $\cdot : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ over $\mathfrak{g}$ such that* (5.2) *and* (5.3) *are satisfied together with $L_{[x,y]} = [L_x, L_y]_{\mathfrak{gl}(\mathfrak{g})}$, for all $x, y \in \mathfrak{g}$.*
3. *There exist a real n-dimensional vector space (V, μ_0) together with a scalar product μ_0 and a Lie group homomorphism $\rho : \widetilde{G} \to OAff(V, \mu_0)$ such that the left action of $\widetilde{G}$ over V defined by $\sigma \cdot v = \rho(\sigma)(v)$ for all $(\sigma, v) \in \widetilde{G} \times V$ admits a point with open orbit and discrete isotropy.*

Proof. We first prove that 1 implies 2. If (G, μ) is a flat pseudo - Riemannian Lie group and ∇ is the Levi-Civita connection associated to μ, from our previous observations we know that μ_ϵ and the Levi-Civita product associated to ∇ satisfy the required identities.

To see 2 implies 3 recall the proof of Theorem 3.2. By hypothesis, the map $\theta\colon \mathfrak{g} \to \mathfrak{g} \rtimes_{id} \mathfrak{o}(\mathfrak{g}, \mu_0)$ defined by $x \mapsto (x, L_x)$ for all $x \in \mathfrak{g}$ is a well defined Lie algebra homomorphism. Therefore, using the exponential map of G, we obtain a homomorphism of Lie groups $\rho\colon \widetilde{G} \to \mathrm{OAff}(\mathfrak{g}, \mu_0)$ for which $0 \in \mathfrak{g}$ is a point with open orbit and discrete isotropy for the left action of $\widetilde{G}$ over $\mathfrak{g}$ defined by $\sigma \cdot x = \rho(\sigma)(x)$ for all $(\sigma, x) \in \widetilde{G} \times \mathfrak{g}$.

Finally 3 implies 1. Let $\rho\colon \widetilde{G} \to \mathrm{OAff}(V, \mu_0)$ be a homomorphism of Lie groups, defined by $\sigma \mapsto (Q(\sigma), F_\sigma)$ for all $\sigma \in \widetilde{G}$, where (V, μ_0) is a real vector space of dimension n together with a scalar product μ_0 such that the orbital map $\pi\colon \widetilde{G} \to \mathrm{Orb}(v)$ defined by $\sigma \mapsto Q(\sigma) + F_\sigma(v)$ is a local diffeomorphism for some $v \in V$. Differentiating on the identity of $\widetilde{G}$, we obtain a Lie algebra homomorphism $\theta\colon \mathfrak{g} \to \mathfrak{g} \rtimes_{id} \mathfrak{o}(V, \mu_0)$ given by $x \mapsto (q(x), f_x)$, where the linear map $\psi_v\colon \mathfrak{g} \to V$ defined by $x \mapsto q(x) + f_x(v)$ is an isomorphism. Now, define on $\mathfrak{g}$ the scalar product $\widetilde{\mu_0}$ and the bilinear map $\cdot$ respectively by

$$\widetilde{\mu_0}(x, y) = \mu_0(\psi_v(x), \psi_v(y))$$

and

$$L_x = \psi_v^{-1} \circ f_x \circ \psi_v, \quad y \mapsto x \cdot y = L_x(y),$$

for all $x, y \in \mathfrak{g}$. If μ denotes the left invariant pseudo-metric on G induced by $\widetilde{\mu_0}$ through Formula (5.1), it is easy to check that $\cdot$ is the Levi-Civita product associated to the Levi-Civita connection determined by μ, given that we have $f_x \in \mathfrak{o}(V, \mu_0)$ for all $x \in \mathfrak{g}$. Therefore, as in the proof of Theorem 3.2 we conclude that μ is a left invariant flat pseudo-metric on G. □

The following result allows us to determine when the Levi-Civita connection associated to a left invariant flat pseudo-metric is geodesically complete. Recall that a linear connection ∇ over a smooth manifold M is **geodesically complete** if for any initial condition $(p, X_p) \in TM$ its geodesics are defined for all $t \in \mathbb{R}$. If (G, ∇) is a flat affine Lie group, J. Helmstetter showed in [8] that ∇ is geodesically complete if and only if $\mathrm{tr}(R_x) = 0$ for all $x \in \mathfrak{g}$, here $R_x\colon \mathfrak{g} \to \mathfrak{g}$ is the linear map defined by

$R_x(y) = (\nabla_{y^+} x^+)(\epsilon)$ for all $x, y \in \mathfrak{g}$. On the other hand, a Lie group G is called **unimodular** if its left invariant Haar measure is also right invariant. J. Milnor showed in [19] that a Lie group G is unimodular if and only if $\det(\mathrm{Ad}_\sigma) = \pm 1$ for all $\sigma \in G$. If G is connected, this is equivalent to requiring $\mathrm{tr}(\mathrm{ad}_x) = 0$ for all $x \in \mathfrak{g}$ (compare [19]). For the next result see [1].

Theorem 5.3 (Aubert-Medina)**.** *Let (G, μ) be a connected flat pseudo-Riemannian Lie group. Then the Levi-Civita connection associated to μ is geodesically complete if and only if G is unimodular.*

Proof. Let ∇ be the Levi-Civita connection associated to the left invariant flat pseudo-metric μ. We denote by $L_x(y) = x \cdot y = (\nabla_{x^+} y^+)(\epsilon)$ the Levi-Civita product on $\mathfrak{g}$ associated to ∇. If μ_ϵ is the scalar product on $\mathfrak{g}$ induced by μ, we have

$$\mu_\epsilon(L_x(y), z) + \mu_\epsilon(y, L_x(z)) = 0,$$

for all $x, y, z \in \mathfrak{g}$. This implies that L_x is antisymmetric with respect to μ_ϵ. Therefore, if $L_x^* \colon \mathfrak{g} \to \mathfrak{g}$ denotes the adjoint operator of L_x with respect to μ_ϵ, we have $L_x^* = -L_x$ for all $x \in \mathfrak{g}$. On the other hand, if $\mathfrak{g}^*$ denotes the dual space associated to $\mathfrak{g}$ and ${}^t L_x \colon \mathfrak{g}^* \to \mathfrak{g}^*$ is the transposed of the linear map L_x, the linear isomorphism $\varphi \colon \mathfrak{g} \to \mathfrak{g}^*$ defined by $\varphi(x) = \mu_\epsilon(x, \cdot)$ where $\varphi(x)(y) = \mu_\epsilon(x, y)$ for all $x, y \in \mathfrak{g}$, fits into the following commutative diagram for all $x \in \mathfrak{g}$

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{L_x^*} & \mathfrak{g} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ \mathfrak{g}^* & \xrightarrow[{}^t L_x]{} & \mathfrak{g}^*; \end{array}$$

thus, we have $L_x^* = \varphi^{-1} \circ^t L_x \circ \varphi$ and so

$$-\mathrm{tr}(L_x) = \mathrm{tr}(L_x^*) = \mathrm{tr}(\varphi^{-1} \circ^t L_x \circ \varphi) = \mathrm{tr}({}^t L_x) = \mathrm{tr}(L_x),$$

for all $x \in \mathfrak{g}$. This implies $\mathrm{tr}(L_x) = 0$ for $x \in \mathfrak{g}$.

Now suppose that ∇ is geodesically complete. Since this is a left invariant flat affine connection, we have $\mathrm{tr}(R_x) = 0$, where $R_x \colon \mathfrak{g} \to \mathfrak{g}$ is the linear map defined by $R_x(y) = y \cdot x$ for all $x, y \in \mathfrak{g}$. On the other hand, identity (5.2) implies $\mathrm{ad}_x = L_x - R_x$, and therefore we get

$$\mathrm{tr}(\mathrm{ad}_x) = \mathrm{tr}(L_x - R_x) = \mathrm{tr}(L_x) - \mathrm{tr}(R_x) = 0,$$

for all $x \in \mathfrak{g}$, which shows that G is a unimodular Lie group.

Reciprocally, if G is a unimodular Lie group, we have $\mathrm{tr}(\mathrm{ad}_x) = 0$ for all $x \in \mathfrak{g}$. As we have $\mathrm{ad}_x = L_x - R_x$ and $\mathrm{tr}(L_x) = 0$, we obtain $\mathrm{tr}(R_x) = 0$ for all $x \in \mathfrak{g}$. From the fact that ∇ is a left invariant flat affine connection and $\mathrm{tr}(R_x) = 0$, we get that it is geodesically complete. □

Example 5.4. The group of affine transformations of the line $\mathrm{Aff}(\mathbb{R})$ has a natural left invariant flat Lorentzian metric given by $\mu = \dfrac{1}{x^2}(dx \otimes dy + dy \otimes dx)$. The Levi-Civita connection associated to μ is determined by the rules

$$\nabla_{e_1^+} e_1^+ = -e_1^+, \quad \nabla_{e_1^+} e_2^+ = e_2^+, \quad \nabla_{e_2^+} e_1^+ = \nabla_{e_2^+} e_2^+ = 0.$$

Since $\mathrm{Aff}(\mathbb{R})$ is not unimodular, we have that ∇ is not geodesically complete. The natural left invariant Riemannian metric on $\mathrm{Aff}(\mathbb{R})$ given by $\overline{\mu} = \dfrac{1}{x^2}(dx \otimes dx + dy \otimes dy)$ is also not flat. The Levi-Civita connection associated to $\overline{\mu}$ is determined by

$$\overline{\nabla}_{e_1^+} e_1^+ = \overline{\nabla}_{e_1^+} e_2^+ = 0, \quad \overline{\nabla}_{e_2^+} e_1^+ = -e_2^+, \quad \overline{\nabla}_{e_2^+} e_2^+ = e_1^+.$$

It is easy to verify that the curvature tensor of $\overline{\nabla}$ is not identically zero. As a consequence of Theorem 7.9 (of next section) it is possible to show that there does not exist left invariant flat Riemannian metrics on $\mathrm{Aff}(\mathbb{R})$.

Example 5.5. Over the Heisenberg group H_3, we can define a left invariant flat Lorentzian metric by

$$\mu = dx \otimes dz + dy \otimes dy + dz \otimes dx - x(dx \otimes dy + dy \otimes dx).$$

The Levi-Civita connection associated to μ is determined by

$$\nabla_{e_1^+} = \nabla_{e_3^+} = 0, \quad \nabla_{e_2^+} e_1^+ = -e_3, \quad \nabla_{e_2^+} e_2^+ = e_1^+, \quad \nabla_{e_2^+} e_3^+ = 0.$$

Since H_3 is unimodular, we have that ∇ is geodesically complete. If H_{2n+1} denotes the Heisenberg group of dimension $2n+1$ for $n \in \mathbb{N}$, then H_{2n+1} is a flat pseudo-Riemannian Lie group if and only if $n = 1$ (see [1]).

6 The classical pseudo-Riemannian cotangent Lie groups of connected flat affine Lie groups

A simple construction that allows us to obtain flat pseudo-Riemannian Lie groups starting out with connected flat affine Lie groups is the following (see [1]).

Let (G, ∇) be a connected affine flat Lie group of dimension n, $\mathfrak{g}$ its Lie algebra, and $\widetilde{G}$ its universal covering Lie group. Since ∇ is a left invariant flat affine connection, the map $L\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ defined by $x \mapsto L_x$, where $L_x(y) = x \cdot y = (\nabla_{x^+} y^+)(\epsilon)$ for all $x, y \in \mathfrak{g}$, is a Lie algebra homomorphism. The dual representation associated to L is the Lie algebra homomorphism $L^*\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}^*)$, defined by $x \mapsto L_x^* = -{}^t L_x$, where $L_x^*(\alpha) = -\alpha \circ L_x$ for all $\alpha \in \mathfrak{g}^*$. Using the exponential map of G, we obtain a Lie group homomorphism $\Phi\colon \widetilde{G} \to \mathrm{GL}(\mathfrak{g}^*)$ via $\exp_G(x) \mapsto \Phi(\exp_G(x)) = \sum_{k=0}^{\infty} \frac{1}{k!} (L_x^*)^k$, namely

$$\Phi_{*,\widetilde{\epsilon}}(x) = \left. \frac{d}{dt} \right|_{t=0} (\Phi(\exp(tx))) = L_x^*,$$

for all $x \in \mathrm{Lie}(\widetilde{G}) = \mathfrak{g}$. Therefore, the product manifold $T^*\widetilde{G} = \widetilde{G} \times \mathfrak{g}^*$ is endowed with the structure of a Lie group given by the semidirect product of $\widetilde{G}$ with the Abelian Lie group $(\mathfrak{g}^*, +)$ through Φ; more precisely

we have

$$(\sigma, \alpha) \cdot (\tau, \beta) = (\sigma\tau, \Phi(\sigma)(\beta) + \alpha),$$

for all $\sigma, \tau \in G$ and $\alpha, \beta \in \mathfrak{g}^*$. The Lie group $T^*\widetilde{G} = \widetilde{G} \ltimes_\Phi \mathfrak{g}^*$ is called the **classical pseudo-Riemannian cotangent Lie group** associated to the flat affine connected Lie group (G, ∇).

Here the term 'classical' stands in contrast to the more general construction of twisted cotangent Lie groups as used by A. Aubert and A. Medina (see [1]). The Lie group $T^*\widetilde{G}$ is then characterized by the following result.

Proposition 6.1 (Aubert-Medina)**.** *The Lie algebra of $T^*\widetilde{G} = \widetilde{G} \ltimes_\Phi \mathfrak{g}^*$ is the product vector space $\mathfrak{g} \ltimes_L \mathfrak{g}^*$ with Lie bracket*

$$[(x, \alpha), (y, \beta)] = ([x, y], L^*_x(\beta) - L^*_y(\alpha)), \tag{6.1}$$

for all $x, y \in \mathfrak{g}$ and $\alpha, \beta \in \mathfrak{g}^*$. Moreover,

$$\widetilde{\omega}((x, \alpha), (y, \alpha)) = \alpha(y) + \beta(x), \tag{6.2}$$

for all $x, y \in \mathfrak{g}$ and $\alpha, \beta \in \mathfrak{g}^*$, is a scalar product over $\mathfrak{g} \ltimes_L \mathfrak{g}^*$ with signature (n, n) which, by formula (5.1)*, defines a left invariant flat pseudo-metric $T^*\widetilde{G}$ whose Levi-Civita connection is determined by*

$$\nabla_{(x,\alpha)^+}(y, \alpha)^+ = (x \cdot y, L^*_x(\beta))^+,$$

for all $x, y \in \mathfrak{g}$ and $\alpha, \beta \in \mathfrak{g}^*$.

Proof. As the Lie group structure of $T^*\widetilde{G}$ is given by a semidirect product, it is simple to check that its Lie algebra is the vector space $\mathfrak{g} \ltimes_L \mathfrak{g}^*$ with Lie bracket given by (6.1). On the other hand, as the Levi-Civita connection is unique, the proof of the last statement is an immediate consequence of Proposition 5.2. □

A more general construction appears in the study of the twisted pseudo-Riemannian cotangent Lie group of a connected flat affine Lie group (see [1, Proposition 2.1]).

Remark 6.2. If G is simply connected flat affine Lie group, then $T^*\widetilde{G}$ is a trivial vector bundle isomorphic to the cotangent bundle T^*G of G. It is well known that there is a natural way to associate a structure of Lie group to T^*G given that it is isomorphic to the trivial bundle $G \times \mathfrak{g}^*$ through the vector bundle isomorphism

$$\begin{aligned} T^*G &\to G \times \mathfrak{g}^* \\ (\sigma, \alpha_\sigma) &\longmapsto (\sigma, \alpha_\sigma \circ (L_\sigma)_{*,\epsilon}). \end{aligned}$$

If $\mathrm{Ad}^*\colon G \to \mathrm{GL}(\mathfrak{g}^*)$ denotes the co-adjoint representation of G, then the product manifold $G \times \mathfrak{g}^*$ has a Lie group structure given by the semi-direct product of G with the Abelian Lie group $\mathfrak{g}^*$ through Ad^*; consequently we have

$$(\sigma, \alpha) \cdot (\tau, \beta) = (\sigma\tau, \mathrm{Ad}^*_\sigma(\beta) + \alpha), \tag{6.3}$$

for all $\sigma, \tau \in G$ and $\alpha, \beta \in \mathfrak{g}^*$. Therefore, T^*G has the structure of a Lie group induced by (6.3). If $\mathrm{ad}^*\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}^*)$ denotes the co-adjoint representation of $\mathfrak{g}$, the Lie algebra of T^*G is the product vector space $\mathfrak{g} \ltimes_{ad^*} \mathfrak{g}^*$ with Lie bracket

$$[(x, \alpha), (y, \beta)] = ([x, y]_\mathfrak{g}, \mathrm{ad}^*_x(\beta) - \mathrm{ad}^*_y(\alpha)), \tag{6.4}$$

for all $x, y \in \mathfrak{g}$ and $\alpha, \beta \in \mathfrak{g}^*$. As G is a simply connected Lie group, then it is elementary to verify that $T^*\widetilde{G}$ is locally isomorphic to the cotangent bundle T^*G as Lie groups if and only if the maps $L^*\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}^*)$ and $\mathrm{ad}^*\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}^*)$ are isomorphic representations, that is, there exists a linear isomorphism $\psi : \mathfrak{g}^* \to \mathfrak{g}^*$ such that $\mathrm{ad}^*_x \circ \psi = \psi \circ L^*_x$ for all $x \in \mathfrak{g}$. In this case, the linear map $l : \mathfrak{g} \ltimes_L \mathfrak{g}^* \to \mathfrak{g} \ltimes_{\mathrm{ad}^*} \mathfrak{g}^*$ defined by $(x, \alpha) \mapsto (x, \psi(\alpha))$ is a Lie algebra isomorphism.

7 Orthogonal Lie groups

In this section we will study some elementary properties of those Lie groups which have bi-invariant pseudo-metrics. These will be called

orthogonal Lie groups (see the definition a few lines down). To study the main characteristic of orthogonal Lie groups we introduce the notion of **orthogonal Lie algebra** which we will be used in the method of **double orthogonal extension** described by A. Medina and Ph. Revoy (see [17]).

We describe how to construct the oscillator Lie algebra of the oscillator Lie group which appears in various branches of Physics and Mathematical Physics and give rise to particular solutions of the Einstein-Yang-Mills equations (see [13]). Finally, we will provide another characterization of flat Riemannian Lie groups due to J. Milnor (compare [19]).

For each $\sigma \in G$, we denote by $R_\sigma \colon G \to G$ the right multiplications by σ in G, which is defined by $R_\sigma(\tau) = \tau\sigma$ for all $\tau \in G$.

A pseudo-metric μ on G is **right invariant** if $R_\sigma^* \mu = \mu$ for all $\sigma \in G$. In other words, μ is right invariant if R_σ is an isometry of (G, μ) for all $\sigma \in G$. A pseudo-metric μ on G is called **bi-invariant** if it is left invariant and right invariant. The pair (G, μ), where μ is a bi-invariant pseudo-metric over G, is called an **orthogonal Lie group**.

If μ is a left invariant pseudo-metric on G, it is easy to show that μ is right invariant if and only if

$$\mu_\epsilon(\mathrm{Ad}_\sigma(x), \mathrm{Ad}_\sigma(y)) = \mu_\epsilon(x, y) \tag{7.1}$$

holds for all $\sigma \in G$ and $x, y \in \mathfrak{g}$. This implies in the context that μ is right invariant if and only if the adjoint representation of $\mathfrak{g}$ is antisymmetric with respect to μ_ϵ. The latter implies $\mathrm{ad}_x \in \mathfrak{o}(\mathfrak{g}, \mu_\epsilon)$ for all $x \in \mathfrak{g}$, namely

$$\mu_\epsilon([x, y], z) + \mu_\epsilon(y, [x, z]) = 0, \tag{7.2}$$

for all $x, y, z \in \mathfrak{g}$.

A scalar product over $\mathfrak{g}$ which satisfies identity (7.2) is called an **invariant scalar product**. A pair $(\mathfrak{g}, \mu_0)$, where $\mathfrak{g}$ is a finite dimensional real Lie algebra and μ_0 is an invariant scalar product over $\mathfrak{g}$ is named an **orthogonal Lie algebra**.

If μ_0 is an invariant scalar product over $\mathfrak{g}$, the left invariant pseudo-metric defined by the formula (5.1) is also right invariant. On the other hand, if (G, μ) is an orthogonal Lie group, the Koszul formula reduced at the identity (5.4) and expression (7.2) imply that the Levi-Civita connection ∇ associated to μ is determined by

$$\nabla_{x^+} y^+ = \frac{1}{2}[x, y]^+, \tag{7.3}$$

for $x, y \in \mathfrak{g}$. Moreover, as a consequence of the Jacobi identity in $\mathfrak{g}$, it follows that the curvature tensor of ∇ is given by the expression

$$R_\nabla(x^+, y^+)z^+ = -\frac{1}{4}[[x, y], z]^+, \tag{7.4}$$

with $x, y, z \in \mathfrak{g}$.

Remark 7.1. A left invariant linear connection on G is called a **Cartan 0-connection** if for all $x \in \mathfrak{g}$, the 1-parameter subgroups of G and the geodesic curves of ∇ determined by the initial condition $(\epsilon, x) \in G \times \mathfrak{g}$ coincide. It is easy to see that every Cartan 0-connection is geodesically complete. Moreover, there exists a unique Cartan 0-connection on G with vanishing torsion, as it is completely determined by Equation (7.3) (see [21, p. 72]).

As an immediate consequence of Identity (7.4) we have the following result (see for instance [1]).

Proposition 7.2 (Aubert-Medina)**.** *Let (G, μ) be an orthogonal Lie group. The bi-invariant pseudo-metric μ is flat if and only if G is a 2-nilpotent Lie group.* □

Example 7.3. *Semisimple Lie groups.* Let G be a semisimple Lie group. It is well known that G is a semisimple if and only if the *Killing form* of $\mathfrak{g}$, which we denote by $k\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, defined by $(x, y) \mapsto k(x, y) = \operatorname{tr}(\operatorname{ad}_x \circ \operatorname{ad}_y)$ for $x, y \in \mathfrak{g}$, is non-degenerate. Direct computation shows

$$k([x, y], z) = -k(y, [x, z]),$$

for $x, y, z \in \mathfrak{g}$. Therefore $(\mathfrak{g}, k)$ is an orthogonal Lie algebra.

Example 7.4. *The cotangent bundle of a Lie group.* Let G be a real connected n-dimensional Lie group, $\mathfrak{g}$ its Lie algebra, T^*G the cotangent bundle of G, and $\mathfrak{g}^*$ the dual vector space of $\mathfrak{g}$. Recall that T^*G is isomorphic to the trivial bundle $G \times \mathfrak{g}^*$ and this is endowed with a natural Lie group structure given by

$$(\sigma, \alpha) \cdot (\tau, \beta) = (\sigma\tau, \mathrm{Ad}^*_\sigma(\beta) + \alpha), \tag{7.5}$$

for all $\sigma, \tau \in G$ and $\alpha, \beta \in \mathfrak{g}^*$. Therefore, the Lie algebra of T^*G is the product vector space $\mathfrak{g} \ltimes_{ad^*} \mathfrak{g}^*$ with Lie bracket

$$[(x, \alpha), (y, \beta)] = ([x, y]_\mathfrak{g}, \mathrm{ad}^*_x(\beta) - \mathrm{ad}^*_y(\alpha)), \tag{7.6}$$

for all $x, y \in \mathfrak{g}$ and $\alpha, \beta \in \mathfrak{g}^*$.

We define over $\mathfrak{g} \ltimes_{ad^*} \mathfrak{g}^*$ the function

$$\mu_0((x, \alpha), (y, \beta)) = \alpha(y) + \beta(x), \tag{7.7}$$

for all $x, y \in \mathfrak{g}$ and $\alpha, \beta \in \mathfrak{g}^*$. It is easy to see that μ_0 defines an invariant scalar product over $\mathfrak{g} \ltimes_{ad^*} \mathfrak{g}^*$, of signature (n, n), so that $(\mathfrak{g} \ltimes_{ad^*} \mathfrak{g}^*, \mu_0)$ is an orthogonal Lie algebra.

Example 7.5. *Oscillator Lie group.* For $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$ with $0 < \lambda_1 \leq \cdots \leq \lambda_n$, the λ-oscillator Lie group, denoted by G_λ, is determined by the product manifold $\mathbb{R}^{2n+2} \cong \mathbb{R} \times \mathbb{R} \times \mathbb{C}^n$ endowed with the product

$$(t, s, z_1, \cdots, z_n) \cdot (t', s', z'_1, \cdots, z'_n) =$$

$$= \left(t + t', s + s' + \frac{1}{2} \sum_{j=1}^{n} \mathrm{Im}(\overline{z_j} z'_j e^{i\lambda_j t}), z_1 + z'_1 e^{i\lambda_1 t}, \cdots, z_n + z'_n e^{i\lambda_n t} \right),$$

where $t, t', s, s' \in \mathbb{R}$ y $z_j, z'_j \in \mathbb{C}$ for all $j = 1, \cdots, n$. The Lie algebra of G_λ, denoted by $\mathfrak{g}_\lambda$, is isomorphic to the vector space $\mathbb{R}e \times \mathbb{R}^{2n} \times \mathbb{R}\hat{e} = \mathrm{Vect}_\mathbb{R}\{e, e_j, \hat{e_j}, \hat{e}\}_{j=1,\cdots,n}$, with Lie bracket

$$[e, e_j] = \lambda_j \hat{e_j}, \qquad [e, \hat{e_j}] = -\lambda_j e_j, \qquad [e_j, \hat{e_j}] = \hat{e},$$

for all $j = 1, \cdots n$.

If $x = \alpha e + \sum_{j=1}^{n} x_j e_j + \sum_{j=1}^{n} y_j \hat{e_j} + \beta \hat{e}$ denotes an element of $\mathfrak{g}_\lambda$, the function μ_0 defined over $\mathfrak{g}_\lambda \times \mathfrak{g}_\lambda$ by

$$\mu_0(x, y) = \sum_{j=1}^{n} \frac{1}{\lambda_j}(x_j x'_j + y_j y'_j) + \alpha\beta' + \alpha'\beta$$

is an invariant scalar product over $\mathfrak{g}_\lambda$. This allow us to conclude that G_λ is an orthogonal Lie group. The signature of the scalar product μ_0 is $(1, 2n+1)$ so that it determines, by means of Formula (5.1), a bi-invariant Lorentzian metric over G_λ.

Remark 7.6. The λ-oscillators Lie groups are the only solvable simply connected non-Abelian Lie groups that admit a bi-invariant Lorentzian metric (see [16]). The oscillator 4-dimensional Lie group has its origin in the study of the harmonic oscillator which is one of the simplest non-relativistic systems where the Schrödinger equation can be completely solved. Moreover, oscillator Lie groups are particular solutions to the Einstein-Yang-Mills equations (see [13]). Over oscillator Lie groups there exist infinitely many solutions to the Yang-Baxter equations (see [3]).

Example 7.7. *A non-orthogonal Lie group: Aff*$(\mathbb{R})$. The Lie group of affine transformations of the line $\mathrm{Aff}(\mathbb{R})$ is a classical example of a non-orthogonal Lie Group. If there were an invariant scalar product μ_0 over $\mathfrak{aff}(\mathbb{R})$, we will get

$$\mu_0([x, y], z) + \mu_0(y, [x, z]) = 0,$$

for all $x, y, z \in \mathfrak{g}$. If we replace here $x = e_1$, $y = e_2$ and $z = e_1$, we obtain $\mu_0(e_1, e_2) = 0$. On the other hand, if we replace $x = e_1$, $y = e_2$ and $z = e_2$ we get $\mu_0(e_2, e_2) = 0$. Therefore, the element e_2 is orthogonal with respect to μ_0 to all elements of $\mathfrak{aff}(\mathbb{R})$, which contradicts the fact that μ_0 is non-degenerate. The bottom line is that there does not exist an invariant scalar product over $\mathfrak{aff}(\mathbb{R})$.

If G is a compact Lie group, using the Haar measure on G we can construct a bi-invariant Riemannian metric over G (see [21, p. 340]). In further generality, a connected Lie group G admits a bi-invariant metric if and only if it is isomorphic to the Cartesian product of a compact group and an additive vector group (see [19]). On the other hand, if $(\mathfrak{g}, \mu_0)$ is an orthogonal Lie algebra with an invariant inner product μ_0 and $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ for each y in the orthogonal complement $\mathfrak{h}^{\perp_{\mu_0}}$ of $\mathfrak{h}$ with respect to μ_0, we have

$$\mu_0([x, y], h) = -\mu_0(y, [x, h]) = 0,$$

for $x \in \mathfrak{g}$ and $h \in \mathfrak{h}$. This implies that $\mathfrak{h}^{\perp_{\mu_0}}$ is also an ideal of $\mathfrak{g}$. Therefore, by induction, have shown that $\mathfrak{g}$ can be expressed as an orthogonal direct sum of simple ideals (see [19]).

Remark 7.8. K. Iwasawa showed in [9] that if G is a connected Lie group, then every compact subgroup is contained in a maximal compact subgroup H, which is also connected. Moreover, topologically G is isomorphic to the Cartesian product of H with an Euclidean space $\mathbb{R}^k$. For (G, μ) a flat Riemannian Lie group, if we ignore for a moment the group structure of G and think of it just as a Riemannian manifold, we have that G is isometric to Euclidean space. Therefore, as a consequence of Iwasawa's theorem, every compact subgroup of (G, μ) is commutative (see [19]).

The following characterization of flat Riemannian Lie groups is due to J. Milnor (see [19]).

Theorem 7.9 (Milnor)**.** *Let (G, μ) be a Riemannian Lie group. The metric μ is flat if and only if the Lie algebra $\mathfrak{g}$ decomposes as an orthogonal direct sum $\mathfrak{b} \oplus \mathfrak{u}$, where $\mathfrak{b}$ is an Abelian subalgebra and $\mathfrak{u}$ is an Abelian ideal such that the linear maps ad_b are antisymmetric with respect to μ_ϵ for all $b \in \mathfrak{b}$.*

Proof. Suppose that (G, μ) is a flat Riemannian Lie group. If ∇ is the Levi-Civita connection associated to μ, then by Proposition 5.2, we

know that the linear map $L\colon \mathfrak{g} \to \mathfrak{o}(\mathfrak{g}, \mu_\epsilon)$ defined by $x \mapsto L_x$, where $L_x(y) = x \cdot y = (\nabla_{x^+} y^+)(\epsilon)$ for all $x, y \in \mathfrak{g}$, is a well defined Lie algebra homomorphism. We denote by $\mathfrak{u}$ the kernel of L. Clearly $\mathfrak{u}$ is an ideal of $\mathfrak{g}$. Since the torsion tensor of ∇ vanishes, we have $[x, y] = L_x(y) - L_y(x)$ for all $x, y \in \mathfrak{g}$. In particular, for $u, v \in \mathfrak{u}$ we have $[u, v] = 0$ and it follows that $\mathfrak{u}$ is an Abelian ideal. Let $\mathfrak{b}$ be the orthogonal complement of $\mathfrak{u}$ with respect to μ_ϵ. For each $b \in \mathfrak{b}$ we have the identity

$$\mathrm{ad}_b(u) = [b, u] = L_b(u) - L_u(b) = L_b(u),$$

for $u \in \mathfrak{u}$. Given that $\mathfrak{u}$ is an ideal of $\mathfrak{g}$, the linear map L_b takes $\mathfrak{u}$ onto itself. Therefore, L_b takes the orthogonal complement $\mathfrak{b}$ to itself, and since this is true for all $b \in \mathfrak{b}$, we conclude that $\mathfrak{b}$ is a Lie subalgebra of $\mathfrak{g}$. On the other hand, since L is a Lie algebra homomorphism and $\mathfrak{u} = \mathrm{Ker}(L)$, we have that $\mathfrak{b}$ is sent isomorphically to a Lie subalgebra $L(\mathfrak{b})$ of $\mathfrak{o}(\mathfrak{g}, \mu_\epsilon)$. For simplicity, we denote $L(\mathfrak{b})$ also by $\mathfrak{b}$. Given that $\mathfrak{o}(\mathfrak{g}, \mu_\epsilon)$ is the Lie algebra of the compact Lie group $O(\mathfrak{g}, \mu_\epsilon)$, which admits a bi-invariant Riemannian metric, we deduce the existence of an invariant inner product μ_0 over $\mathfrak{o}(\mathfrak{g}, \mu_\epsilon)$. Since $\mathfrak{b}$ is a Lie subalgebra of $\mathfrak{o}(\mathfrak{g}, \mu_\epsilon)$, it is easy to verify that μ_0 restricts naturally to an invariant inner product over $\mathfrak{b}$. Therefore, $\mathfrak{b}$ can be written as an orthogonal direct sum $\mathfrak{b}_1 \oplus \cdots \oplus \mathfrak{b}_k$ of simple ideals. If any of these simple ideal, say $\mathfrak{b}_j$, were non Abelian, then the corresponding simple Lie group G_j must be compact (see [19, Thm 2.2]) and the inclusion $\mathfrak{b}_j \subset \mathfrak{b} \subset \mathfrak{g}$ would imply the existence of a nontrivial Lie group homomorphism $G_j \to G$. Hence, G must contain a non-trivial compact subgroup, which is a contradiction. Therefore, each $\mathfrak{b}_j$ must be Abelian and accordingly $\mathfrak{b}$ is an Abelian Lie subalgebra. Finally, since for each $b \in \mathfrak{b}$ the restriction of ad_b to $\mathfrak{b}$ is the trivial map, whereas we have $\mathrm{ad}_b = L_b$ when restricting ad_b to $\mathfrak{u}$, we obtain $\mathrm{ad}_b \in \mathfrak{o}(\mathfrak{g}, \mu_\epsilon)$ for all $b \in \mathfrak{b}$.

Reciprocally, suppose that the Lie algebra $\mathfrak{g}$ decomposes as an orthogonal direct sum $\mathfrak{b} \oplus \mathfrak{u}$, where $\mathfrak{b}$ is an Abelian subalgebra and $\mathfrak{u}$ is an Abelian ideal such that $\mathrm{ad}_b \in \mathfrak{o}(\mathfrak{g}, \mu_\epsilon)$ for all $b \in \mathfrak{b}$. As μ_ϵ is non-degenerate, the Koszul formula reduced to the identity (5.4) and both

formulas (5.2) and (5.2) imply that the Levi-Civita product associated to μ_ϵ satisfies the identities

$$L_u = 0, \qquad L_b = \mathrm{ad}_b,$$

for all $u \in \mathfrak{u}$ and $b \in \mathfrak{b}$. It is easy to verify that this implies $L_{[x,y]} = [L_x, L_y]_{\mathfrak{gl}(\mathfrak{g})}$ for all $x, y \in \mathfrak{g}$. Therefore, by Proposition 5.2 we have that μ is a left-invariant flat Riemannian metric. □

Example 7.10. *Aff*$(\mathbb{R})$ *does not admit a left-invariant flat Riemannian metric.* Recall that the Lie Algebra of $\mathrm{Aff}(\mathbb{R})$ is $\mathfrak{aff}(\mathbb{R}) = \mathrm{Vect}_{\mathbb{R}}\{e_1, e_2\}$ with Lie bracket $[e_1, e_2] = e_2$. Suppose that $\mathrm{Aff}(\mathbb{R})$ admits a left-invariant flat Riemannian metric μ. Let ∇ be the Levi-Civita connection associated to μ and $L_x(y) = (\nabla_{x^+} y^+)(\epsilon)$ the Levi-Civita product determined by ∇. By Theorem 7.9 $\mathfrak{aff}(\mathbb{R})$ decomposes as an orthogonal direct sum $\mathfrak{b} \oplus \mathfrak{u}$ where $\mathfrak{u} = \mathrm{Ker}(L)$ is an Abelian ideal of $\mathfrak{aff}(\mathbb{R})$ and $\mathfrak{b}$ is an Abelian subalgebra of $\mathfrak{aff}(\mathbb{R})$ such that $\mathrm{ad}_b \in \mathfrak{o}(\mathfrak{aff}(\mathbb{R}), \mu_\epsilon)$ for all $b \in \mathfrak{b}$. Given these conditions, it is clear that we have $\mathfrak{u} = \mathbb{R}e_2$ and $\mathfrak{b} = \mathbb{R}e_1$. Therefore, from $\mathrm{ad}_{e_1} \in \mathfrak{o}(\mathfrak{aff}(\mathbb{R}), \mu_\epsilon)$ we get $\mu_\epsilon(e_1, e_2) = 0$, and since

$$e_2 = [e_1, e_2] = L_{e_1}(e_2) - L_{e_2}(e_1) = L_{e_1}(e_2),$$

the condition $L_{e_1} \in \mathfrak{o}(\mathfrak{aff}(\mathbb{R}), \mu_\epsilon)$ implies $\mu_\epsilon(e_2, e_2) = 0$. Consequently, e_2 is orthogonal to every element of $\mathfrak{aff}(\mathbb{R})$ with respect to μ_ϵ, which contradicts the fact that μ_ϵ is non-degenerate.

8 The double orthogonal extension

In what follows we describe a construction method known by the name of **double orthogonal extension** which is due to A. Medina and Ph. Revoy (compare [17]). This method provides, among other things, a way to construct all finite-dimensional orthogonal Lie algebras. As an application of the double orthogonal extension we indicate how to construct the Lie algebra of the λ-oscillator Lie group.

Given an orthogonal Lie algebra $(\mathfrak{g}, \mu_0)$, the space of **skew - symmetric derivations** of $\mathfrak{g}$ with respect to μ_0, denoted by $\mathrm{Der}_a(\mathfrak{g}, \mu)$, is defined as the set of derivations $D\colon \mathfrak{g} \to \mathfrak{g}$ that verify $\mu_0(D(x), y) = -\mu_0(x, D(y))$ for all $x, y \in \mathfrak{g}$. It is easy to check that $\mathrm{Der}_a(\mathfrak{g}, \mu)$ is a Lie subalgebra of $\mathrm{Der}(\mathfrak{g})$. Suppose that there exists a Lie algebra homomorphism $\psi\colon \mathfrak{h} \to \mathrm{Der}_a(\mathfrak{g}, \mu)$ for some Lie algebra $\mathfrak{h}$. Define the map $\Phi\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{h}^*$ by $\Phi(x, y)(z) = \mu_0(\psi_z(x), y)$ for $x, y \in \mathfrak{g}$ and $z \in \mathfrak{h}$. Such a map is clearly bilinear. Moreover, since $\psi_z \in \mathrm{Der}_a(\mathfrak{g}, \mu)$ for all $z \in \mathfrak{h}$, we have that Φ is skew-symmetric and satisfies

$$\Phi([x, y], w) + \Phi([y, w], x) + \Phi([w, x], y) = 0, \tag{8.1}$$

for all $x, y, w \in \mathfrak{g}$.

The properties of Φ together with identity (8.1) tell us that Φ defines a 2-cocycle of the Lie algebra $\mathfrak{g}$ with values in the vector space $\mathfrak{h}^*$ with respect to the trivial representation of $\mathfrak{g}$ by $\mathfrak{h}^*$ (see [6]). Therefore, the product vector space $\mathfrak{g}_\Phi = \mathfrak{g} \times_\Phi \mathfrak{h}^*$ is a Lie algebra with Lie bracket given by

$$[(x, \alpha), (y, \beta)]_c = ([x, y]_\mathfrak{g}, \Phi(x, y)),$$

for all $x, y \in \mathfrak{g}$ and $\alpha, \beta \in \mathfrak{h}^*$. In what follows we denote by $\pi^*\colon \mathfrak{h} \to \mathfrak{gl}(\mathfrak{h}^*)$ the co-adjoint representation of $\mathfrak{h}$. For each $z \in \mathfrak{h}$, we define the map $\Theta_z\colon \mathfrak{g} \times_\Phi \mathfrak{h}^* \to \mathfrak{g} \times_\Phi \mathfrak{h}^*$ by $(x, \alpha) \mapsto (\psi_z(x), \pi_z^*(\alpha))$ for all $x \in \mathfrak{g}$ and $\alpha \in \mathfrak{h}^*$. Since $\psi\colon \mathfrak{h} \to \mathrm{Der}_a(\mathfrak{g}, \mu)$ is a Lie algebra homomorphism, it is easy to verify they satisfy

$$\pi_z^*(\Phi(x, y)) = \Phi(\psi_z(x), y) + \Phi(x, \psi_z(y)), \tag{8.2}$$

for all $x, y \in \mathfrak{g}$ and $z \in \mathfrak{h}$. Formula (8.2) aids us to show that Θ_z is a derivation of the Lie algebra $(\mathfrak{g} \times_\Phi \mathfrak{h}^*, [\cdot, \cdot]_c)$ for each $z \in \mathfrak{h}$, namely, we get

$$\Theta_z([(x, \alpha), (y, \beta)]_c) = [\Theta_z(x, \alpha), (y, \beta)]_c + [(x, \alpha), \Theta_z(y, \beta)]_c,$$

for all $x, y \in \mathfrak{g}$ and $\alpha, \beta \in \mathfrak{h}^*$. Therefore, the map $\Theta\colon \mathfrak{h} \to \mathrm{Der}(\mathfrak{g} \times_\Phi \mathfrak{h}^*, [\cdot, \cdot]_c)$, defined by $z \mapsto \Theta_z$, is a well behaved Lie algebra homomorphism. The vector space $\widetilde{\mathfrak{g}} := \mathfrak{h} \ltimes_\Theta (\mathfrak{g} \times_\Phi \mathfrak{h}^*)$ has the structure of a Lie

algebra given by the semidirect product of $\mathfrak{h}$ with $\mathfrak{g} \oplus_\Phi \mathfrak{h}^*$ through the Lie algebra homomorphism Θ; in other words, the Lie bracket on $\widetilde{\mathfrak{g}}$ is given explicitely by

$$[(z,x,\alpha),(z',y,\beta)] = ([z,z']_{\mathfrak{h}}, \psi_z(y) - \psi_{z'}(x) + [x,y]_{\mathfrak{g}}, \\ \pi_z^*(\beta) - \pi_{z'}^*(\alpha) + \Phi(x,y)), \tag{8.3}$$

for all $x, y \in \mathfrak{g}$, $z, z' \in \mathfrak{h}$ and $\alpha, \beta \in \mathfrak{h}^*$. Finally, over $\widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}}$ we define the function $\widetilde{\mu_0}$ as

$$\widetilde{\mu_0}((z,x,\alpha),(z',y,\beta)) = \mu_0(x,y) + \alpha(z') + \beta(z), \tag{8.4}$$

for all $x, y \in \mathfrak{g}$, $z, z' \in \mathfrak{h}$ and $\alpha, \beta \in \mathfrak{h}^*$. Since μ_0 is an invariant scalar product over $\mathfrak{g}$, a direct calculation shows that $\widetilde{\mu_0}$ is an invariant scalar product on $\widetilde{\mathfrak{g}}$ so that $(\widetilde{\mathfrak{g}}, \widetilde{\mu_0})$ is an orthogonal Lie algebra called the **double orthogonal extension** of $(\mathfrak{g}, \mu_0)$ by $\mathfrak{h}$ via ψ.

Remark 8.1. If the signature of the invariant scalar product μ_0 is (p,q), then the signature of $\widetilde{\mu_0}$ is $(p + \dim(\mathfrak{h}), q + \dim(\mathfrak{h}))$.

Example 8.2. *The cotangent bundle of a Lie group.* If in the method of double orthogonal extension we set $\mathfrak{g} = \{0\}$, it is easy to see that we get $\widetilde{\mathfrak{g}} = \mathfrak{h} \ltimes_{\pi^*} \mathfrak{h}^*$ with Lie bracket given by (7.6) and $\widetilde{\mu_0}((x,\alpha),(y,\beta)) = \alpha(y) + \beta(x)$ for all $x, y \in \mathfrak{h}$ and $\alpha, \beta \in \mathfrak{h}^*$. Therefore, the orthogonal Lie algebra obtained is the Lie algebra of the cotangent bundle of the connected and simply connected Lie group H with Lie algebra $\mathfrak{h}$.

Example 8.3. *The Lie algebra of the λ-oscillator Lie group.* Let $\mathfrak{g} = \mathbb{R}^{2n}$ be considered as an Abelian Lie algebra and $\mu_0 = \langle \cdot, \cdot \rangle$ the usual inner product on $\mathbb{R}^{2n}$. Clearly $(\mathbb{R}^{2n}, \mu_0)$ is an orthogonal Lie algebra. We define the linear map $\delta \colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by

$$x = (x^1, \cdots, x^{2n}) \mapsto (-x^{n+1}, \cdots, -x^{2n}, x_1, \cdots, x_n)$$

which satisfies

$$\mu_0(\delta(x), y) = -\sum_{j=1}^{n} x_{j+n} y_j + \sum_{j=1}^{n} x_j y_{j+n} = -\mu_0(x, \delta(y)),$$

for all $x, y \in \mathbb{R}^{2n}$. If $\mathfrak{h} = \mathbb{R}e$ is a unidimensional Lie algebra, then the map $\psi\colon \mathbb{R}e \to \mathrm{Der}_a(\mathbb{R}^{2n}, \mu_0)$ defined by $te \mapsto \psi(te) = t\delta$, is a well defined Lie algebra homomorphism. Direct calculation shows that $\mathbb{R}^{2n}_{\Phi} = (\mathbb{R}^{2n} \times_{\Phi} \mathbb{R}e^*, [\cdot,\cdot]_c)$ is isomorphic to the Heisenberg Lie algebra of dimension $2n+1$ and that $\widetilde{\mathbb{R}^{2n}} = \mathbb{R}e \ltimes_{\Theta} (\mathbb{R}^{2n} \times_{\Phi} \mathbb{R}e^*)$ is the Lie algebra with bracket

$$[e, e_j] = \delta(e_j) = \hat{e_j}, \qquad [e, \hat{e_j}] = \delta(\hat{e_j}) = -e_j, \qquad [e_j, \hat{e_j}] = e^*,$$

for all $j = 1, \cdots, n$. The invariant product $\widetilde{\mu_0}$ defined by $\widetilde{\mathbb{R}^{2n}}$ is given by

$$\widetilde{\mu_0}(\gamma e + x + \alpha e^*, \gamma' e + y + \beta e^*) = \mu_0(x, y) + \alpha\gamma' + \beta\gamma,$$

for all $x, y \in \mathbb{R}^{2n}$ and $\alpha, \beta, \gamma, \gamma' \in \mathbb{R}$. The orthogonal Lie algebra $(\widetilde{\mathbb{R}^{2n}}, \widetilde{\mu_0})$ is isomorphic to the λ-oscillator Lie algebra with $\lambda_j = 1$ for all $j = 1, \cdots, n$.

A slight modification of this construction allows us to obtain the Lie algebra $\mathfrak{g}_\lambda$ for $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$ with arbitrary $0 < \lambda_1 \leq \cdots \leq \lambda_n$.

Remark 8.4. A. Medina and Ph. Revoy proved in [17] that one can inductively produce all orthogonal Lie algebras starting out with simple and unidimensional ones by taking direct sums and double extensions. More precisely, let $\mathfrak{g}$ be an indecomposable orthogonal Lie algebra, that is, an orthogonal Lie algebra that cannot be written as the direct sum of two non-trivial orthogonal Lie algebras. Then either $\mathfrak{g}$ is simple, or $\mathfrak{g}$ is unidimensional, or else $\mathfrak{g}$ is a double extension of an orthogonal Lie algebra $\widetilde{\mathfrak{g}}$ by a unidimensional or a simple Lie algebra $\mathfrak{h}$. As an application, it is possible to show that any indecomposable non-simple Lie algebra of a Lorentzian Lie group with dimension greater than 1 is the double orthogonal extension of an Abelian Lie algebra with inner product by a unidimensional Lie algebra. This provides a classification of Lorentzian orthogonal Lie algebras up to isomorphism (see [16]).

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Resumen

En estas notas estudiamos algunos conceptos básicos de geometría afín y su relación con la geometría Riemanniana. Proporcionamos una caracterización para variedades afines que posee una contraparte válida para variedades pseudo-Riemannianas cuya conexión de Levi-Civita es plana. Se prueba que ningún grupo de Lie semisimple conexo puede admitir una conexión afín plana invariante a izquierda. Caracterizamos grupos de Lie pseudo-Riemannianos planos y se demuestra que la unimodularidad del grupo de Lie es condición necesaria y suficiente para que la conexión de Levi-Civita asociada a una pseudo-métrica plana invariante a izquierda sobre este resulte geodésicamente completa. Adicionalmente, se describe la forma de cómo obtener métricas hiperbólicas planas invariantes a izquierda sobre el fibrado cotangente de un grupo de Lie afín plano simplemente conexo. Por último, se exponen algunas propiedades de grupos de Lie dotados de pseudo-métricas bi-invariantes, además exhibimos la construcción de álgebras de Lie ortogonales mediante el uso del método de doble extensión ortogonal.

Palabras clave: Estructura afín plana, pseudo-métrica plana, grupo de Lie pseudo-Riemanniano plano, grupo de Lie ortogonal, álgebra de Lie ortogonal.

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The Jacobian conjecture: Approximate roots and intersection numbers

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Abstract

We compute the intersection number of a Jacobian pair in two different ways following Yansong Xu, but using the language of [5]. We obtain nearly the same formulas, but with an inequality instead of an equality.

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Keywords: Jacobian conjecture, Newton-Puisseux method.

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1 Introduction

The Jacobian Conjecture in dimension two stated by Keller in [10] claims that any pair of polynomials $P, Q \in L = K[x, y]$, with $[P, Q] = \partial_x P \partial_y Q - \partial_x Q \partial_y P \in K^\times$, defines an invertible automorphism of L. If this conjecture is false, then we can find a counterexample such that the shape of the support of the components $P = f(x)$, $Q = f(y)$ is contained in rectangles $(0,0)$, $m(a,0)$, $m(a,b)$, $m(0,b)$ and $(0,0)$, $n(a,0)$, $n(a,b)$, $n(0,b)$, where $m(a,b)$ is in the support of P and $n(a,b)$ is in the support of Q. In a recent paper [14], Yangsong Xu gives two formulas for the intersection number of possible counterexamples, which we call I_M and I_m. If these formulas were true, we would be able to discard several infinite families of possible counterexamples as described in [7].

When we translated the result and proofs of [14] into the language of [12], we obtained the same formula for I_M (Theorem 6.2), but for I_m we achieved only an inequality (Theorem 7.3). Consequently, we cannot discard the infinite families as desired.

Hence, the main result of the present article is the translation of the concept of approximate roots into our language (see [12], also [5] and [7]), which requires a dictionary from Moh's language into our own. This is interesting by itself, and the modified formulas help understand some features of Moh's methods.

Along this paper we freely use the notation of [12].

2 General lower side corners

For $l \in \mathbb{N}$ let $(P, Q) \in L^{(l)}$ be an (m, n)-pair (see [12, Definition 4.3]). In this section we take $(\rho, \sigma) \in]0, -1), (1, 1)]$ subject to

$$\frac{1}{m} \operatorname{en}_{\rho,\sigma}(P) = \frac{1}{n} \operatorname{en}_{\rho,\sigma}(Q) = (a/l, b) \quad \text{with} \quad a/l > b > 0$$

(assuming that such a direction exists). Note that $\rho > 0$ is true by assumption. Suppose $u_p = v_{\rho,\sigma}(P) > 0$. Then the points $(a/l, b)$ and

$(c/l, d) = \dfrac{1}{m}\,\mathrm{st}_{\rho,\sigma}(P)$ must satisfy certain conditions. The purpose of this section is to analyse them.

Proposition 2.1. *For P, Q and (ρ, σ) as described above we have $[\ell_{\rho,\sigma}(P), \ell_{\rho,\sigma}(Q)] = 0$.*

Proof. By [12, Proposition 1.13] we need to prove $v_{\rho,\sigma}(P)+v_{\rho,\sigma}(Q) > \rho+\sigma$. If $\rho+\sigma \leq 0$, then this is true, since we have $v_{\rho,\sigma}(Q) = \dfrac{n}{m}v_{\rho,\sigma}(P) > 0$; while for $\rho + \sigma > 0$, because of $\dfrac{a}{l} > b \geq 1$ and $\rho > 0$, we have

$$v_{\rho,\sigma}(P) + v_{\rho,\sigma}(Q) = (m+n)\left(\rho\frac{a}{l} + \sigma b\right) > (m+n)b(\rho+\sigma) > \rho+\sigma,$$

as desired. □

Proposition 2.2. *Under the above assumptions, if $\rho + \sigma > 0$, then ρ divides l and there exist $\lambda, \mu \in K^\times$ such that $\ell_{\rho,\sigma}(P) = \lambda x^{u_p/\rho}(z-\mu)^{mb}$, here $z = x^{-\sigma/\rho}y$.*

Proof. By [12, Theorem 2.6], there exists a (ρ, σ)-homogeneous element $F \in L^{(l)}$ such that

- $v_{\rho,\sigma}(F) = \rho + \sigma$,
- $[F, \ell_{\rho,\sigma}(P)] = \ell_{\rho,\sigma}(P)$,
- $\mathrm{st}_{\rho,\sigma}(P) \sim \mathrm{st}_{\rho,\sigma}(F)$ or $\mathrm{st}_{\rho,\sigma}(F) = (1,1)$,
- $\mathrm{en}_{\rho,\sigma}(P) \sim \mathrm{en}_{\rho,\sigma}(F)$ or $\mathrm{en}_{\rho,\sigma}(F) = (1,1)$.

If $\mathrm{en}_{\rho,\sigma}(P) = m(a/l, b) \sim \mathrm{en}_{\rho,\sigma}(F)$, then we can find $\lambda > 0$ such that $\mathrm{en}_{\rho,\sigma}(F) = \lambda(a/l, b)$. Therefore

$$\rho + \sigma = v_{\rho,\sigma}(F) = \rho\lambda\frac{a}{l} + \lambda\sigma b > \lambda b(\rho+\sigma)$$

implies $0 < \lambda b < 1$, which is impossible since $\lambda b = v_{0,1}(\mathrm{en}_{\rho,\sigma}(F)) \in \mathbb{Z}$. Consequently we have $\mathrm{en}_{\rho,\sigma}(F) = (1,1)$, and hence $\mathrm{st}_{\rho,\sigma}(F) = (1 +$

$\sigma/\rho, 0)$ by [12, Proposition 2.11(2)]. Thus ρ divides l and we have $\mathrm{st}_{\rho,\sigma}(P) \sim \mathrm{st}_{\rho,\sigma}(F)$, which readily implies $v_{0,1}(\mathrm{st}_{\rho,\sigma}(P)) = 0$. Write

$$F = x^{\frac{u}{l}} y^v f(z) \quad \text{and} \quad \ell_{\rho,\sigma}(P) = x^{\frac{c}{l}} y^d p(z), \quad \text{with } p(0), f(0) \neq 0.$$

Note that here $v = d = 0$, $\rho c/l = u_p$, $v_{0,1}(\mathrm{en}_{\rho,\sigma}(P)) = mb$ and $f(z) = \lambda_1(z - \mu)$ for some $\lambda_1, \mu \in K^\times$. By [12, Proposition 2.11(1)] we have then $\ell_{\rho,\sigma}(P) = \lambda x^{u_p/\rho}(z - \mu)^{mb}$ for some $\lambda \in K^\times$, which concludes the proof. □

By [12, Proposition 2.1(2)] (which applies thanks to Proposition 2.1) there exist $\lambda_P, \lambda_Q \in K^\times$ and a (ρ, σ)-homogeneous element $R \in L^{(l)}$ such that

$$\ell_{\rho,\sigma}(P) = \lambda_P R^m \qquad \text{and} \qquad \ell_{\rho,\sigma}(Q) = \lambda_Q R^n.$$

Take $\lambda \in K^\times$ and let $R_0 \in L^{(l)}$ be a (ρ, σ)-homogeneous element such that $\ell_{\rho,\sigma}(P) = \lambda R_0^h$ with h maximal (consequently m divides h and we can assume $R = R_0^{h/m}$ and $\lambda_P = \lambda$). Arguing as in [5, Corollary 2.6] we obtain a certain $i \geq 0$ and a (ρ, σ)-homogeneous element $G \in L^{(l)}$ subject to $[G, R] = R^i$.

Let $(a/l, b), (c/l, d) \in \frac{1}{l}\mathbb{Z} \times \mathbb{Z}$ be such that $a/l > b > d \geq 0$ and $a > c > 0$. Assume also $b - d < a/l - c/l$ (we do not claim the existence of P and Q at this point). It is well known that for each $(r/l, s) \in \frac{1}{l}\mathbb{Z} \times \mathbb{Z} \setminus \mathbb{Z}(1, 1)$ there exists a unique $(\varrho, \varsigma) \in \mathfrak{V}_{>0}$ (see (3.2) at page 29 of [12]), which we denote by $\mathrm{dir}(r/l, s)$, such that $v_{\varrho,\varsigma}(r/l, s) = 0$. Set $(\rho, \sigma) = -\mathrm{dir}((a/l, b) - (c/l, d))$ and note the inequality $(0, -1) < (\rho, \sigma) < (1, -1)$. We will analyse the existence of $i \in \mathbb{N}$ and (ρ, σ)-homogeneous elements $R, G \in L^{(l)}$, such that

$$v_{\rho,\sigma}(R) > 0, \quad [G, R] = R^i, \quad (a/l, b) = \mathrm{en}_{\rho,\sigma}(R) \quad \text{and} \quad (c/l, d) = \mathrm{st}_{\rho,\sigma}(R). \tag{2.1}$$

Let $\ell \in \mathbb{N}$ be minimal with $\ell v_{\rho,\sigma}(R) + \rho + \sigma > 0$. By [5, Proposition 3.12], if there exist $i \in \mathbb{N}$ and $R, G \in L^{(l)}$ satisfying (2.1), and such

that

$$R \neq \lambda x^{\frac{u}{\rho}} h^j(z) \quad \text{for all } \lambda \in K^\times,\ j \in \mathbb{N} \text{ and all linear polynomials } h, \tag{2.2}$$

where $z = x^{-\frac{\sigma}{\rho}} y$, then either there exist $\vartheta, t' \in \mathbb{N}$ subject to

$$\vartheta \leq N_1, \qquad t' < \ell\vartheta \qquad \text{and} \qquad (\rho,\sigma) = -\operatorname{dir}\Big(t'\Big(\frac{c}{l}, d\Big) + \vartheta(1,1)\Big), \tag{2.3}$$

where $N_1 = \gcd(a-c, b-d)$, or

$$d > 0,\ \ \vartheta \text{ divides } N_2,\ \ t' < \ell\vartheta \quad \text{and} \quad (\rho,\sigma) = -\operatorname{dir}\Big(t'\Big(\frac{c}{l}, d\Big) + \vartheta(1,1)\Big), \tag{2.4}$$

where $N_2 = \gcd(c, d)$. By [5, Remark 3.13] we have then

$$\frac{\vartheta}{t'} = -\frac{\rho a/l + \sigma b}{\rho + \sigma}.$$

Therefore defining

$$s = \frac{\rho a + \sigma b l}{\gcd(\rho l + \sigma l, \rho a + \sigma b l)}\Big|\vartheta,$$

we can take (and we do take it) $\vartheta = s$ in (2.3) and (2.4).

We suspect that the existence of ϑ and t' satisfying the conditions in (2.3) or in (2.4) is enough for the existence of $i \in \mathbb{N}$ and two (ρ,σ)-homogeneous elements $R, G \in L^{(l)}$ such that the conditions in (2.1) and (2.2) are fullfiled (with $(c/l, d) = \mathrm{st}_{\rho,\sigma}(R)$), but at the moment we have no proof of this claim.

Remark 2.3. Since $N_2 < b$, if $s = b$, then necessarily $b \leq N_1$. So, by [5, Proposition 3.12(2)], there exists a linear factor with multiplicity b. As this contradicts (2.2), we have consequently $s < b$.

Remark 2.4. By [5, Theorem 3.4], in (2.1) we can assume that i is the minimal element subject to

$$v_{\rho,\sigma}(R)(i-1) + \rho + \sigma \geq 0,$$

or, equivalently, $i = \Big\lceil 1 - \frac{\rho+\sigma}{v_{\rho,\sigma}(R)} \Big\rceil$.

For the case $b = 2$, we can establish necessary and sufficient conditions on a, l for the existence of $c \in \mathbb{N}$, $d \in \{0, 1\}$ and two (ρ, σ)-homogeneous elements $R, G \in L^{(l)}$ satisfying the conditions of (2.1) as soon as we impose that R satisfies (2.2). This additional requirement corresponds to the existence of split roots (see Definition 3.5). Before we establish the result we note that $(0, -1) < (\rho, \sigma) < (1, -1)$ and $(\rho, \sigma) = -\operatorname{dir}\Big(\frac{a-c}{l}, b - d\Big) \sim (lb - ld, c - a)$ imply $c < a$ and $b - d < a/l - c/l$.

Proposition 2.5. *Let $a, l \in \mathbb{N}$ be such that $a/l > 2$. Set $b = 2$. Let $(\rho, \sigma) \in](0, -1), (1, -1)[$ be a direction, and define the number*

$$\vartheta = \frac{\rho a + \sigma b l}{\gcd(\rho l + \sigma l, \rho a + \sigma b l)}.$$

Then the following assertions are equivalent.

(1) *There exist $c \in \mathbb{N}$, $d \in \{0, 1\}$ and two (ρ, σ)-homogeneous elements $R, G \in L^{(l)}$ satisfying conditions (2.1) and (2.2).*

(2) *There exist $c \in \mathbb{N}$ and two (ρ, σ)-homogeneous elements $R, G \in L^{(l)}$ satisfying conditions (2.1) and (2.2) with $d = 1$.*

(3) *We have $\vartheta = 1$, $v_{\rho,\sigma}(a/l, 2) > 0$ and there exist $c \in \mathbb{N}$ such that*

$$(\rho, \sigma) = -\operatorname{dir}\Big(\frac{a - c}{l}, 1\Big) = -\operatorname{dir}\Big(t'\Big(\frac{c}{l}, 1\Big) + (1, 1)\Big), \qquad (2.5)$$

for some $0 < t' < \ell$, here $\ell \in \mathbb{N}$ is minimal with the property $\ell v_{\rho,\sigma}(a/l, 2) + \rho + \sigma > 0$.

(4) *There exists $\Delta \in \mathbb{N}$ with $l < \Delta < a/2$ such that $a - 2\Delta \mid \Delta - l$. Moreover, $(\rho, \sigma) \sim (l, -\Delta)$.*

Proof. We first prove that 1) implies 2). Suppose $d = 0$ and write

$$R = \lambda x^{\frac{u}{t}}(z - \alpha_1)(z - \alpha_2) \quad \text{with } z = x^{-\frac{\sigma}{\rho}} y.$$

Note that by (2.2) we have $\alpha_1 \neq \alpha_2$. Also $\rho u/l = 2\sigma + \rho a/l$ implies $u = (2l\sigma + \rho a)/\rho$. Moreover, since $b - d = 2$, we have

$$(2l, c-a) \cdot \Big(\frac{a}{l} - \frac{c}{l}, b-d\Big) = 2(a-c) - (c-a)(b-d) = 0,$$

and consequently $(\rho, \sigma) \sim (2l, c-a)$. Also, since $d = 0$, necessarily (2.3) is satisfied. We claim that 2 divides $a - c$. In fact, this follows from

$$0 = (2l, c-a) \cdot \Big(t'\Big(\frac{c}{l}, 0\Big) + \vartheta(1,1)\Big) = 2ct' + (c-a)\vartheta,$$

for otherwise 2 divides $\vartheta \leq N_1 = \gcd(a-c, 2) = 1$. Set $\Delta = (a-c)/2$ and consider the automorphism φ of $L^{(l)}$ defined by $\varphi(x^{1/l}) = x^{1/l}$ and $\varphi(y) = y + \alpha_1 x^{-\Delta/l}$. Using $(\rho, \sigma) \sim (l, -\Delta)$ it is easy to conclude

$$\varphi(R) = \lambda x^{\frac{u}{l}} z(z - (\alpha_2 - \alpha_1)).$$

By [12, Proposition 3.10], we have $[\varphi(G), \varphi(R)] = \varphi(R)^i$. An easy computation gives $\mathrm{en}_{\rho,\sigma}(\varphi(R)) = (a/l, b)$ and $\mathrm{st}_{\rho,\sigma}(\varphi(R)) = ((a-\Delta)/l, 1)$. So, replacing R by $\varphi(R)$ yields $d = 1$.

That 2) implies 1) is a trivial fact.

Now we prove that 2) implies 3). Since $d = 1$, we get $N_1 = N_2 = 1$. Hence we have $\vartheta = 1$, and Equality (2.5) is satisfied for some $0 < t' < \ell$. Moreover, it is clear that

$$v_{\rho,\sigma}\Big(\frac{a}{l}, 2\Big) = v_{\rho,\sigma}(R) > 0$$

is verified and we also have

$$(\rho, \sigma) = -\operatorname{dir}\big(\mathrm{en}_{\rho,\sigma}(R) - \mathrm{en}_{\rho,\sigma}(R)\big) = -\operatorname{dir}\Big(\frac{a-c}{l}, 1\Big).$$

For 3) implies 4), since

$$(l, c-a) \cdot \Big(\frac{a}{l} - \frac{c}{l}, 1\Big) = 0,$$

we have $(\rho,\sigma)\sim(l,-\Delta)$, with $\Delta=a-c$. Thus, by (2.5), we obtain

$$0=(l,-\Delta)\cdot\Big(t'\Big(\frac{a-\Delta}{l},1\Big)+(1,1)\Big)=t'a-2t'\Delta+l-\Delta,$$

which implies $a-2\Delta|l-\Delta$, as desired. But $(\rho,\sigma)\sim(l,-\Delta)$ and $v_{\rho,\sigma}(a/l,2)>0$ yield

$$a-2\Delta=(l,-\Delta)\cdot\Big(\frac{a}{l},2\Big)=\frac{l}{\rho}(\rho,\sigma)\cdot\Big(\frac{a}{l},2\Big)>0,$$

and so $\Delta<a/2$. Finally, the relation $l-\Delta=\dfrac{l}{\rho}(\rho+\sigma)<0$ forces $\Delta>l$.

To show that 4) implies 2) we set $c=a-\Delta$, $z=x^{\Delta/l}y$ and $(\rho,\sigma)=-\operatorname{dir}((a/l,2)-(c/l,1))$. Since $0<l<\Delta$, the inequalities $(0,-1)<(\rho,\sigma)<(1,-1)$ hold. Let $k_1\in\mathbb{N}$ be such that $k_1(a-2\Delta)=\Delta-l$ and let $g(z)$ be a polynomial with derivative $g'(z)=z^{k_1}(1+z)^{k_1}$. A straightforward computation shows that

$$R=x^{\frac{a-2\Delta}{l}}z(1+z)=x^{\frac{c}{l}}y(1+z)\qquad\text{and}\qquad G=\frac{l}{2\Delta-a}g(z),$$

satisfy $\Big(\dfrac{a}{l},2\Big)=\operatorname{en}_{\rho,\sigma}(R)$, $\Big(\dfrac{c}{l},1\Big)=\operatorname{st}_{\rho,\sigma}(R)$, $v_{\rho,\sigma}(R)>0$ and $[G,R]=R^{k_1+1}$, as desired. □

3 Approximate π-roots

Recall that the **intersection number** of two bivariate polynomials P and Q is defined by $I(P,Q)=\deg_x(\operatorname{Res}_y(P,Q))$, where $\operatorname{Res}_y(P,Q)$ denotes the resultant of P and Q as polynomials in y. In [14], the author defines for a Jacobian pair (P,Q) the polynomial $P_\xi=P(x,y)-\xi$, where ξ is a generic element of the field K, and proposes two different formulas for $I(P_\xi,Q)$: one in terms of the major roots (see [14, Theorem 5.1]), and the other in terms of the minor roots (as in [14, Theorem 4.7]). We will prove the first formula using our language (see Theorem 6.2), however, instead of equality in the formula for I_m we recover an inequality

in Theorem 7.3. In order to achieve these results, it will be convenient to provide a proof of the preparatory results of [14] in the language of [12].

We first define approximate roots, final major roots and final minor roots using our language.

In this section we consider a polynomial $P \in L$, monic in y. For $l \in \mathbb{N}$ we take the following algebras:

$$L = K[x,y] \subsetneq K[x^{\pm\frac{1}{l}},y] \subsetneq K((x^{-1/l}))[y] \subsetneq K[\pi]((x^{-1/l}))[y],$$

where π is a variable (a "symbol" in [14]). We also will use the subring $L_\pi^{(l)} = K[\pi][x^{\pm 1/l},y]$ of $K[\pi]((x^{-1/l}))[y]$. Note that $\deg_x = v_{1,0}$ is well defined in $K[\pi]((x^{-1/l}))[y]$.

Unless otherwise indicated, the elements P of the above mentioned algebras are polynomials in y with coefficients in one of the algebras $K[x]$, $K[x^{\pm\frac{1}{l}},y]$, $K[\pi]((x^{-1/l}))$,.... Consequently, expressions like $P(\tau)$, $P(\alpha)$,..., will mean P with y replaced by τ, by α, and so on.

By the Newton-Puiseux theorem (see [4, Corollary 13.15, page 295]) there exist $l \in \mathbb{N}$ and $\alpha_i, \beta_i \in K((x^{-1/l}))$ such that

$$P = \prod_{i=1}^{M}(y - \alpha_i).$$

We set $\mathcal{R}(P) = \{\alpha_i : i = 1,\dots,M\}$. Let $\alpha \in \mathcal{R}(P)$ and write $\alpha = \sum_j a_j x^j$ with $j \in \frac{1}{l}\mathbb{Z}$. The **$\pi$-approximation of α up to x^{j_0}** is the element

$$\tau = \sum_{j>j_0} a_j x^j + \pi x^{j_0} \in K[\pi, x^{\pm\frac{1}{l}}].$$

Note the equality $\deg_x(\tau - \alpha) = j_0$.

Let $\tau = \sum_{j>j_0} a_j x^j + \pi x^{j_0} \in K[\pi, x^{\pm\frac{1}{l}}]$. Set

$$D_\tau^P = \{\alpha \in \mathcal{R}(P) : \tau \text{ is the } \pi\text{-approximation of } \alpha \text{ up to } x^{j_0}\}.$$

If $\alpha \in D_\tau^P$, we say that τ **approximates α up to x^{j_0}**.

Note that the element $\alpha_i = \sum_j b_j x^j \in \mathcal{R}(P)$ belongs to D_τ^P if and only if $\deg_x(\hat{\alpha}_i) \leq j_0$, where $\hat{\alpha}_i = \alpha_i - \sum_{j>j_0} a_j x^j$, that is, if and only if we have $a_j = b_j$ for all $j > j_0$.

We say that $\tau = \sum_{j>j_0} a_j x^j + \pi x^{j_0} \in K[\pi, x^{\pm\frac{1}{l}}]$ is a **π-root of P** if there exists $\alpha \in \mathcal{R}(P)$ such that τ approximates α up to x^{j_0}. In that case we say that j_0 is the **order** of τ. When we want to underline the dependence of j_0 on τ we will write $\delta_\tau = j_0$.

Notation 3.1. Let $\tau = \sum_{j>j_0} a_j x^j + \pi x^{j_0}$ be a π-root of P. In what follows denote by φ_τ the automorphism of $L^{(l)}$ determined by $\varphi_\tau(x^{1/l}) = x^{1/l}$ and $\varphi_\tau(y) = y + \sum_{j>j_0} a_j x^j$.

Remark 3.2. Let $\alpha \in \mathcal{R}(P)$. Assume that τ approximates α up to j_0 and τ_1 approximates α up to j_1. If $j_0 > j_1$, then we have $D_{\tau_1}^P \subseteq D_\tau^P$.

In the sequel, for each $j \in \frac{1}{l}\mathbb{Z}$, we let $\mathrm{dir}(j)$ denote the unique direction (ρ, σ) such that $\rho > 0$ and $j = \frac{\sigma}{\rho}$. Moreover, given a polynomial $\tau = \sum_{i>j_0} a_i x^i + \pi x^{j_0}$, we set $z = x^{-\sigma/\rho} y$, where $(\rho, \sigma) = \mathrm{dir}(j_0)$.

The following proposition shows that our definition of π-root coincides with that given in [11, Definition 1.3] with x^{-1} replaced by t.

Proposition 3.3. *Let $\tau = \sum_{j>j_0} a_j x^j + \pi x^{j_0}$ and $f_{P,\tau}(\pi) \in K[\pi]$ be the polynomial determined by the equality*

$$P(\tau) = f_{P,\tau}(\pi) x^{\lambda_\tau} + \textit{terms with lower order in } x, \tag{3.1}$$

where $\lambda_\tau = \deg_x(P(\tau)) \in \frac{1}{l}\mathbb{Z}$. Set $\varphi = \varphi_\tau$ and $(\rho, \sigma) = \mathrm{dir}(j_0)$. Then we have

$$|D_\tau^P| = \deg(f_{P,\tau}) = v_{0,1}(\mathrm{en}_{\rho,\sigma}(\varphi(P))) \tag{3.2}$$

and

$$\ell_{\rho,\sigma}(\varphi(P)) = x^{\lambda_\tau} f_{P,\tau}(z). \tag{3.3}$$

Consequently τ is a π-root of P if and only if we have $\deg(f_{P,\tau}) > 0$.

Proof. Let $\mathrm{ev}_{\pi x^{j_0}} : L_\pi^{(l)} \to L_\pi^{(l)}$ be the evaluation of y at πx^{j_0}. For example we have, $\mathrm{ev}_{\pi x^{j_0}}(y) = \pi x^{j_0} = \pi x^{\sigma/\rho}$ and $\mathrm{ev}_{\pi x^{j_0}}(x^{1/l}) = x^{1/l}$. Note the relation $P(\tau) = \mathrm{ev}_{\pi x^{j_0}}(\varphi(P))$. Since $\mathrm{ev}_{\pi x^{j_0}}$ is (ρ,σ)-homogeneous, we get

$$\ell_{\rho,\sigma}(\mathrm{ev}_{\pi x^{j_0}}(\varphi(P))) = \mathrm{ev}_{\pi x^{j_0}}(\ell_{\rho,\sigma}(\varphi(P))).$$

On the other hand, since ρ divides l, we get

$$\ell_{\rho,\sigma}(\varphi(P)) = x^{r/l} g(z) \quad \text{for some } r \in \mathbb{Z} \text{ and } g(z) \in K[z]. \tag{3.4}$$

Using $\mathrm{ev}_{\pi x^{j_0}}(z) = \pi$ we obtain

$$\ell_{\rho,\sigma}(\mathrm{ev}_{\pi x^{j_0}}(\varphi(P))) = x^{r/l} g(\pi).$$

Therefore we have

$$P(\tau) = \mathrm{ev}_{\pi x^{j_0}}(\varphi(P)) = x^{r/l} g(\pi) + \text{terms with lower order in } x,$$

because $v_{\rho,\sigma}(x^j) = j\rho < \rho r/l = v_{\rho,\sigma}(x^{r/l})$ if and only if $j < r/l$. So we have $f_{P,\tau}(\pi) = g(\pi)$, $\lambda_\tau = r/l$, and Equality (3.4) becomes Equality (3.3). Since $\deg_z(\ell_{\rho,\sigma}(\varphi(P))) = \deg_y(\ell_{\rho,\sigma}(\varphi(P)))$, we also have $\deg(f_{P,\tau}) = v_{0,1}(\mathrm{en}_{\rho,\sigma}(\varphi(P)))$. Consequently, in order to conclude the proof, it suffices to prove $|D_\tau^P| = v_{0,1}(\mathrm{en}_{\rho,\sigma}(\varphi(P)))$. In the chain of equalities

$$v_{0,1}(\mathrm{en}_{\rho,\sigma}(\varphi(P))) = \sum_{i=1}^{M} v_{0,1}(\mathrm{en}_{\rho,\sigma}(\varphi(y-\alpha_i))) = \sum_{i=1}^{M} v_{0,1}(\mathrm{en}_{\rho,\sigma}(y-\hat{\alpha}_i))),$$

where $\hat{\alpha}_i = \alpha_i - \sum_{j>j_0} a_j x^j$, when we replace

$$\mathrm{en}_{\rho,\sigma}(y-\hat{\alpha}_i)) = \begin{cases} (0,1) & \text{if } \deg_x(\hat{\alpha}_i) \le \sigma/\rho = j_0, \\ (\deg_x(\hat{\alpha}_i), 0) & \text{if } \deg_x(\hat{\alpha}_i) > \sigma/\rho = j_0, \end{cases}$$

we finish up with

$$\sum_{i=1}^{M} v_{0,1}(\mathrm{en}_{\rho,\sigma}(y-\hat{\alpha}_i))) = \#\{\alpha_i \in \mathcal{R}(P) : \deg_x(\hat{\alpha}_i) \le j_0\} = |D_\tau^P|,$$

as desired. □

Remark 3.4. Note that if $|D_\tau^P| > 0$, then $|D_\tau^{P_y}| = |D_\tau^P| - 1 \geq 0$. In fact, we get $\varphi(P_y) = (\varphi(P))_y$, and it is straightforward to check that if $\rho > 0$ and $v_{0,1}(\mathrm{en}_{\rho,\sigma}(P)) > 0$, then $v_{0,1}(\mathrm{en}_{\rho,\sigma}(P_y)) = v_{0,1}(\mathrm{en}_{\rho,\sigma}(P)) - 1$. The assertion then follows from (3.2).

Definition 3.5. We say that a π-root τ of P is a **final π-root of** P if $f_{P,\tau}(\pi)$ has no multiple roots and $\deg_\pi(f_{P,\tau}(\pi)) > 1$, here $f_{P,\tau}(\pi)$ is defined by Equality (3.1).

Remark 3.6. Let τ be a final π-root of P. Since the support of $f_{P,\tau}$ has more than one point, from Equality (3.3) we conclude $(\rho, \sigma) \in \mathrm{Dir}(\varphi_\tau(P))$.

Proposition 3.7. *Let $\tau = \sum_{j>j_0} a_j x^j + \pi x^{j_0}$ be a π-root of P and let $\lambda \in K$. Consider the automorphism $\varphi_1 \colon L^{(l)} \to L^{(l)}$ given by $\varphi_1(x^{1/l}) = x^{1/l}$ and $\varphi_1(y) = y + \sum_{j>j_0} a_j x^j + \lambda x^{j_0}$. Assume that $\varphi_1(P)$ is not a monomial and set $(\rho', \sigma') = \mathrm{Pred}_{\varphi_1(P)}(\rho, \sigma)$ (see [12, Definition 3.4]), where $(\rho, \sigma) = \mathrm{dir}(j_0)$. If $\rho' > 0$, then set $j_1 = \dfrac{\sigma'}{\rho'}$, else take any $j_1 \in \dfrac{1}{l}\mathbb{Z}$ with $j_1 < j_0$. In both cases set $(\rho_1, \sigma_1) = \mathrm{dir}(j_1)$. If $\pi - \lambda$ has multiplicity $r > 0$ in $f_{P,\tau}(\pi)$, then*

$$\tau_1 = \sum_{j>j_0} a_j x^j + \lambda x^{j_0} + \pi x^{j_1}$$

is a π-root of P and we get $|D_{\tau_1}^P| = r$ (remember that $j_1 < j_0$). Moreover, we have

$$(\rho_1, \sigma_1) \in [\mathrm{Pred}_{\varphi_1(P)}(\rho, \sigma), (\rho, \sigma)[. \tag{3.5}$$

Proof. Write $\varphi_1 = \widetilde{\varphi} \circ \varphi$, where φ is as in Proposition 3.3, $\widetilde{\varphi}(y) = y + \lambda x^{j_0}$, and $\widetilde{\varphi}(x) = x$. By Equality (3.3), the fact that $\widetilde{\varphi}$ is (ρ, σ)-homogeneous together with $\widetilde{\varphi}(z) = z + \lambda$ lead us to

$$\begin{aligned} \ell_{\rho,\sigma}(\varphi_1(P)) &= \widetilde{\varphi}(\ell_{\rho,\sigma}(\varphi(P))) = \widetilde{\varphi}(x^{\lambda_\tau} f_{P,\tau}(z)) \\ &= x^{\lambda_\tau} \widetilde{\varphi}(f_{P,\tau}(z)) = x^{\lambda_\tau} z^r g_1(z), \end{aligned}$$

for some $g_1(z) \in K[z]$ with $g_1(0) \neq 0$. By construction we have then $(\rho_1, \sigma_1) \in [\mathrm{Pred}_{\varphi_1(P)}(\rho, \sigma), (\rho, \sigma)[$, and so, by Proposition 3.3, we get

$$r = v_{0,1}(\mathrm{st}_{\rho,\sigma}(\varphi_1(P))) = v_{0,1}(\mathrm{en}_{\rho_1,\sigma_1}(\varphi_1(P))) = |D^P_{\tau_1}|,$$

as desired. □

Corollary 3.8. *Let $\tau = \sum_{j>j_0} a_j x^j + \pi x^{j_0}$ be a π-root of P and let $\lambda \in K$. If $\pi - \lambda$ does not divide $f_{P,\tau}(\pi)$, then there exists no root $\alpha \in \mathcal{R}(P)$ such that $\deg_x(\alpha - (\lambda x^{j_0} + \sum_{j>j_0} a_j x^j)) < j_0$.*

Proof. Let $f_{P,\tau}(\pi) = \prod_{i=1}^k (\pi - \lambda_i)^{m_i}$. By Proposition 3.7, for each i there exist $\tau_1(i)$ and m_i roots in $D^P_{\tau_1(i)} \subset D^P_\tau$, for which $\mathrm{Coeff}_{x^{j_0}} = \lambda_i$. Since we have

$$|D^P_\tau| = \deg(f_{P,\tau}(\pi)) = \sum_{i=1}^k m_i = \sum_{i=1}^k |D^P_{\tau_1(i)}|$$

and the sets $D^P_{\tau_1(i)}$ are pairwise disjoint, we obtain $D^P_\tau = \bigcup_{i=1}^k D^P_{\tau_1(i)}$. Consequently, the coefficient of x^{j_0} in each element of D^P_τ is a root of $f_{P,\tau}$. Since λ is not a root of $f_{P,\tau}$, this finishes the proof. □

Remark 3.9. The proof of the corollary shows that if the multiplicity of $\pi - \lambda$ in $f_{P,\tau}(\pi)$ is r, then any π-root τ_2 of P which begins with $\lambda x^{j_0} + \sum_{j>j_0} a_j x^j$ satisfies $|D^P_{\tau_2}| \leq r$.

Remark 3.10. Let $\alpha = \sum_j a_j x^j \in K((x^{-1/l}))$, $j_0 \in \frac{1}{l}\mathbb{Z}$, $(\rho, \sigma) = \mathrm{dir}(j_0)$ and $\tau = \sum_{j>j_0} a_j x^j + \pi x^{j_0}$. Define $T = \sum_{j \leq j_0} a_j x^j$. Since

$$P(\alpha) = \mathrm{ev}_{y=T}(\varphi_\tau(P)),$$

we have $\ell_{\rho,\sigma}(P(\alpha)) = \ell_{\rho,\sigma}(\mathrm{ev}_{y=\lambda x^{j_0}}(\varphi_\tau(P)))$ whenever the right hand side of the equality is nonzero.

Proposition 3.11. *Let $\alpha = \sum_{j>j_0} a_j x^j + \lambda x^{j_0} + \sum_{j<j_0} a_j x^j$ and set $\tau = \sum_{j>j_0} a_j x^j + \pi x^{j_0}$. If $f_{P,\tau}(\lambda) \neq 0$, then $\lambda^P_\tau = \deg_x(P(\tau)) = \deg_x(P(\alpha))$.*

Proof. By Remark 3.10, Equality (3.3) and the fact that $\mathrm{ev}_{y=\lambda x^{j_0}}$ is (ρ,σ)-homogeneous we get

$$\ell_{\rho,\sigma}(P(\alpha)) = \ell_{\rho,\sigma}(\mathrm{ev}_{y=\lambda x^{j_0}}(\varphi(P))) = \mathrm{ev}_{y=\lambda x^{j_0}}(\ell_{\rho,\sigma}(\varphi(P))) = x^{\lambda_\tau^P} f_{P,\tau}(\lambda).$$

Thus, we achieve

$$\deg_x(P(\alpha) = \deg_x(\ell_{\rho,\sigma}(P(\alpha))) = \lambda_\tau^P = \deg_x(P(\tau)),$$

as needed. □

4 Approximate roots for Jacobian pairs

For the rest of the section (P_0, Q_0) will be a Jacobian pair in L satisfying the conditions required in [12, Corollary 5.21]. This in particular means that (P_0, Q_0) is a minimal pair and a standard (m,n)-pair for some coprime integers $m, n > 1$. By [12, Proposition 4.6(3)], there exist $a < b$ in $\mathbb{N}$ such that $\mathrm{en}_{1,0}(P_0) = m(a,b)$ and $\mathrm{en}_{1,0}(Q_0) = n(a,b)$. So, by [12, Corollary 5.21(4)], we know that $\ell_{1,1}(P_0) = \lambda x^{am} y^{bm}$ and $\ell_{1,1}(Q_0) = \lambda' x^{an} y^{bn}$ hold for some $\lambda, \lambda' \in K^\times$. Replacing P_0 by $\frac{1}{\lambda} P_0$ and Q_0 by $\frac{1}{\lambda'} Q_0$, we can further assume $\lambda = \lambda' = 1$. Let ψ be the automorphism of L characterized by $\psi(y) = y$ and $\psi(x) = x + y$. Set $P = \psi(P_0)$ and $Q = \psi(Q_0)$ (see Figure 1). Since ψ is $(1,1)$-homogeneous we have

$$\ell_{1,1}(P) = \psi(\ell_{1,1}(P_0)) = (x+y)^{ma} y^{mb} \quad \text{and} \quad \ell_{1,1}(Q) = (x+y)^{na} y^{nb}. \tag{4.1}$$

Hence, P and Q are monic polynomials in y and, moreover, a straightforward computation yields

$$\mathrm{en}_{1,0}(P) = m(a,b) \quad \text{and} \quad \mathrm{en}_{1,0}(Q) = n(a,b). \tag{4.2}$$

Remark 4.1. We will establish several results about P, but, since by [12, Proposition 4.6] we know that (Q,P) is an (n,m)-pair, the same results remain valid, *mutatis mutandis*, for Q.

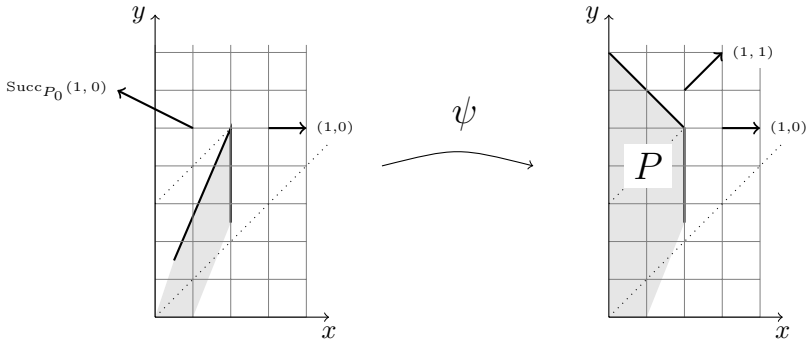

Figure 1: The shapes of P_0 according to [12, Corollary 5.21(4)] and of P according to (4.1) and (4.2).

Proposition 4.2. *Let $\alpha \in \mathcal{R}(P)$ and let τ be the π-approximation of α up to x^{j_0}. Assume $\lambda_\tau = \deg_x(P(\tau)) > 0$, and take φ and (ρ, σ) as in Proposition 3.3. Then the following facts hold.*

(1) *If $f_{P,\tau}$ has multiple roots, then $[\ell_{\rho,\sigma}(\varphi(P)), \ell_{\rho,\sigma}(\varphi(Q))] = 0$.*

(2) *If $[\ell_{\rho,\sigma}(\varphi(P)), \ell_{\rho,\sigma}(\varphi(Q))] = 0$, then there exists $\beta \in \mathcal{R}(Q)$ such that $\deg_x(\alpha - \beta) < j_0$.*

Proof. Write $\tau = \sum_{j>j_0} a_j x^j + \pi x^{j_0}$. By Proposition 3.7 there exists $j_1 < j_0$ such that

$$\tau_1 = \sum_{j>j_0} a_j x^j + \lambda x^{j_0} + \pi x^{j_1}$$

is a π-root of P.

(1) Since we have $\ell_{\rho,\sigma}(\varphi(P)) = x^{\lambda_\tau} f_{P,\tau}(z)$ (see Equality (3.3)), by hypothesis there exist $k > 1$ and $\lambda \in K$ such that $(z - \lambda)^k$ divides $\ell_{\rho,\sigma}(\varphi(P))$. Consequently $(z-\lambda)^{k-1}$ divides $[\ell_{\rho,\sigma}(\varphi(P)), \ell_{\rho,\sigma}(\varphi(Q))]$. As we have $[\ell_{\rho,\sigma}(\varphi(P)), \ell_{\rho,\sigma}(\varphi(Q))] \in K$, this yields $[\ell_{\rho,\sigma}(\varphi(P)), \ell_{\rho,\sigma}(\varphi(Q))] = 0$.

(2) Let $(z-\lambda)$ be a linear factor of $\ell_{\rho,\sigma}(\varphi(P))$. Since we have $v_{\rho,\sigma}(P) = \rho\lambda_\tau > 0$, from [12, Proposition 2.1(2)b)] it follows that $(z-\lambda)$ divides $\ell_{\rho,\sigma}(\varphi(Q))$. Hence, by Proposition 3.3 we know that τ is a π-root of Q and so, by Proposition 3.7, there exists $j_2 < j_0$ such that

$$\tau_2 = \sum_{j>j_0} a_j x^j + \lambda x^{j_0} + \pi x^{j_2}$$

is a π-root of Q. We conclude that for any $\alpha \in D^P_{\tau_1}$ and $\beta \in D^Q_{\tau_2}$ the inequality $\deg_x(\alpha-\beta) < j_0$ holds, as claimed. □

Remark 4.3. Let $\alpha = \sum a_j x^j \in \mathcal{R}(P)$. Assume $j_0 > j_1$ and that τ approximates α up to x^{j_0} and τ_1 approximates α up to x^{j_1}. Then we must have $\lambda_\tau > \lambda_{\tau_1}$. In fact, setting $(\rho,\sigma) = \mathrm{dir}(j_0)$ and $(\rho_1,\sigma_1) = \mathrm{dir}(j_1)$, Equality (3.3) and [12, Proposition 3.9] show

$$v_{\rho,\sigma}(x^{\lambda_\tau}) = v_{\rho,\sigma}(\varphi(P)) = v_{\rho,\sigma}(\varphi_1(P)) \geq v_{\rho,\sigma}(\mathrm{en}_{\rho_1,\sigma_1}\varphi_1(P)),$$

with $\varphi = \varphi_\tau$ and $\varphi_1 = \varphi_{\tau_1}$. A direct computation using $(\rho,\sigma) > (\rho_1,\sigma_1)$, $v_{\rho_1,\sigma_1}(\varphi_1(P)) = v_{\rho_1,\sigma_1}(x^{\lambda_{\tau_1}})$ and $v_{0,1}(\mathrm{en}_{\rho_1,\sigma_1}(\varphi_1(P))) > v_{0,1}(x^{\lambda_{\tau_1}})$ derives in

$$v_{\rho,\sigma}(\mathrm{en}_{\rho_1,\sigma_1}(\varphi_1(P))) > v_{\rho,\sigma}(x^{\lambda_{\tau_1}}).$$

Since $\rho > 0$, the result follows.

Proposition 4.4. *Let $\alpha \in \mathcal{R}(P)$. Then there exists j_0 such that $\lambda_\tau = 0$ for the π-approximation τ of α up to x^{j_0}.*

Proof. Let $\varphi_0 \in \mathrm{Aut}(K((x^{-1/l}))[y])$ be given by $\varphi_0(x^{1/l}) = x^{1/l}$ and $\varphi_0(y) = y + \alpha$. We will construct a direction $(\rho_0,\sigma_0) \in](0,-1),(0,1)[$ such that $v_{\rho_0,\sigma_0}(\varphi_0(P)) = 0$. In order to achieve this, for each point of $\mathrm{Supp}(\varphi_0(P))$ we consider the direction $(\rho,\sigma) \in](0,-1),(0,1)[$ orthogonal to the segment that joins that point to the origin. The minimum (ρ_0,σ_0) of these directions satisfies $v_{\rho_0,\sigma_0}(\varphi_0(P)) = 0$. Set $j_0 = \dfrac{\sigma_0}{\rho_0}$. We assert that the π-approximation

$$\tau = \sum_{j>j_0} a_j x^j + \pi t^{j_0}$$

of α up to x^{j_0} satisfies $\lambda_\tau = 0$. In fact, we have

$$0 = v_{\rho_0,\sigma_0}(\varphi_0(P)) = v_{\rho_0,\sigma_0}(\varphi_\tau(P)) = v_{\rho_0,\sigma_0}(x^{\lambda_\tau}) = \rho_0\lambda_\tau,$$

where the second equality follows from [12, Proposition 3.9], and the third, from (3.3). □

Proposition 4.5. *Let $\tau = \sum_{j>j_0} a_j x^j + \pi x^{j_0}$ be a π-root of P, and let (ρ,σ), λ_τ and φ be as in Proposition 3.3. If τ is also a π-root of Q and we have $\lambda_\tau \geq 0$, then we get*

$$\mathrm{en}_{\rho,\sigma}(\varphi(Q)) = \frac{n}{m}\,\mathrm{en}_{\rho,\sigma}(\varphi(P)) \quad \textit{and} \quad \frac{|D^Q_\tau|}{|D^P_\tau|} = \frac{n}{m}.$$

Proof. Write $\mathrm{Dir}(\varphi(P)) \cap [(\rho,\sigma),(1,1)] = \{(\rho,\sigma) = (\rho_0,\sigma_0) < (\rho_1,\sigma_1) < \cdots < (\rho_k,\sigma_k) = (1,1)\}$. Take $\alpha \in D^P_\tau$ and $0 \leq i \leq k$. Let $j_i = \dfrac{\sigma_i}{\rho_i}$ and let τ_i be the π-approximation of α up to x^{j_i}. Set $\lambda_{\tau_i} = \deg_x(P(\tau_i))$ and $\varphi_i = \varphi_{\tau_i}$. Since we have

$$[\ell_{\rho_i,\sigma_i}(\varphi(P)), \ell_{\rho_i,\sigma_i}(\varphi(Q))] \in K,$$

if $[\ell_{\rho_i,\sigma_i}(\varphi(P)), \ell_{\rho_i,\sigma_i}(\varphi(Q))] \neq 0$, then $v_{0,-1}([\ell_{\rho_i,\sigma_i}(\varphi(P)), \ell_{\rho_i,\sigma_i}(\varphi(Q))]) = 0$, and next, by [12, Proposition 1.13], we obtain

$$\begin{aligned} 0 =& v_{0,-1}([\ell_{\rho_i,\sigma_i}(\varphi(P)), \ell_{\rho_i,\sigma_i}(\varphi(Q))]) \\ \leq& v_{0,-1}(\ell_{\rho_i,\sigma_i}(\varphi(P))) + v_{0,-1}(\ell_{\rho_i,\sigma_i}(\varphi(Q))) - (-1+0). \end{aligned}$$

This in turn implies

$$v_{0,-1}(\mathrm{st}_{\rho_i,\sigma_i}(\varphi(P)) + v_{0,-1}(\mathrm{st}_{\rho_i,\sigma_i}(\varphi(Q)) \geq -1,$$

or, equivalently,

$$v_{0,1}(\mathrm{st}_{\rho_i,\sigma_i}(\varphi(P)) + v_{0,1}(\mathrm{st}_{\rho_i,\sigma_i}(\varphi(Q)) \leq 1;$$

hence in this case we obtain $i = 0$. The bottom line is that, if we want $i > 0$, we must have $[\ell_{\rho_i,\sigma_i}(\varphi(P)), \ell_{\rho_i,\sigma_i}(\varphi(Q))] = 0$, where we must take $\lambda_{\tau_i} > 0$ by Remark 4.3. So we can conclude

$$v_{\rho_i,\sigma_i}(\varphi(P)) = v_{\rho_i,\sigma_i}(\varphi_i(P)) = \rho_i\lambda_{\tau_i} > 0;$$

here the first equality follows from [12, Proposition 3.9] and the second from (3.3). Now, an inductive argument using (4.1), [12, Remark 3.1] and $\mathrm{en}_{\rho_i,\sigma_i}(\varphi(P))) = \mathrm{st}_{\rho_{i+1},\sigma_{i+1}}(\varphi(P)))$, for $i = k, \dots, 1$, proves

$$v_{\rho_i,\sigma_i}(\varphi(Q)) > 0 \text{ and } \mathrm{st}_{\rho_i,\sigma_i}(\varphi(Q)) = \frac{n}{m}\,\mathrm{st}_{\rho_i,\sigma_i}(\varphi(P)), \text{ for } i = k, \dots, 1.$$

So we get

$$\mathrm{en}_{\rho_0,\sigma_0}(\varphi(Q)) = \frac{n}{m}\,\mathrm{en}_{\rho_0,\sigma_0}(\varphi(P)) \quad \text{and} \quad \frac{v_{0,1}(\mathrm{en}_{\rho_0,\sigma_0}(\varphi(Q)))}{v_{0,1}(\mathrm{en}_{\rho_0,\sigma_0}(\varphi(P)))} = \frac{n}{m}.$$

This finishes the proof, as we have $\dfrac{|D_\tau^Q|}{|D_\tau^P|} = \dfrac{v_{0,1}(\mathrm{en}_{\rho,\sigma}(\varphi(Q)))}{v_{0,1}(\mathrm{en}_{\rho,\sigma}(\varphi(P)))}$, by Proposition 3.3. □

In [14] the author chooses a generic element $\xi \in K$ and analyses the roots of $P_\xi = P + \xi$. Instead of speaking of a generic element ξ, we will assume (summing eventually to P an element $\xi \in K$) that any π-root τ of P with $\lambda_\tau = 0$ is such that

(1) $f_{P,\tau}$ has no multiple roots;

(2) $f_{P,\tau}$ and $f_{Q,\tau}$ have no common roots (thus are coprime).

This is possible, since, by (3.3), in the case $\lambda_\tau = 0$ adding ξ to P is the same as adding ξ to the univariate polynomial $f_{P,\tau}(z)$. We also can, and will, assume $(0,0) \in \mathrm{Supp}(P) \cap \mathrm{Supp}(Q)$.

Remark 4.6. Suppose that τ is a π-root of P with $\lambda_\tau < 0$. Then, by Proposition 3.7, Remark 3.2 and Item (1) we have $|D_\tau^P| = 1$. Moreover, we also get $|D_\tau^Q| = 0$. In fact, take $\alpha \in D_\tau^P$. By Proposition 4.4 there exists j_1 and a π-approximation τ_1 of α up to x^{j_1} such that $\lambda_{\tau_1} = 0$. By Remark 4.3 necessarily $j_1 > j_0$, where j_0 is the order of τ. Let λ be the coefficient of α at x^{j_1}. Then $\pi - \lambda$ divides f_{P,τ_1} and so, by item (2), $\pi - \lambda$ does not divide f_{Q,τ_1}. If τ_1 is not a π-root of Q, then clearly $|D_\tau^Q| = 0$. Otherwise, by Corollary 3.8 applied to τ_1 and Q, we also reach $|D_\tau^Q| = 0$.

Remark 4.7. From the first assertion in the previous remark it follows that for any final π-root τ of P we must have $\lambda_\tau \geq 0$.

Notation 4.8. Let $\alpha = \sum_j a_j x^j \in \mathcal{R}(P)$ and set

$$\delta_\alpha = \min\{\deg_x(\alpha - \beta) : \beta \in \mathcal{R}(Q)\}.$$

Remark 4.9. Note that the π-approximation of α up to x^{δ_α} is also a π-root of Q.

Proposition 4.10. *(Compare [14, Lemma 4.2]) Set* $\tau = \sum_{j>\delta} a_j x^j + \pi x^{\delta_\alpha}$. *Then* τ *is a final* π*-root of* P.

Proof. Since clearly τ is a π-root of P, we only need to prove that τ is a final π-root of P, *i.e.*, that we have $\deg(f_{P,\tau}) > 1$ and that $f_{P,\tau}$ has no multiple roots. By Remark 4.6 we know $\lambda_\tau \geq 0$. By Item (1) above, we also know that when $\lambda_\tau = 0$, the polynomial $f_{P,\tau}$ has no multiple roots. If $\lambda_\tau > 0$, then $f_{P,\tau}$ also does not have multiple roots: otherwise by Proposition 4.2 there exists $\beta \in \mathcal{R}(Q)$ such that $\deg_x(\alpha - \beta) < \delta_\alpha$, contradicting the definition of δ_α. Finally, by Proposition 4.5 we know that m divides $|D_\tau^P| = \deg(f_{P,\tau})$, and so, we obtain $\deg(f_{P,\tau}) > 1$, which concludes the proof. □

5 Major and minor final π-roots

A final π-root τ of P is called a **minor final π-root of** P if $\lambda_\tau = 0$, and it is called a **major final π-root of** P if $\lambda_\tau > 0$. The set of minor final π-roots of P is denoted by P_m, while the set of final major π-roots of P is denoted by P_M.

Note that we have

$$\mathcal{R}(P) = \bigcup_{\tau \in P_m \cup P_M} D_\tau^P,$$

since, by Proposition 4.10, every root $\alpha \in \mathcal{R}(P)$ is associated with a final π-root of P (that we will call the **final π-root of P associated with**

α), here $\lambda_\tau \geq 0$ by Remark 4.7. Note also that if $\tau \neq \tau_1$ are final π-roots, then $D^P_\tau \cap D^P_{\tau_1} = \emptyset$. In fact, assume by contradiction $\alpha \in D^P_\tau \cap D^P_{\tau_1}$, and take for example $\delta_\tau < \delta_{\tau_1}$, which means that τ is a better approximation of α than τ_1. Then, since the multiplicity of any factor of f_{P,τ_1} is one, by Remark 3.9 we get $|D^P_\tau| \leq 1$, which contradicts the fact that τ is a final π-root of P.

Remark 5.1. Given a final π-root τ of P take $\alpha \in D^P_\tau$. Then, by Proposition 4.10, the π-approximation of α up to x^{δ_α} is a final π-root, and, since $D^P_\tau \cap D^P_{\tau_1} = \emptyset$ for any other final π-root τ_1 of P, necessarily τ is the π-approximation of α up to x^{δ_α}. This is equivalent to $\delta_\tau = \delta_\alpha$.

Proposition 5.2. *Let τ be a final π-root of P, let $\varphi = \varphi_\tau$ and set $\lambda^Q_\tau = \deg_x(Q(\tau))$. Then we have the following.*

(1) *If τ is a minor final π-root of P, then we have*

   a) $\lambda^Q_\tau = 0$,
   b) $[\ell_{\rho,\sigma}(\varphi(Q)), \ell_{\rho,\sigma}(\varphi(P))] = 0$,
   c) $\delta_\tau < -1$.

(2) *If τ is a major final π-root of P, then we have*

   a) $[\ell_{\rho,\sigma}(\varphi(Q)), \ell_{\rho,\sigma}(\varphi(P))] \neq 0$,
   b) τ *is a major final π root of Q,*
   c) $\lambda^Q_\tau = \dfrac{n}{m} deg_x(P(\tau))$,
   d) $\delta_\tau > -1$.

Proof. By Remarks 4.9 and 5.1, any final π-root τ of P is also a π-root of Q. We will use this fact for (1)a) and (2)b).

(1) By Proposition 4.5, since $\lambda_\tau \geq 0$, we have

$$m \operatorname{en}_{\rho,\sigma}(\varphi(Q)) = n \operatorname{en}_{\rho,\sigma}(\varphi(P)),$$

and so we obtain

$$\rho\lambda^Q_\tau = v_{\rho,\sigma}(\varphi(Q)) = \frac{n}{m} v_{\rho,\sigma}(\varphi(P)) = \frac{n}{m}\rho\lambda^P_\tau = 0,$$

where the first and third equality follow from (3.3). This implies $\lambda_\tau^Q = \deg_x(Q(\tau)) = 0$, thus proving a). Moreover, by [12, Proposition 2.1(1)] we know that the vanishing of $v_{\rho,\sigma}(\varphi(Q))$ and $v_{\rho,\sigma}(\varphi(P))$ implies

$$[\ell_{\rho,\sigma}(\varphi(Q)), \ell_{\rho,\sigma}(\varphi(P))] = 0,$$

this proves item b). Now assume by contradiction $\dfrac{\sigma}{\rho} = \delta_\tau \geq -1$, which implies $\rho + \sigma \geq 0$. Then, by [12, Proposition 1.13], we have

$$0 = v_{\rho,\sigma}([\varphi(P), \varphi(Q)]) \leq v_{\rho,\sigma}(\varphi(Q)) + v_{\rho,\sigma}(\varphi(P)) - (\rho+\sigma) = -(\rho+\sigma) \leq 0,$$

so we have equality and, again by [12, Proposition 1.13], we also obtain $[\ell_{\rho,\sigma}(\varphi(Q)), \ell_{\rho,\sigma}(\varphi(P))] \neq 0$. But this contradicts item b) and thus proves $\delta_\tau < -1$, that is, part c).

(2) By Remarks 4.9 and 5.1, we know that τ is a π-root of Q and, that for any $\alpha \in D_\tau^P$, we have

$$\delta_\tau = \min\{\deg_x(\alpha - \beta) | \beta \in \mathcal{R}(Q)\}.$$

Hence, by Proposition 4.2(2), we obtain $[\ell_{\rho,\sigma}(\varphi(Q)), \ell_{\rho,\sigma}(\varphi(P))] \neq 0$, which proves a). Moreover, by Proposition 4.2(1) with Q and P interchanged, $f_{Q,\tau}$ has no multiple roots. On the other hand, by Proposition 4.5, we have

$$|D_\tau^Q| = \frac{n}{m}|D_\tau^P| > 1,$$

and so τ is a final π-root of Q. Again Proposition 4.5 and Equality (3.3) yield

$$\begin{aligned}\rho \deg_x Q(\tau) =& \rho\lambda_\tau^Q = v_{\rho,\sigma}(\varphi(Q)) = \frac{n}{m} v_{\rho,\sigma}(\varphi(P)) \\ =& \frac{n}{m}\rho\lambda_\tau^P = \rho\frac{n}{m}\deg_x(P(\tau)),\end{aligned}$$

and so $\deg_x Q(\tau) = \dfrac{n}{m}\deg_x P(\tau)) > 0$, which finishes the proof of b) and c). It remains to check the condition $\delta_\tau > -1$. Assume by contradiction $\dfrac{\sigma}{\rho} = \delta_\tau \leq -1$. Then $\rho + \sigma \leq 0$, and so

$$v_{\rho,\sigma}(\varphi(Q)) + v_{\rho,\sigma}(\varphi(P)) - (\rho+\sigma) \geq \rho\lambda_\tau^P \left(1 + \frac{n}{m}\right) > 0 = v_{\rho,\sigma}[\varphi(P), \varphi(Q)],$$

which, by [12, Proposition 1.13], implies $[\ell_{\rho,\sigma}(\varphi(Q)), \ell_{\rho,\sigma}(\varphi(P))] = 0$. This contradicts item a) and thus finishes the proof of part d). □

6 Intersection number and major roots

In this section we first obtain in Theorem 6.2 the same formula for I_M as in [14, Theorem 5.1]. Then we explain how to compute I_M for the families found in [7].

Lemma 6.1. *For τ a final π-root of P, we have $\lambda_\tau^Q = \deg_x(Q(\tau)) = \deg_x(Q(\alpha))$ whenever $\alpha \in D_\tau^P$.*

Proof. We assert that $f_{P,\tau}(z)$ and $f_{Q,\tau}(z)$ have no common roots. In fact, assume on the contrary that $z - s$ is a common factor. If τ is a major final root, then

$$z - s \mid [\lambda_\tau f_{P,\tau}(z), \lambda_\tau^Q f_{Q,\tau}(z)] = [\ell_{\rho,\sigma}(\varphi(Q)), \ell_{\rho,\sigma}(\varphi(P))] \in K^\times,$$

a contradiction. Whereas, if τ is a minor root, then the choice of ξ guarantees that $f_{P,\tau}$ and $f_{Q,\tau}$ have no common roots.

Note that if the coefficient of x^{j_0} in α is s, then $f_{P,\tau}(s) = 0$, since otherwise $\pi - s$ does not divide $f_{P,\tau}(\pi)$ and Corollary 3.8 leads to a contradiction. Hence, by the assertion, we get $f_{Q,\tau}(s) \neq 0$, and from Proposition 3.11, we obtain $\deg_x(Q(\tau)) = \deg_x(Q(\alpha))$. □

Theorem 6.2. *For $I_M = \sum_{\tau \in P_M} |D_\tau^{P_\xi}| \lambda_\tau^Q$ we have $I_M = I(P, Q)$.*

Proof. From the well know equality $\mathrm{Res}_y(P, Q) = \prod_{\alpha \in \mathcal{R}(P)} Q(\alpha)$ we pass to

$$I(P,Q) = \deg_x \prod_{\alpha \in \mathcal{R}(P)} Q(\alpha) = \sum_{\alpha \in \mathcal{R}(P)} \deg_x(Q(\alpha)) \tag{6.1}$$

$$= \sum_{\tau \in P_m \cup P_M} \sum_{\alpha \in D_\tau^P} \deg_x(Q(\alpha)). \tag{6.2}$$

Using Lemma 6.1 we arrive to

$$\begin{aligned} I(P,Q) &= \sum_{\tau\in P_m\cup P_M}\sum_{\alpha\in D_\tau^P}\deg_x(Q(\alpha)) \\ &= \sum_{\tau\in P_M}|D_\tau^{P_\xi}|\lambda_\tau^Q + \sum_{\tau\in P_m}|D_\tau^{P_\xi}|\lambda_\tau^Q = \sum_{\tau\in P_M}|D_\tau^{P_\xi}|\lambda_\tau^Q, \end{aligned}$$

since $\lambda_\tau^Q = 0$ if $\tau \in P_m$. □

A root $\alpha \in \mathcal{R}(P)$ is called a **minor root** if the associated final π-root τ is a minor final π-root; it is called a **major root** if τ is a major final π-root.

Proposition 6.3. *Let τ be an approximate π-root of P of order $j_0 \le 0$, with $\lambda_\tau \ge 0$, and let $(\rho,\sigma) = \mathrm{dir}(j_0)$. If $v_{1,-1}(\mathrm{en}_{\rho,\sigma}(\varphi_\tau(P))) > 0$, then any root $\alpha \in D_\tau^P$ is a minor root.*

Proof. The hypotheses guarantee that $(\varphi_\tau(P),\varphi_\tau(Q))$ and (ρ,σ) satisfy the hypotheses of Proposition 2.1 (for instance $(\rho,\sigma) \in](0,-1),(1,0)]$, because of $j_0 \le 0$). If $v_{\rho,\sigma}(\varphi_\tau(P)) = \rho\lambda_\tau = 0$, then τ is a minor final π-root and the result is true. Else we have $v_{\rho,\sigma}(\varphi_\tau(P)) = \rho\lambda_\tau > 0$, since $\lambda_\tau \ge 0$. Take $\alpha \in D_\tau^P$. By Proposition 5.2 it suffices to prove $\delta_\alpha < -1$. By Propositions 2.1 and 4.2 we have $\delta_\alpha < \delta_\tau = j_0$, so the result is clear when $\delta_\tau \le -1$. Assume $\delta_\tau > -1$. In this case we have $\rho+\sigma > 0$. Using Proposition 2.2 and Equality (3.3) we obtain $f_{P,\tau}(z) = \varsigma(z-\mu)^{mb}$ for some $\varsigma,\mu \in K^\times$, where $b = \dfrac{1}{m}v_{0,1}(\mathrm{en}_{\rho,\sigma}(\varphi_\tau(P))) = \dfrac{|D_\tau^P|}{m}$ (see Proposition 3.3). Hence, by Proposition 3.7, there exists $j_1 < j_0$ such that for the π-root

$$\tau_1 = \sum_{j>j_0} a_j x^j + \mu x^{j_0} + \pi x^{j_1}$$

we have $D_{\tau_1}^P = D_\tau^P$. If $j_1 \le -1$, then we finish the proof immediately after applying the above argument with τ replaced by τ_1 since we must have $\lambda_{\tau_1} \ge 0$ (in fact, if $\lambda_{\tau_1} < 0$, then, by Remark 4.6, we get $|D_{\tau_1}^P| = 1$,

which is impossible because of $bm = |D^P_\tau|$). Assume now $j_1 > -1$ and set $(\rho_1, \sigma_1) = \mathrm{dir}(j_1)$. By Proposition 2.2 ρ_1 divides l, and so $j_1 \in \frac{1}{l}\mathbb{Z}$. Hence, if $j_0 = -\frac{k}{l}$ for some $0 \leq k \leq l$, then $-j_1 \in \{\frac{k+1}{l}, \frac{k+2}{l}, \ldots, \frac{l-1}{l}, \frac{l}{l}\}$, so, after repeating the same procedure a finite number of times, say t, we arrive at $\delta_\alpha < j_t \leq -1$, as desired. □

Proposition 6.4. *Let a, b satisfy Equalities* (4.1)*. There exist ma minor roots α of P with $\deg_x(\alpha) = 1$ and leading term $-x$, and mb roots β of P with $\deg_x(\beta) \leq 0$.*

Proof. Take $\tau_0 = \pi x^0$. Then we have $j_0 = 0$, $\mathrm{dir}(j_0) = (1, 0)$ and $\varphi_{\tau_0} = \mathrm{id}$. From the first equality in (4.2), we get

$$\mathrm{en}_{\rho,\sigma}(\varphi_{\tau_0}(P)) = \mathrm{en}_{1,0}(P) = m(a, b),$$

and by Proposition 3.3, we obtain $|D^P_{\tau_0}| = mb$. Since $\deg_x(\beta) \leq j_0 = 0$ for all $\beta \in D^P_{\tau_0}$, this yields mb roots with $\deg_x(\beta) \leq 0$. On the other hand, by Proposition 3.7, with $\tau = \pi x$, $\lambda = -1$ and $\varphi_1(y) = y - x$, there exists $j_1 < 1$ such that the π-root $\tau_1 = -x + \pi x^{j_1}$ satisfies $|D^P_{\tau_1}| = ma$ since $f_{P,\tau}(z) = (z+1)^{ma} z^{mb}$; therefore the multiplicity of $\lambda = -1$ is ma. Moreover, by (3.5) and the first equality in (4.1), we have

$$\mathrm{en}_{\rho_1,\sigma_1}(\varphi_1(P)) = \mathrm{st}_{1,1}(\varphi_1(P)) = m(b, a),$$

and so $v_{1,-1}(\mathrm{en}_{\rho_1,\sigma_1}(\varphi_1(P))) > 0$. Then every root $\alpha \in D^P_{\tau_1}$ is a minor root. □

Following [14], the minor roots in Proposition 6.4 are called **top minor roots**.

Proposition 6.5. *Let $\alpha \in \mathcal{R}(P)$ be a major root, let τ be the associated final π-root and let $(\rho, \sigma) = \mathrm{dir}(\delta_\alpha)$. Then $\left(\frac{1}{m}\,\mathrm{en}_{\rho,\sigma}(\varphi_\tau(P)), (\rho, \sigma)\right)$ is a regular corner of type I of $(\varphi_\tau(P), \varphi_\tau(Q))$ (see [12, Definition 5.5] and the discussion above Remark 5.9 in [12] for the classification of regular corners).*

Proof. Item (3) of [12, Definition 5.5] holds by hypothesis, Item (1) holds by the very definition of π-root, Proposition 6.3 and [12, Theorem 2.6(4)], and Item (2) holds by Remark 3.6. Moreover, Proposition 5.2(2)a) proves that $\left(\frac{1}{m}\operatorname{en}_{\rho,\sigma}(\varphi_\tau(P)),(\rho,\sigma)\right)$ is of type I. □

Proposition 6.6. *Let $j_0 < j_1 < \cdots < j_k \in \frac{1}{l}\mathbb{Z}$ and let $(\rho,\sigma) = \operatorname{dir}(j_0)$. Consider the automorphism φ of $L^{(l)}$ defined by*

$$\varphi(x^{1/l}) = x^{1/l} \quad \text{and} \quad \varphi(y) = y + \sum_{i=1}^{k} a_i x^{j_i}.$$

Let $A = ((a/l,b),(\rho,\sigma))$ be a regular corner of $(\varphi(P),\varphi(Q))$. Then the following facts hold.

(1) *$\tau = \sum_{i=1}^{k} a_i x^{j_i} + \pi x^{j_0}$ is a π-root of P and Q.*

(2) *If A is of type Ib, then τ is a final major π-root of P and Q, with*

$$|D_\tau^P| = mb \quad \text{and} \quad |D_\tau^Q| = nb; \tag{6.3}$$

moreover, if $\operatorname{st}_{\rho,\sigma}(\varphi(Q)) = (k/l,0)$ for some $1 \le k < l - a/b$, then $\lambda_\tau^Q = \frac{k}{l}$.

Proof. (1) By Items (1) and (3) of [12, Definition 5.5], we have $A = \frac{1}{m}\operatorname{en}_{\rho,\sigma}(\varphi(P))$ and $b \ge 1$. Hence, by Equalities (3.2) and (3.3) we conclude $\deg(f_{P,\tau}) > 0$ and that τ is a π-root of P. Since by [12, Corollary 5.7] and Remark 4.1 the equality $A = \frac{1}{n}\operatorname{en}_{\rho,\sigma}(\varphi(Q))$ holds and (Q,P) is an (n,m)-pair, we infer that τ is also a π-root of Q.

(2) The two expressions for A obtained in the proof of Item (1), combined with Equality (3.2) and the corresponding equality for Q yield the equalities in (6.3). Since A is of type Ib, we have

$$[\ell_{\rho,\sigma}(\varphi(P)),\ell_{\rho,\sigma}(\varphi(Q))] \neq 0,$$

and so, by Proposition 4.2(1), the polynomial $f_{P,\tau}$ has no multiple root. Moreover, using again Equality (3.2) and Equality (6.3) we obtain $\deg(f_{P,\tau}) = mb > 1$. This proves that τ is a major final π-root of P, and then, by Proposition 5.2(2)b), also of Q. Finally, assuming $\mathrm{st}_{\rho,\sigma}(\varphi(Q)) = (k/l, 0)$, Equality (3.3) for Q implies $\rho\lambda_\tau^Q = v_{\rho,\sigma}(\varphi(Q)) = \rho\dfrac{k}{l}$, from which the last assertion follows, as $\rho \neq 0$. □

Example 6.7. Consider the family F_1 of [7], corresponding to an (m, n)-pair (P_0, Q_0) as in [12, Corollary 5.21]:

$$A_0 = (4, 12),\ A_0' = (1, 0),\ A_1 = (7/4, 3),\ k = 1,\ m = 2j+3,\ n = 3j+4. \tag{6.4}$$

Then (P_0, Q_0) has the shape given in Figure 2 and we get $\ell_{-2,1}(P) = R^{4m}$ for a $(-2, 1)$-homogeneous element $R = \lambda_0(\lambda_1 y - xy^3)$ with $\lambda_0, \lambda_1 \neq 0$.

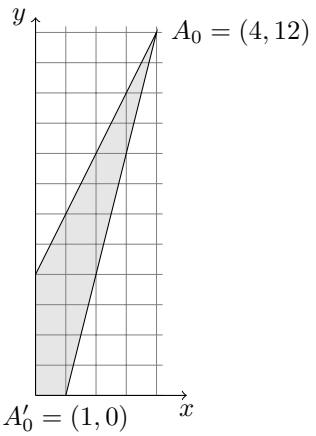

Figure 2: The shape of (P_0, Q_0)

In fact, by (6.4), the edge from A_0 to A_0' is determined. So we only need to prove

$$(\rho, \sigma) = \mathrm{Succ}_{P_0}(1, 0) = \mathrm{Succ}_{Q_0}(1, 0) = (-2, 1)$$

and $\frac{1}{m}\operatorname{en}_{-2,1}(P) = (0,4)$. From [12, Corollary 5.21(4)] we obtain $(-1,1) < (\rho,\sigma) < (-1,0)$. Moreover, by the second equality in [7, (2.13)] we have

$$q_0 = \frac{v_{4,-1}(4,12)}{\gcd(v_{4,-1}(4,12), 4-1)} = \frac{4}{\gcd(4,3)} = 4.$$

On the other hand, at the beginning of [5, Subsection 2.4] it is shown the equality

$$\operatorname{en}_{\rho,\sigma}(F_0) = \frac{p_0}{q_0}\frac{1}{m}\operatorname{en}_{\rho_0,\sigma_0}(P_0),$$

and therefore, by [12, Corollary 7.2], there exists a (ρ,σ)-homogeneous element R such that $\ell_{\rho,\sigma}(P) = R^{4m}$. This is only possible if $(\rho,\sigma) = (-k,1)$ for some $k \in \mathbb{N}$, with $k \geq 2$. But $k \geq 3$ leads to $v_{\rho,\sigma}(P_0) \leq 0$ and then to $\deg_y(P_0(0,y)) \leq 0$, which contradicts [13, Proposition 10.2.6]. Therefore we have $k = 2$, $R = \lambda_0(\lambda_1 y - xy^3)$ and hence we get

$$\frac{1}{m}\operatorname{en}_{-2,1}(P) = 4\operatorname{en}_{-2,1}(R) = (0,4),$$

as desired. Since $P = \psi(P_0)$ and $Q = \psi(Q_0)$, where $\psi(y) = y$ and $\psi(x) = x + y$ (see the beginning of Section 4), the shape of P is as in Figure 3, and P is a monic polynomial in y of degree $16m$.

Write $\ell_{4,-1}(P) = x^m g(z)^m$, where $z = x^{1/4}y$. By [5, Theorem 2.20(6)] and the condition $v_{1,-1}(A_0') > 0$, we know that $(A_0, (\rho,\sigma)) = ((4,12), (4,-1))$ is a regular corner of type IIb) of (P,Q). Hence, by item (8) of the same theorem, we get $v_{0,1}(A_1) = \frac{m_\lambda}{m}$, where m_λ is the multiplicity of $z-\lambda$ in $\mathfrak{p}_0(z) = g(z)^m$. Since we have $v_{0,1}(A_1) = 3$, by [5, Remarks 3.8 and 3.9] we get

$$g(z) = \lambda_0(z^4 - \lambda_1^4)^3,$$

for some $\lambda_0, \lambda_1 \in K^\times$. We obtain then

$$\ell_{4,-1}(P_\xi) = \lambda_0 x^m (z-\lambda_1)^{3m}(z-i\lambda_1)^{3m}(z+\lambda_1)^{3m}(z+i\lambda_1)^{3m},$$

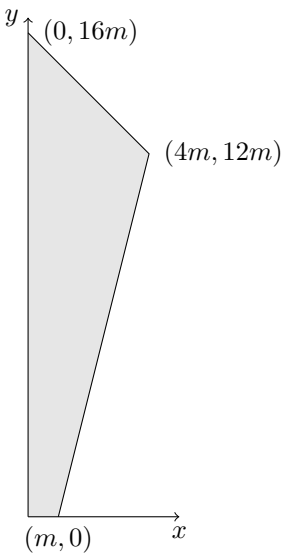

Figure 3: The shape of P_ξ

and so, we have four final major π-roots

$$\tau_0 = \lambda_1 x^{1/4} + \pi x^\delta, \qquad \tau_1 = i\lambda_1 x^{1/4} + \pi x^\delta,$$
$$\tau_2 = -\lambda_1 x^{1/4} + \pi x^\delta, \qquad \tau_3 = -i\lambda_1 x^{1/4} + \pi x^\delta,$$

where $\delta = \sigma/\rho$, with $(\rho, \sigma) = \operatorname{dir}\left(m\left(\frac{7}{4}, 3\right) - \left(\frac{3}{4}, 1\right)\right)$. Here $A_1 = \left(\frac{7}{4}, 3\right)$ is the same final corner (see [5, Definition 2.18]) for all major final roots, corresponding to the regular corner $(A_1, (\rho, \sigma))$ of type Ib) of each of the four (m, n)-pairs $(\varphi_{\tau_j}(P), \varphi_{\tau_j}(Q))$. By the first equality in (6.3), there are $3m$ roots of P associated to each of these major roots, and by Proposition 6.4, the remaining $4m$ roots of P are minor roots. Now we

can compute

$$I_M = \sum_{\tau \in P_M} |D_\tau^P| \lambda_\tau^Q = \sum_{j=0}^{3} |D_{\tau_j}^P| \lambda_{\tau_j}^Q$$
$$= 4 \cdot mb \cdot \frac{k}{l} = 4 \cdot m \cdot 3 \cdot \frac{1}{4} = 3m = 3(2j+3).$$

7 Intersection number and minor roots

In this section we obtain an inequality for I_m in Theorem 7.3, as opposed to the equality in [14, Theorem 4.7], whose proof has a serious gap. We also show how to compute I_m for the families of [7]. For the sake of brevity we write P_x, Q_x, P_y and Q_y instead of the partial derivatives $\partial_x P$, $\partial_x Q$, $\partial_y P$ and $\partial_y Q$.

Lemma 7.1. *Let (P, Q) be as above, (ρ, σ) be a direction, with $\rho \neq 0$, and $\alpha \in \mathcal{R}(P)$. Write $\ell_{\rho,\sigma}(P) = x^u g(z)$ with $z = x^{-\sigma/\rho} y$. The following facts hold.*

(1) *If $\deg(g) > 0$, then $\ell_{\rho,\sigma}(P_y) = x^{u-\sigma/\rho} g'(z)$.*

(2) *α is a minor root if and only if $\deg_x(Q(\alpha)) = 0$.*

(3) *Let $\beta \in \mathcal{R}(P_y)$. There exists $\tau \in P_m$ such that $\beta \in D_\tau^{P_y}$ if and only if $\deg_x(P(\beta)) = 0$.*

(4) *If α is a minor root, then $\deg_x(P_y(\alpha)) = -\delta_\alpha$.*

(5) *Let $\tau \in P_m$ and assume that $f_{P_y,\tau}$ and $f_{Q_y,\tau}$ are coprime. Then $\deg_x(Q_y(\beta)) = -\delta_\tau$ for all $\beta \in D_\tau^{P_y}$.*

(6) *Let $\tau \in P_m$ and assume that $f_{P_y,\tau}$ and $f_{Q_y,\tau}$ are not coprime. Then there exists $\beta \in D_\tau^{P_y}$ such that $\deg_x(Q_y(\beta)) < -\delta_\tau$.*

Proof. (1) This follows from the fact that the morphism ∂_y satisfies $\partial_y(x^i y^j) = j x^i y^{j-1}$ for $j > 0$: in fact we have

$$v_{\rho,\sigma}(\partial_y(x^i y^j)) = v_{\rho,\sigma}(x^i y^j) - \sigma.$$

Hence $\ell_{\rho,\sigma}(\partial_y P) = \partial_y \ell_{\rho,\sigma}(P)$ when $\partial_y \ell_{\rho,\sigma}(P) \neq 0$, and so

$$\ell_{\rho,\sigma}(P_y) = \partial_y(x^u g(z)) = x^{u-\sigma/\rho} g'(z),$$

because of $\deg(g) > 0$.

(2) By Items (1)a) and (2)c) of Proposition 5.2, we know that α is a minor root if and only if we have $\lambda_\tau^Q = 0$ for the π-root τ associated to α. This proves (2), since we have $\lambda_\tau^Q = \deg_x(Q(\alpha))$ by Lemma 6.1.

(3) If we define

$$\delta_\beta = \min\{\deg_x(\alpha - \beta) | \alpha \in \mathcal{R}(P)\},$$

and

$$\beta = \sum_{j>\delta_\beta} a_j x^j + \lambda x^{\delta_\beta} + \sum_{j<\delta_\beta} a_j x^j,$$

then $\tau = \sum_{j>\delta_\beta} a_j x^j + \pi x^{\delta_\beta}$ is a π-root of P. From Remark 3.4 we obtain $0 < |D_\tau^{P_y}| = |D_\tau^P| - 1$, and so, by Remark 4.6 we get $\lambda_\tau^P \geq 0$. Take $\alpha \in D_\tau^P$ and let τ_1 be the final π-root of P associated with α. We have $\delta_\alpha \leq \delta_\beta$ (since $\delta_\beta < \delta_\alpha$ implies $|D_\tau^P| = 1$) and hence $\lambda_{\tau_1} \leq \lambda_\tau$, so $\lambda_\tau^P = 0$ if and only if $\tau = \tau_1$ is a final minor π-root of P.

We claim the equality $\lambda_\tau^P = \deg_x(P(\beta))$. In fact, we have $f_{P,\tau}(\lambda) \neq 0$ since otherwise, by Proposition 3.7, there exists $j_1 < \delta_\beta$ such that the π-approximation of β up to j_1 is a π-root of P, contradicting the minimality of δ_β. Hence, by Proposition 3.11, we have $\deg_x(P(\beta)) = \lambda_\tau^P \geq 0$. Therefore, if $\deg_x(P(\beta)) = 0$, then $\beta \in D_\tau^{P_y}$ and $\tau \in P_m$. On the other hand, if $\beta \in D_{\tau_2}^{P_y}$ for some $\tau_2 \in P_m$, then $\delta_\beta \leq \delta_{\tau_2}$, hence $0 \leq \lambda_\tau \leq \lambda_{\tau_2} = 0$, and so $0 = \lambda_\tau^P = \deg_x(P(\beta))$, as desired.

(4) Let $\tau = \sum_{j>\delta_\alpha} a_j x^j + \pi x^{\delta_\alpha}$ be the minor final π-root of P associated with α. Write

$$\alpha = \sum_{j>\delta_\alpha} a_j x^j + \lambda x^{\delta_\alpha} + \sum_{j<\delta_\alpha} a_j x^j.$$

Since $f_{P,\tau}(\lambda) = 0$, and $f_{P,\tau}$ has no multiple roots, we have $f'_{P,\tau}(\lambda) \neq 0$. But by Item (1) we have $f_{P_y,\tau} = f'_{P,\tau}$, and, by Proposition 3.11, we

obtain $\lambda_\tau^{P_y} = \deg_x(P_y(\tau)) = \deg_x(P_y(\alpha))$. Using again Item (1) we get $\lambda_\tau^{P_y} = \lambda_\tau - \sigma/\rho$, and the result follows immediately since $\lambda_\tau = 0$.

(5) Let $\tau = \sum_{j>j_0} a_j x^j + \pi x^{j_0}$ and take $\beta \in D_\tau^{P_y}$. Set

$$\beta = \sum_{j>j_0} a_j x^j + \lambda x^{j_0} + \sum_{j<j_0} a_j x^j.$$

By Corollary 3.8, we have $f_{P_y,\tau}(\lambda) = 0$. As $f_{P_y,\tau}$ is coprime with $f_{Q_y,\tau}$, we get $f_{Q_y,\tau}(\lambda) \neq 0$. By Proposition 3.11, we obtain $\lambda_\tau^{Q_y} = \deg_x(Q_y(\tau)) = \deg_x(Q_y(\beta))$ and by Item (1) we have $\lambda_\tau^{Q_y} = \lambda_\tau^Q - \sigma/\rho$. The result follows immediately from the equality $\lambda_\tau^Q = 0$.

(6) Write $\tau = \sum_{j>j_0} a_j x^j + \pi x^{j_0}$. Let $\lambda \in K$ be such that $f_{Q_y,\tau}(\lambda) = 0 = f_{P_y,\tau}(\lambda)$. By Proposition 3.7, there exist $j_1, j_2 < j_0$ such that $\tau_1 = \sum_{j>j_0} a_j x^j + \lambda x^{j_0} + \pi x^{j_1}$ is a π-root of P_y and $\tau_2 = \sum_{j>j_0} a_j x^j + \lambda x^{j_0} + \pi x^{j_2}$ is a π-root of Q_y. Take $j_3 = \max\{j_1, j_2\}$. Then $\tau_3 = \sum_{j>j_0} a_j x^j + \lambda x^{j_0} + \pi x^{j_3}$ is a π-root of Q_y and P_y. If $\beta \in D_{\tau_3}^{P_y}$, then

$$\beta = \sum_{j>j_0} a_j x^j + \lambda x^{j_0} + \sum_{j<j_0} a_j x^j.$$

With $T = \lambda x^{j_0} + \sum_{j<j_0} a_j x^j$ we have

$$\begin{aligned} Q_y(\beta) &= \mathrm{ev}_{y=T}(\varphi_\tau(Q_y)) = \mathrm{ev}_{y=\lambda x^{j_0}}(\ell_{\rho,\sigma}(\varphi_\tau(Q_y))) + R \\ &= x^{\lambda_\tau^{Q_y}} f_{Q_y,\tau}(\lambda) + R \end{aligned}$$

for some R with $v_{\rho,\sigma}(R) < v_{\rho,\sigma}(\varphi_\tau(Q_y))) = \rho\lambda_\tau^{Q_y}$. Since $f_{Q_y,\tau}(\lambda) = 0$, we obtain

$$\rho \deg_x(Q_y(\beta)) = v_{\rho,\sigma}(Q_y(\beta)) < \rho\lambda_\tau^{Q_y}.$$

However, from Item (1) we have $\lambda_\tau^{Q_y} = \lambda_\tau^Q - \sigma/\rho$, and since $\lambda_\tau^Q = 0$ we conclude

$$\deg_x(Q_y(\beta)) < -\sigma/\rho,$$

as desired. □

Lemma 7.2. *For any $\alpha \in K((x^{-1/l}))$ we have*

$$Q_y(\alpha)\frac{d}{dx}P(\alpha) - P_y(\alpha)\frac{d}{dx}Q(\alpha) \in K^{\times}.$$

Proof. This follows directly from $\frac{d}{dx}P(\alpha) = P_x(\alpha)+P_y(\alpha)\frac{d\alpha}{dx}$, $\frac{d}{dx}Q(\alpha) = Q_x(\alpha) + Q_y(\alpha)\frac{d\alpha}{dx}$ and the Jacobian condition. □

Theorem 7.3. *For $I_m = 1 - \sum_{\tau\in P_m}(\delta_\tau + 1)$ we get $I_m \leq I(P,Q)$. We also have*

$$I(P, P_yQ) = \deg(P) - \sum_{\tau\in P_m} |D_\tau^P|(1+\delta_\tau). \tag{7.1}$$

Proof. It suffices to prove (7.1) and

$$I(P, P_y) \leq \deg(P) - 1 - \sum_{\tau\in P_m} (|D_\tau^P| - 1)(\delta_\tau + 1). \tag{7.2}$$

In fact, Equality (7.1) and Inequality (7.2) yield

$$I(P,Q) = I(P, P_yQ) - I(P, P_y) \geq 1 - \sum_{\tau\in P_m} (\delta_\tau + 1),$$

as desired.

Proof of Equality (7.1). By Lemma 7.2, for each $\alpha \in \mathcal{R}(P)$ we have $P_y(\alpha)\frac{d}{dx}Q(\alpha) \in K^{\times}$. Moreover, by Lemma 7.1(2), if α is a major root, then $\deg_X(Q(\alpha)) = \lambda_\tau^Q > 0$, and so $\deg_x(P_y(\alpha)Q(\alpha)) = 1$. On the other hand, if α is a minor root, then by Proposition 5.2(1)a), Lemma 6.1 and Lemma 7.1(4) we obtain

$$\deg_x(P_y(\alpha)Q(\alpha)) = \deg_x(P_y(\alpha)) = -\delta_\alpha = -\delta_\tau,$$

where τ is the minor final π-root associated with α. Using these facts

we obtain

$$\begin{aligned} I(P, P_yQ) &= \sum_{\alpha\in\mathcal{R}(P)} \deg_x(P_y(\alpha)Q(\alpha)) \\ &= \sum_{\tau\in P_m}\sum_{\alpha\in D_\tau^P} \deg_x(P_y(\alpha)Q(\alpha)) + \sum_{\tau\in P_M}\sum_{\alpha\in D_\tau^P} \deg_x(P_y(\alpha)Q(\alpha)) \\ &= \sum_{\tau\in P_m} |D_\tau^P|(-\delta_\tau) + \sum_{\tau\in P_M} |D_\tau^P| + \sum_{\tau\in P_m} |D_\tau^P| - \sum_{\tau\in P_m} |D_\tau^P| \\ &= \deg(P) - \sum_{\tau\in P_m} |D_\tau^P|(1+\delta_\tau), \end{aligned}$$

where the first equality is justified as in the proof of Theorem 6.2.

Proof of Inequality (7.2). By Lemma 7.2, for each $\beta \in \mathcal{R}(P_y)$, we have $Q_y(\beta)\dfrac{d}{dx}P(\beta) \in K^\times$. Define

$$P_{y,m} = \{\beta \in \mathcal{R}(P_y) : \text{ there is a minor final } \pi\text{-root } \tau \text{ with } \beta \in D_\tau^{P_y}\}.$$

Then, by Lemma 7.1(3), if β is not in $P_{y,m}$, we have $\deg_x(Q_y(\beta)P(\beta)) = 1$. On the other hand, if β is in $P_{y,m}$, then by Items (3), (5) and (6) of Lemma 7.1 we obtain

$$\deg_x(P(\beta)Q_y(\beta)) = \deg_x(Q_y(\beta)) \le -\delta_\tau,$$

where τ is the minor final π-root associated with β. Using these facts

we obtain

$$\begin{aligned} I(P_y, PQ_y) &= \sum_{\beta\in\mathcal{R}(P_y)} \deg_x(P(\beta)Q_y(\beta)) \\ &= \sum_{\tau\in P_m}\sum_{\beta\in D_\tau^{P_y}} \deg_x(P(\beta)Q_y(\beta)) + \sum_{\beta\notin P_{y,m}} \deg_x(P(\beta)Q_y(\beta)) \\ &\le \sum_{\tau\in P_m} |D_\tau^{P_y}|(-\delta_\tau) + \deg(P_y) - \sum_{\tau\in P_m} |D_\tau^{P_y}| \\ &= \deg(P) - 1 - \sum_{\tau\in P_m} |D_\tau^{P_y}|(1+\delta_\tau) \\ &= \deg(P) - 1 - \sum_{\tau\in P_m} (|D_\tau^{P}| - 1)(1+\delta_\tau), \end{aligned} \tag{7.3}$$

where the last equality follows from Remark 3.4.

Since the Jacobian condition implies

$$\operatorname{Res}_y(P_y, Q_y)\operatorname{Res}_y(P_y, P_x) = \operatorname{Res}_y(P_y, Q_yP_x) = \prod_{\beta\in\mathcal{R}(P_y)} Q_y(\beta)P_x(\beta) = 1,$$

we have $I(P_y, Q_y) = 0$, and thus (7.3) yields Inequality (7.2). □

Example 6.7 (continuation). In the case of family F_1 of [7] there is only one minor root τ corresponding to the remaining $4m$ roots of P_ξ, and $\delta_\tau = -3$. Hence we have $I_m = 1-(-3+1) = 3$. Since $I_M = 3(2j+3)$, we have $I_m < I_M$ (which is compatible with Theorem 7.3 and doesn't allow us to disregard this family).

In fact, consider the diagram in Figure 4, where flip : $K[x,y] \to K[x,y]$ is given by $\text{flip}(x) = y$ and $\text{flip}(y) = -x$, and the three morphisms $\varphi, \varphi_1, \widetilde{\varphi} : K[x, x^{-1}, y] \to K[x, x^{-1}, y]$ are characterized by

$$\begin{aligned} &\varphi(x) = x + y, && \varphi_1(x) = x, && \widetilde{\varphi}(x) = x, \\ &\varphi(y) = y, && \varphi_1(y) = y - x, && \widetilde{\varphi}(y) = y + \lambda_1 x^{-2}. \end{aligned}$$

In order to define G set $u = x + y + \lambda_1 x^{-2}$. The morphism $G : K[y]((x^{-1})) \to K[y]((x^{-1}))$ is given by $G(x) = u$ and $G(y) = y +$

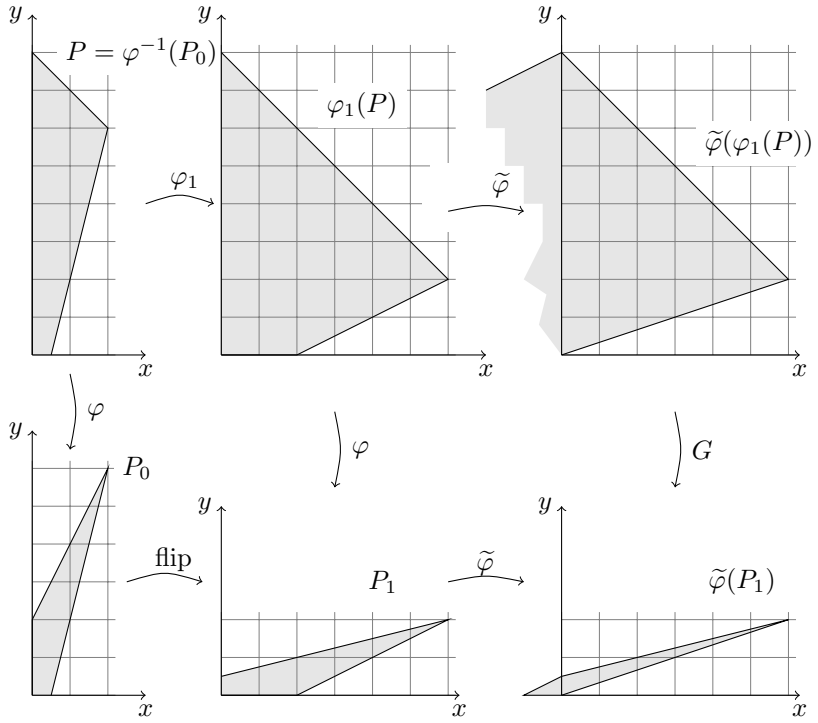

Figure 4: Finding minor roots: $\ell_{1,-3}(\widetilde{\varphi}(\varphi_1(P))) = \ell_{1,-3}(\widetilde{\varphi}(P_1))$

$\lambda_1(x^{-2} - u^{-2})$. Note that u is invertible in $K[y]((x^{-1}))$ with

$$u^{-1} = x^{-1}\left(1 - x^{-1}(y + \lambda_1 x^{-2}) + x^{-2}(y + \lambda_1 x^{-2})^2 - \dots\right).$$

We also have $G \circ \widetilde{\varphi} = \widetilde{\varphi} \circ \varphi$ and $\ell_{1,-3}(G(S)) = \ell_{1,-3}(S)$ for all $S \in K[y]((x^{-1}))$, which implies $\ell_{1,-3}(\widetilde{\varphi}(S)) = \ell_{1,-3}(\widetilde{\varphi}(\varphi(S)))$.

Now, the previous computations yield $\ell_{1,-2}(P_1) = R_1^{4m}$ where

$$R_1 = \mathrm{flip}(R) = \lambda_0 x^3(y - \lambda_1 x^{-2}).$$

Moreover, we also have $\ell_{-1,4}(\widetilde{\varphi}(P_1)) = \ell_{-1,4}(P_1)$ and the element F

of [12, Corollary 7.4] satisfies

$$\mathrm{st}_{-1,4}(F) = (9,3) = \frac{3}{4}\frac{1}{m}\,\mathrm{st}_{-1,4}(\widetilde{\varphi}(P_1)),$$

and so we must have $q = 4$ in [12, Corollary 7.4]. Hence, if we assume by contradiction $(\rho_1, \sigma_1) = \mathrm{Pred}_{\widetilde{\varphi}(P_1)}(1,-2) > (1,-3)$, from [12, Corollary 7.4] we obtain a (ρ_1, σ_1)-homogeneous element $R_2 \in K[x, x^{-1}, y]$ such that $\ell_{\rho_1,\sigma_1}(\widetilde{\varphi}(P_1)) = R_2^{4m}$, which is impossible.

It follows that

$$\mathrm{Pred}_{\widetilde{\varphi}(P_1)}(1,-2) = (1,-3) = \mathrm{Pred}_{\widetilde{\varphi}(Q_1)}(1,-2)$$

holds, as we also know that $(0,0)$ belongs to $\mathrm{Supp}(\widetilde{\varphi}(P_1)) \cap \mathrm{Supp}(\widetilde{\varphi}(Q_1))$. But we have $\ell_{1,-3}(\widetilde{\varphi}(P_1)) = \ell_{1,-3}(\widetilde{\varphi}(\varphi_1(P)))$ and so, from Proposition 3.3 we know that $\tau = -x + \lambda_1 x^{-2} + \pi x^{-3}$ is a π-root of P, since $\varphi_\tau = \widetilde{\varphi} \circ \varphi_1$. Moreover we get $\lambda_\tau = 0$; and $f_{\pi,P}(z) = \ell_{1,-3}(\varphi_\tau(P))$ has no multiple roots (eventually replace P by $P + \xi$ for an adequate constant ξ). Consequently τ is a final minor root of P and $|D_\tau^P| = v_{0,1}(\mathrm{en}_{1,-3}(\varphi_\tau(P))) = 4m$, where $4m$ is the number of remaining minor roots of P. So we conclude that τ is the only minor root of P and we also obtain $\delta_\tau = \dfrac{\sigma}{\rho} = \dfrac{-3}{1} = -3$, as claimed.

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Resumen

En [14] Yansong Xu calcula el número de intersección de un par jacobiano usandos dos igualdades diferentes. Probamos la primera de estas desigualdades usando el lenguaje de [12], pero en lugar de la segunda solamente obtenemos una desigualdad.

Palabras clave: Conjetura del jacobiano, teorema de Newton-Puisseux

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A simplified proof of the Granja-Merle factorization theorem

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Abstract

In [G] Granja proved a factorization theorem for power series in two variables which generalized the Merle theorem [M] on polar curves. Our aim is to reprove his result without resorting to Hamburger-Noether expansions and Apéry sequences. We base our proof on the method developed by us in [GB-Pł].

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Keywords: Plane algebroid curves, semigroup associated with a branch, key polynomials, contact coefficients.

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1 Introduction

Let $\mathbf{K}$ be an algebraically closed field of characteristic $p \geq 0$. Let $f \in \mathbf{K}[[x,y]]$ be a non-zero power series with no constant term. An **algebroid curve** $\{f=0\}$ is the ideal generated by $f \in \mathbf{K}[[x,y]]$. We say that $\{f=0\}$ is **irreducible** (respectively reduced) if f is irreducible (respectively f has no multiple factors). The irreducible curves are also called **branches**. The **order** ord f of the power series f is the **multiplicity** of the curve $\{f=0\}$.

For any power series $f, g \in \mathbf{K}[[x,y]]$ we define the **intersection multiplicity** (also called **intersection number**) by

$$i_0(f,g) = \dim_{\mathbf{K}} \mathbf{K}[[x,y]]/(f,g),$$

where (f,g) is the ideal of $\mathbf{K}[[x,y]]$ generated by f and g. If f, g are non-zero power series with no constant term we have $i_0(f,g) < +\infty$ if and only if $\{f=0\}$ and $\{g=0\}$ have no common branch.

For any irreducible power series $f \in \mathbf{K}[[x,y]]$ we set

$$\Gamma(f) = \{i_0(f,g) : g \text{ runs over all power series such that } g \not\equiv 0 \pmod{f}\}.$$

A subset of $\mathbb{N}$ is a **semigroup** if it is closed under addition and it contains 0. Since we have $i_0(f,gh) = i_0(f,g) + i_0(f,h)$, then $\Gamma(f)$ should be a semigroup. We call $\Gamma(f)$ the **semigroup associated with the branch** $\{f=0\}$.

Suppose that the branch $\{f=0\}$ is different from $\{x=0\}$. Write $n = i_0(f,x)$. Let $(\overline{b_0}, \ldots, \overline{b_h})$, where $\overline{b_0} = n$, be a n-minimal system of generators of $\Gamma(f)$ defined by the conditions

- $\overline{b_0} = n$,
- $\overline{b_k} = \min\left(\Gamma(f) - (\mathbb{N}\overline{b_0} + \cdots + \mathbb{N}\overline{b_{k-1}})\right)$ for $1 \leq k \leq h$ and,
- $\Gamma(f) = \mathbb{N}\overline{b_0} + \cdots + \mathbb{N}\overline{b_h}$

(cf. [4, Preliminaries]). Then we write $\overline{\text{char}_x} f = (\overline{b_0}, \ldots, \overline{b_h})$. Note that the semigroup $\Gamma(f)$ is a numerical semigroup, i.e., it satisfies $\gcd(\Gamma(f)) =$

1. Let $c(f)$ be the smallest element of $\Gamma(f)$ such that $c(f)+N \in \Gamma(f)$ for any integer $N \geq 0$ (compare [1, p. 136]). The number $c(f)$ is called the **conductor** of $\Gamma(f)$.

Put $e_i = \gcd(\overline{b_0}, \ldots, \overline{b_i})$ for $0 \leq i \leq h$ and $n_i = \dfrac{e_{i-1}}{e_i}$ for $1 \leq i \leq h$. We have $n_k > 1$ for $1 \leq k < h$ and $e_h = 1$. The following theorem is well-known, its proof employs Puiseux series when they are available, that is, when the characteristic of the field $\mathbf{K}$ is zero or n is not a multiple of it. We present a formulation of the theorem in use of *key polynomials* instead of Puiseux series, since they are available without any restriction.

Theorem 1.1 (Semigroup theorem)**.** *With the assumptions and notations introduced above, there exists a sequence of monic polynomials $f_0, f_1, \ldots, f_{h-1} \in \mathbf{K}[[x]][y]$ such that for $1 \leq k \leq h$ we have*

(a_k) $\deg_y(f_{k-1}) = \dfrac{n}{e_{k-1}}$,

(b_k) $i_0(f_{k-1}, f) = \overline{b_k}$,

(c_k) *if $k > 1$ then* $n_{k-1}\overline{b_{k-1}} < \overline{b_k}$.

Proof. See [4, Theorem 3.2]. □

The polynomials $f_0, f_1, \ldots, f_{h-1} \in \mathbf{K}[[x]][y]$ are called the **key polynomials** of f. They are not uniquely determined by f.

Theorem 1.2 (Granja factorization theorem)**.** *Let $\{f = 0\}$ be a branch different from $\{x = 0\}$. Let $\overline{\operatorname{char}_x} f = (\overline{b_0}, \ldots, \overline{b_h})$, where $\overline{b_0} = n = \operatorname{ord} f(0, y) > 1$. Fix k between 1 and h. Let $g = g(x, y)$ be a power series with no constant term, subject to $i_0(g, x) < \dfrac{n}{e_k}$ and $i_0(f, g) = \sum_{i=1}^{k} \lambda_i \overline{b_i}$, with $0 \leq \lambda_i < n_i$, for $1 \leq i \leq k$. Then there is a factorization $g = g_1 \cdots g_k \in \mathbf{K}[[x, y]]$ such that*

- $i_0(g_i, x) = \lambda_i \dfrac{n}{e_{i-1}}$ *for* $1 \leq i \leq k$,
- *if $\lambda_i > 0$ and $\phi \in \mathbf{K}[[x, y]]$ is an irreducible factor of g_i, then*

(a) $\dfrac{i_0(f,\phi)}{i_0(\phi,x)} = \dfrac{e_{i-1}\overline{b_i}}{n}$,

(b) $i_0(\phi,x) \equiv 0 \pmod{n/e_{i-1}}$.

We postpone the proof of Theorem 1.2 until Section 3.

A generalization of the Granja factorization theorem is due to Delgado [2, Section 2], who employs, like Granja, Hamburger-Noether expansions and Apéry sequences.

Example 1.3. Let $f_0, \dots, f_{h-1}$ be a sequence of key polynomials of f. Fix k between 1 and h. Take $g = f_0^{\lambda_1} \cdots f_{k-1}^{\lambda_k}$. Then g satisfies the assumptions of Theorem 1.2; here $g_i = f_{i-1}^{\lambda_i}$, for $1 \le i \le k$. Also, if ϕ is an irreducible factor of g_i then $\phi = f_{i-1}$.*unit*. Clearly in this case we should have $\dfrac{i_0(f,\phi)}{i_0(\phi,x)} = \dfrac{e_{i-1}\overline{b_i}}{n}$ and $i_0(\phi,x) = \dfrac{n}{e_{i-1}}$.

Example 1.4. Suppose $\mathbf{K}$ is a field of characteristic different from 2. The power series $f(x,y) = (y^2 - x^3)^2 - 4x^5y - x^7$ is irreducible and we have $\overline{\operatorname{char}_x} f = (4, 6, 13)$ (see [4, page 246]). Let $k = 2$. Obviously we have $\lambda_1 = \lambda_2 = 1$. Take $g(x,y) = y^3 - x^3y$. Then we get $i_0(g,x) = 3 < \dfrac{n}{e_2} = 4$ and $i_0(f,g) = \lambda_1\overline{b}_1 + \lambda_2\overline{b}_2$. For $g_1 = y$ and $g_2 = y^2 - x^3$ we get $g = g_1g_2$ and $i_0(g_1,x) = 1$, $i_0(g_2,x) = 2$. Now, for $\tilde{g} = y^3 - x^2y - x^5$ we get $i_0(\tilde{g},x) = 3$ and $i_0(f,\tilde{g}) = 19$. We have $\tilde{g} = \tilde{g}_1\tilde{g}_2 = (y + \cdots)(y^2 - x^3 + \cdots)$, where the dots denote higher order terms.

As a corollary of Theorem 1.2 we present the following result.

Theorem 1.5 (Merle factorization theorem)**.** *(See [7],[8],[3] for the case* $\mathbf{K} = \mathbb{C}$*) Suppose* $\overline{\operatorname{char}_x} f = (\overline{b_0}, \dots, \overline{b_h})$, $\overline{b_0} = \operatorname{ord} f(0,y) = n > 1$ *and* $n \not\equiv 0 \pmod{\operatorname{char} \mathbf{K}}$*. Then we have* $\dfrac{\partial f}{\partial y} = g_1 \cdots g_h$ *in* $\mathbf{K}[[x,y]]$*, where*

1. $i_0(g_i,x) = \dfrac{n}{e_i} - \dfrac{n}{e_{i-1}}$ *for* $1 \le i \le h$*;*

2. *if* $\phi \in \mathbf{K}[[x,y]]$ *is an irreducible factor of* g_i*, then* $\dfrac{i_0(f,\phi)}{i_0(\phi,x)} = \dfrac{e_{i-1}\overline{b_i}}{n}$ *and* $i_0(\phi,x) \equiv 0 \pmod{n/e_{i-1}}$.

Proof. Since $n \not\equiv 0 \pmod{\text{char } \mathbf{K}}$ we have $i_0\left(\dfrac{\partial f}{\partial y}, x\right) = n-1$. Let $\mathcal{O} = \mathbf{K}[[x,y]]/(f)$ be the local ring of the branch $\{f=0\}$; write $\overline{\mathcal{O}}$ for its integral closure and set $\mathcal{C} = \overline{\mathcal{O}}$ for its conductor.

It is well-known (see [5, Lemma 3.1]) that we have $i_0\left(f, \frac{\partial f}{\partial y}\right) = \dim_{\mathbf{K}} \overline{\mathcal{O}}/\mathcal{C} + n - 1$ and $\dim_{\mathbf{K}} \overline{\mathcal{O}}/\mathcal{C} = c(f)$ (see [1, pp. 136-139]).

On the other hand, as the conductor satisfies $c(f) = \sum_{k=1}^{h}(n_k - 1)\overline{b_k} - \overline{b_0} + 1$ (see [4, Corollary 3.5]), we obtain $i_0\left(f, \dfrac{\partial f}{\partial y}\right) = \displaystyle\sum_{k=1}^{h}(n_k - 1)\overline{b_k}$. We apply Theorem 1.2 to the series f and $g = \dfrac{\partial f}{\partial y}$ for $\lambda_k = n_k - 1$ to get the result. □

The first result on factorization of the derivative was proved by Henry J.S. Smith in [8]. Nevertheless his work fell into oblivion for a long time. Merle proved the factorization theorem in the case where the vertical axis is tranverse to the branch and Ephraim proved the theorem in any coordinates [3]. The Granja theorem is a natural generalization of the result due to Merle. That is the reason for the title of this note.

The Granja theorem was originally formulated in terms of Apéry sequences and its proof employs Hamburger-Noether expansions (see [6]). In [4] we developed a new approach to the theory of plane branches. We used the logarithmic distance on the set of branches without resorting to Hamburger-Noether expansions or a resolution process. Our proof of the Granja factorization theorem follows the spirit of [4].

2 The contact coefficient

For any pair of branches $\{f=0\}$ and $\{g=0\}$ different from $\{x=0\}$ set

$$d_x(f,g) = \frac{i_0(f,g)}{i_0(f,x)i_0(g,x)}.$$

The function d_x satisfies the **strong triangle inequality (STI)**: for any branches $\{f=0\}$, $\{g=0\}$ and $\{h=0\}$ different from $\{x=0\}$ we

get

$$d_x(f,g) \geq \inf\{d_x(f,h), d_x(g,h)\},$$

that is, at least two of the numbers $d_x(f,g), d_x(f,h), d_x(g,h)$ are equal among them and the third is not smaller than the equal two (see [4, Section 2, Theorem 2.8]).

Let $\{f=0\}$ and $\{g=0\}$ be two branches. Then we write

$$h_x(f,g) = \frac{i_0(f,g)}{i_0(g,x)}$$

and call the number $h_x(f,g)$ the **contact coefficient** of the branches $\{f=0\}$ and $\{g=0\}$. For f_{k-1} a $(k-1)$-th key polynomial of f with $\overline{\mathrm{char}_x} f = (\overline{b_0}, \ldots, \overline{b_h})$ we have

$$h_x(f, f_{k-1}) = \frac{e_{k-1}\overline{b_k}}{n}.$$

Proposition 2.1. *With the assumptions and notations introduced above for the contact coefficient, we have the following possibilities.*

(a) If $h_x(f,\phi) > \dfrac{e_{k-1}\overline{b_k}}{n}$ *then* $i_0(\phi,x) \equiv 0 \pmod{n/e_k}$.

(b) If $h_x(f,\phi) < \dfrac{e_{k-1}\overline{b_k}}{n}$ *then* $i_0(f,\phi) \in \mathbb{N}\overline{b_0} + \cdots + \mathbb{N}\overline{b_{k-1}}$.

(c) If $h_x(f,\phi) = \dfrac{e_{k-1}\overline{b_k}}{n}$ *then* $i_0(\phi,x) \equiv 0 \pmod{n/e_{k-1}}$ *and* $i_0(f,\phi) \equiv 0 \pmod{\overline{b_k}}$.

Proof. For a) see [4, Lemma 5.6]. By assumption we have $h_x(f,\phi) < h_x(f,f_{k-1})$, and this lead us to $d_x(f,\phi) < d_x(f,f_{k-1})$. Using the STI with the power series ϕ, f_{k-1} and f we get $d_x(f,\phi) = d_x(f_{k-1},\phi)$, which implies $i_0(f,\phi) = \dfrac{i_0(f,x)}{i_0(f_{k-1},x)} i_0(f_{k-1},\phi) = e_{k-1} i_0(f_{k-1},\phi) \in \mathbb{N}\overline{b_0} + \cdots + \mathbb{N}\overline{b_{k-1}}$ since $\overline{\mathrm{char}_x} f_{k-1} = \left(\dfrac{\overline{b_0}}{e_{k-1}}, \ldots, \dfrac{\overline{b_{k-1}}}{e_{k-1}}\right)$ (see [4, Proposition 4.2]). This proves b). For c), first we check $i_0(\phi,x) \equiv 0 \pmod{n/e_{k-1}}$. If

$k = 1$ then it is obvious, so assume $k > 1$. We have $\dfrac{i_0(f,\phi)}{i_0(\phi,x)} = \dfrac{e_{k-1}\overline{b_k}}{n} > \dfrac{e_{k-2}\overline{b_{k-1}}}{n}$, and hence $i_0(\phi,x) \equiv 0 \pmod{n/e_{k-1}}$ by the first statement of this same proposition. All these facts yield $i_0(f,\phi) = \dfrac{e_{k-1}\overline{b_k}}{n} i_0(\phi,x) \equiv 0 \pmod{\overline{b_k}}$. □

3 Proof of Theorem 1.2

Fix k and choose $g \in \mathbf{K}[[x,y]]$ so that the assumptions of Theorem 1.2 hold. Let $g = \phi_1 \cdots \phi_s$ with irreducible $\phi_j \in \mathbf{K}[[x,y]]$. First we check

$$\frac{i_0(f,\phi)}{i_0(\phi,x)} \leq \frac{e_{k-1}\overline{b_k}}{n} \tag{3.1}$$

if ϕ is an irreducible factor of g. Indeed, if not, we have an irreducible factor ϕ of g such that $\dfrac{i_0(f,\phi)}{i_0(\phi,x)} > \dfrac{e_{k-1}\overline{b_k}}{n}$. Proposition 2.1 lead us then to $i_0(\phi,x) \equiv 0 \pmod{\dfrac{n}{e_k}}$, a contradiction since $i_0(\phi,x) \leq i_0(g,x) < \dfrac{n}{e_k}$. Now if $\lambda_k \neq 0$, then there exists at least one irreducible factor ϕ of g such that $\dfrac{i_0(f,\phi)}{i_0(\phi,x)} = \dfrac{e_{k-1}\overline{b_k}}{n}$. If we have $\dfrac{i_0(f,\phi)}{i_0(\phi,x)} \neq \dfrac{e_{k-1}\overline{b_k}}{n}$ for all irreducible factors of g, Equation 3.1 take us to $\dfrac{i_0(f,\phi_j)}{i_0(\phi_j,x)} < \dfrac{e_{k-1}\overline{b_k}}{n}$ for $1 \leq j \leq s$. In use of Proposition 2.1 we have $i_0(f,\phi_j) \in \mathbb{N}\overline{b_0} + \cdots + \mathbb{N}\overline{b_{k-1}}$ and consequently $i_0(f,g) = \sum_{j=1}^{s} i_0(f,\phi_j) \in \mathbb{N}\overline{b_0} + \cdots + \mathbb{N}\overline{b_{k-1}}$. This is impossible because of $i_0(f,g) = \sum_{i=1}^{k} \lambda_i \overline{b_i} \not\equiv 0 \pmod{e_{k-1}}$ (if we have $\sum_{i=1}^{k} \lambda_i \overline{b_i} \equiv 0 \pmod{e_{k-1}}$, then we would get $\lambda_k \overline{b_k} \equiv 0 \pmod{e_{k-1}}$ and $\lambda_k \dfrac{\overline{b_k}}{e_k} \equiv 0 \pmod{n_{k-1}}$, which is impossible since $0 < \lambda_k < n_k$).

Now, we prove Theorem 1.2 by induction on $k > 0$. If $\lambda_k = 0$, then the theorem reduces to the case $k-1$. Therefore we assume $\lambda_k \neq 0$. Let g_k be the product of all factors ϕ_j of g satisfying $\dfrac{i_0(f,\phi_j)}{i_0(\phi_j,x)} = \dfrac{e_{k-1}\overline{b_k}}{n}$.

Therefore we have $\dfrac{i_0(f,g_k)}{i_0(g_k,x)} = \dfrac{e_{k-1}\overline{b_k}}{n}$ and $g = \tilde{g}g_k$ in $\mathbf{K}[[x,y]]$. By Proposition 2.1 (b) we obtain $i_0(f,\tilde{g}) \in \mathbb{N}\overline{b_0} + \cdots + \mathbb{N}\overline{b_{k-1}}$. Similarly, by Proposition 2.1 (c), we get $i_0(f,g_k) \equiv 0 \pmod{\overline{b_k}}$ since we have $i_0(f,\phi) \equiv 0 \pmod{\overline{b_k}}$ for any irreducible factor ϕ of g_k. From $\dfrac{i_0(f,g_k)}{i_0(g_k,x)} = \dfrac{e_{k-1}\overline{b_k}}{n}$ we get

$$i_0(f,g_k) = \frac{e_{k-1}\overline{b_k}}{n} i_0(g_k,x) \leq \frac{e_{k-1}\overline{b_k}}{n} i_0(g,x) < \frac{e_{k-1}\overline{b_k}}{n}\frac{n}{e_k} = n_k\overline{b_k}.$$

With $i_0(f,g_k) = a_k\overline{b_k}$, we have $a_k < n_k$. Any element of the semigroup $\mathbb{N}\overline{b_0}+\cdots+\mathbb{N}\overline{b_{k-1}}$ has a representation $a_0\overline{b_0}+\cdots+a_{k-1}\overline{b_{k-1}}$, where $a_0 \geq 0$ and $0 \leq a_i < n_i$. Therefore we have $i_0(f,g) = i_0(f,\tilde{g}) + i_0(f,g_k) = a_0\overline{b_0} + a_1\overline{b_1} + \cdots + a_{k-1}\overline{b_{k-1}} + a_k\overline{b_k}$, where $0 \leq a_i < n_i$ for $1 \leq i \leq k$ and $a_0 \geq 0$. On the other hand, by assumption we have $i_0(f,g) = \lambda_1\overline{b_1} + \cdots + \lambda_k\overline{b_k}$. The unicity of the representation of $i_0(f,g)$ implies $a_0 = 0$ and $a_i = \lambda_i$. In particular $a_k = \lambda_k$ and $i_0(f,g_k) = \lambda_k\overline{b_k}$ hold. Since $\dfrac{i_0(f,g_k)}{i_0(g_k,x)} = \dfrac{e_{k-1}\overline{b_k}}{n}$ we have $i_0(g_k,x) = \dfrac{n}{e_k}\lambda_k$. If $k = 1$ we are done (in that case we conclude $i_0(f,\tilde{g}) = 0$, so $\tilde{g}$ is a unit and we work with $\tilde{g}g_1$ instead of g_1). If $k > 1$ then $\tilde{g} = \dfrac{g}{g_k}$ satisfies the assumptions of Theorem 1.2 with $k-1$. By induction we are done. □

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Resumen

En [6] Granja provó una generalización del teorema de Merle [7] para curvas polares de ramas planas. El trabajo presenta una prueba de este resultado sin recurrir a la expansión de Hamburger-Noether o secuencias de Apéry (presentes en la prueba original), sino basándonos en el método desarrollado en [4].

Keywords: Curvas algebroides planas, semigrupo asociado a una rama, polinomios llave, coeficientes de contacto.

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On arcs and plane curves

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Abstract

We investigate complete plane arcs which arise from the set of rational points of certain Frobenius non-classical plane curves over finite fields. We also point out direct consequences of the Griesmer bound for some linear codes.

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Keywords: Finite fields, plane arcs, Frobenius non-classical curves.

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1 Introduction

In all that follows $\mathbb{F} = \mathbb{F}_q$ will denote the finite field of order q. Let $n \geq 2$, $r \geq 2$ be integers. In this paper we are interested in certain subsets $\mathcal{A}$ of the projective plane $PG(\mathbb{F}) = PG(q)$; we follow closely Giulietti *et al.* [5]. In what follows # will denote cardinality of a set.

Let us consider the following conditions:

(A_0) the number of points in $\mathcal{A}$ is n;

(A_1) there is no line ℓ in $PG(q)$ such that $\#\mathcal{A} \cap \ell > r$;

(A_2) for any point $P_0 \in PG(q) \setminus \mathcal{A}$, there exists a line ℓ_0 in $PG(q)$ subject to $P_0 \in \ell_0$ and $\#\mathcal{A} \cap \ell_0 = r$.

If (A_0) and (A_1) hold, $\mathcal{A}$ is called an (n, r)**-arc**. If in addition (A_2) is true, $\mathcal{A}$ is said to be a **complete** (n, r)**-arc**.

These objects are mainly studied in the context of Finite Geometry, where many results can be reformulated in terms of Curve Theory over Finite Fields, Coding Theory or Cryptography; see e.g. Hirschfeld's work [8], [9], [1]. The general problem we are dealing with is the following.

Problem 1.1. Giving q and r as above, for which n there exist a complete (n, r)-arc in $PG(q)$?

Remark 1.2. Let $\mathcal{A}$ be an (n, r)-arc in $PG(q)$ and $P \in \mathcal{A}$. Then each line ℓ in $PG(q)$ that satisfies $P \in \ell$ contains at most $(r-1)$ points of $\mathcal{A}$. Thus (see [8, Corollary 2.15]) we have

$$n \leq (r-1)(q+1) + 1 = (r-1)q + r\,.$$

In particular, we have $n \leq q^2 + q + 1$ as $r \leq \#\ell_0 = q + 1$. We observe then that trivially the plane $PG(q)$ is a (complete) $(q^2+q+1, q+1)$-arc. Further upper bounds can be found in [9, Table 5.2] or [1].

Therefore it makes sense to explore arcs related to plane curves over finite fields; our main reference on curves is the book by Hirschfeld *et al.* [7]. In this sense, we present examples and results related to this problem, see for instance Proposition 3.3 and Example 3.6 in Section 3. But, first, we present some preliminary examples.

Example 1.3. Let $F(X,Y,Z) \in \mathbb{F}[X,Y,Z]$ be an absolutely irreducible, homogeneus polynomial of degree $r \geq 2$. We consider the projective plane curve $\mathcal{C} : F = 0$ over the algebraic closure $\bar{\mathbb{F}}$ of $\mathbb{F}$, where in addition (by simplicity) $\mathcal{C}$ will be assumed to be non-singular. Let $(X : Y : Z)$ be homogeneus coordinates on the projective plane $PG(\bar{\mathbb{F}})$. Then we have $\phi_q(\mathcal{C}) \subseteq \mathcal{C}$, where $\phi_q : (a : b : c) \mapsto (a^q : b^q : c^q)$ is the $\mathbb{F}$-Frobenius map on $PG(\bar{\mathbb{F}})$.

Then we are led to consider $\mathcal{C}(\mathbb{F})$ as being the set of **$\mathbb{F}$-rational points** of $\mathcal{C}$ (namely, the points $P \in \mathcal{C}$ such that $\phi_q(P) = P$). We assume $n = \#\mathcal{C}(\mathbb{F}) \geq 2$.

For a line ℓ in $PG(q)$ set $r_\ell = \#\mathcal{C}(\mathbb{F}) \cap \ell$. We have $r_\ell \leq r$ by Bezout's theorem (see e.g. [7]) and hence $\mathcal{C}(\mathbb{F})$ is in fact an (n,r)-arc.

Now concerning $n = \#\mathcal{C}(\mathbb{F})$ we have the following key obstruction (Hasse-Weil bound):

$$n \leq q + 1 + 2g\sqrt{q}\,,$$

where $g = (r-1)(r-2)/2$ is the genus of $\mathcal{C}$ (see e.g. [7, Theorem 9.18]).

Example 1.4. Let $\mathcal{A} \subseteq PG(q)$ be a $(n,2)$-arc. We get $n \leq q+2$ by Remark 1.2. Notice that if $\mathcal{A} = \mathcal{C}(\mathbb{F})$ were defined as in Example 1.3 (i.e. from a non-singular plane curve of degree $n = 2$), then $n \leq q+1$.

As a matter of fact, we work on plane curves with a special geometrical property from which an "easy" solution to Problem 1.1 is expected (cf. Question 2.5 below).

2 Number of $\mathbb{F}$-rational ponts

Here we use the notation and assumptions of Example 1.3. Let $\mathcal{C}$ be a projective, non-singular plane curve defined by $F = F(X,Y,Z) = 0$. For $P \in \mathcal{C}$ the relation

$$F_X(P)X + F_Y(P)Y + F_Z(P)Z = 0$$

defines the tangent line $T_P\mathcal{C}$ of $\mathcal{C}$ at P, where F_X, F_Y, F_Z are, respectively, the partial derivative of F with respect to the indeterminates X, Y, Z.

It is a fundamental observation that a "large number" of $\mathbb{F}$-rational points of $\mathcal{C}$ are related to the property "$\phi_q(P) \in T_P\mathcal{C}$ for almost all $P \in \mathcal{C}$" (cf. Remark 3.5 below). In this case $\mathcal{C}$ is called **Frobenius non-classical**, otherwise it is called **Frobenius classical**; cf. [12], [6] (compare Propositions 2.1, 2.4 below).

Proposition 2.1. *For $\mathcal{C}$ a projective, non-singular, Frobenius classical plane curve over $\mathbb{F}$ of degree r, we have*

$$\#\mathcal{C}(\mathbb{F}) \leq r(r+q-1)/2\,.$$

Proof. Let $\mathcal{C}$ be defined by $F = F(X,Y,Z) = 0$. For $P = (a:b:c) \in \mathcal{C}$ we notice that $\phi_q(P) \in T_P\mathcal{C}$ holds if and only if we have

$$a^q F_X(P) + b^q F_Y(P) + c^q F_Z(P) = 0\,.$$

This led us to consider the (possible singular) curve $\mathcal{C}_1$ defined for

$$G(X,Y,Z) = X^q F_X(X,Y,Z) + Y^q F_Y(X,Y,Z) + Z^q F_Z(X,Y,Z)\,.$$

Now, by Bezout's theorem, we obtain

$$\sum_{P \in \mathcal{C}(\mathbb{F})} I(P; \mathcal{C} \cap \mathcal{C}_1) \leq r(q+r-1)\,,$$

since $\mathcal{C} \not\subseteq \mathcal{C}_1$ as $\mathcal{C}$ is Frobenius classical; here $I(P; \mathcal{C} \cap \mathcal{C}_1)$ is the intersection multiplicity of $\mathcal{C}$ and $\mathcal{C}_1$ at P. We have then

$$G(X,Y,Z) = F_X(X^q - X) + F_Y(Y^q - Y) + F_Z(Z^q - Z) + rF(X,Y,Z)\,,$$

and hence $P \in \mathcal{C}_1$ for $P \in \mathcal{C}(\mathbb{F})$. This way we get $I(P; \mathcal{C} \cap \mathcal{C}_1) \geq 2$ for $P \in \mathcal{C}(\mathbb{F})$, and the result follows. □

Remark 2.2. ([2, Theorem 2]) Let $q = p$ be a prime with $p \equiv 1 \pmod 4$, and c a non-square in $\mathbb{F}$. The following plane curve $\mathcal{C}$ over $\mathbb{F}$ defined by

$$(Y + cZ)^{(p-1)/2} + Y^{(p-1)/2} - Z^{(p-1)/2} - X^{(p-1)/2} = 0$$

is non-singular, Frobenius classical and satisfies the upper bound in Proposition 2.1, that is $\#\mathcal{C}(\mathbb{F}) = 3(p-1)^2/8$.

Question 2.3. Is the set of $\mathbb{F}$-rational points of the curve $\mathcal{C}$ in Remark 2.2 above a complete $(n,(p-1)/2)$-arc with $n=3(p-1)^2/8$, that is, does Condition (A_2) hold?

We have the following complementary result.

Proposition 2.4. ([6, Theorem 1]) *Let $\mathcal{C}$ a projective, non-singular, Frobenius non-classical plane curve over $\mathbb{F}$ of degree r. Then we have*

$$\#\mathcal{C}(\mathbb{F}) = r(q-r+2)\,.$$

□

Question 2.5. Is the set of $\mathbb{F}$-rational points of the curve $\mathcal{C}$ in Proposition 2.4 a complete (n,r)-arc with $n=r(q-r+2)$?

Remark 2.6. According to [1, Section 5], an (n,r)-arc in $PG(q)$ is said to be **large** whenever $n/q > r-2$. Thus, arcs related to Question 2.5 would be large if and only if $q > r(r-2)/2$. On the other hand, those related to Question 2.3 would not be so.

3 Complete arcs: Property A_2

Throughout this section $\mathcal{C}: F(X,Y,Z)=0$ will be a projective, non-singular, Frobenius non-classical plane curve of degree $r\geq 2$ defined over $\mathbb{F}$.

If $P\in\mathcal{C}$ and ℓ is a line in $PG(\bar{\mathbb{F}})$, then there are three possibilities: $P\notin\ell$, ℓ is transversal to $\mathcal{C}$ at P or ℓ is the tangent line $T_P\mathcal{C}$ of $\mathcal{C}$ at P. In each of these cases, we will write the intersection multiplicity $I(P;\mathcal{C}\cap\ell)$ at $P\in\mathcal{C}$ as $j_0(P)$, $j_1(P)$ and $j_2(P)$ to stress the fact that they are, respectivelly, 0, 1 or greater than 1; see e.g. [12]. Moreover, $j_2(P)$ is the same for almost all P; this common value will be denoted by $\epsilon=\epsilon(\mathcal{C})$. The finitely many points P where $j_2(P)\neq\epsilon$ are the so-called **inflexion points of** $\mathcal{C}$ (or the **Weierstrass points** of $\mathcal{C}$ with respect to the embedding $\mathcal{C}\subseteq PG(\bar{\mathbb{F}})$). These points include the $\mathbb{F}$-rational points since for such points P we have $j_2(P)\geq\epsilon+1$ (see [12]).

Observe that for $P\notin\mathcal{C}(\mathbb{F})$, $T_P\mathcal{C}$ is determinated by P and $\phi_q(P)$.

Lemma 3.1. ([6, Section 3]) *Suppose $\epsilon>2$. Then*

(1) ϵ *is a power of the characteristic of* $\mathbb{F}$ *and satisfies* $\epsilon \leq \sqrt{q}$;

(2) $r \equiv 1 \pmod{\epsilon}$;

(3) $\sqrt{q}+1 \leq r \leq (q-1)/(\epsilon-1)$. □

Now we study property (A_2) for $\mathcal{C}(\mathbb{F})$.

Lemma 3.2. *Let* $\mathcal{C}$ *be a projective, non-singular, Frobenius non-classical plane curve. Let* ℓ_0 *be a line in* $PG(q)$ *such that* $\ell_0 \neq T_P$ *for any* $P \in \mathcal{C}$. *Then we have* $\ell_0 \cap \mathcal{C} \subseteq \mathcal{C}(\mathbb{F})$.

Proof. Suppose there exists $P \in \ell_0 \cap \mathcal{C}$ with $\phi_q(P) \neq P$. Since $\phi_q(\ell_0) = \ell_0$, then $\phi_q(P) \in \ell_0$. Thus we get $\ell_0 = T_P\mathcal{C}$ as $T_P\mathcal{C}$ is determined by P and $\phi_q(P)$. □

Proposition 3.3. *Let* $\mathcal{C}$ *be a projective, non-singular, Frobenius non-classical plane curve of degree* r. *Suppose that for any* $P_0 \in PG(q) \setminus \mathcal{C}(\mathbb{F})$ *there is a line* ℓ_0 *in* $PG(q)$ *such that* $P_0 \in \ell_0$ *and* $\ell_0 \neq T_P$ *for any* $P \in \mathcal{C}$. *Then* $\mathcal{C}(\mathbb{F})$ *is a complete* $(r(q-r+2), r)$*-arc.*

Proof. We have $\#\mathcal{C}(\mathbb{F}) = r(q-r+2)$ by Proposition 2.4 above. Let $P_0 \in PG(q) \setminus \mathcal{C}(\mathbb{F})$ and ℓ_0 be as in the hypothesis. By Lemma 3.1 we have $\ell_0 \cap \mathcal{C}(\mathbb{F}) \subseteq \mathcal{C}(\mathbb{F})$. If $\#\ell_0 \cap \mathcal{C} < n$, we obtain $I(P, \ell_0 \cap \mathcal{C}) > 1$ for some $P \in \ell_0 \cap \mathcal{C}$ and we reach $\ell_0 = T_P\mathcal{C}$, a contradiction. □

We have the following numerical sufficient condition.

Corollary 3.4. *Consider* $\mathcal{C}$ *a projective, non-singular, Frobenius non-classical plane curve of degree* r *and let* ϵ *be the generic order of contact of* $\mathcal{C}$ *with tangent lines. If* $r(r-1) < \epsilon(q+1)$, *then* $\mathcal{C}(\mathbb{F})$ *is a complete* $(r(q-r+2), r)$*-arc.*

Proof. By a result of Kaji [10], the dual curve of $\mathcal{C}$ has degree $r^* = r(r-1)/\epsilon$. Thus the hypothesis $r^* < q+1$ allows us to apply Proposition 3.3. □

Remark 3.5. For $\mathcal{C}$, r and ϵ as in Corollary 3.4, we have $\#\mathcal{C}(\mathbb{F}) = r(q-r+2)$ (Proposition 2.4). Therefore, after some elementary computations we obtain

$$r^* = \frac{r(r-1)}{\epsilon} < q+1 \text{ if and only if } \#\mathcal{C}(\mathbb{F}) > (q+1)(r-\epsilon)\,.$$

Example 3.6. Let $q = m^2$ be a perfect square. We consider the Hermitian curve $\mathcal{C} \subseteq PG(\overline{\mathbb{F}})$ over $\mathbb{F}$ defined by the equation

$$X^{m+1} + Y^{m+1} + Z^{m+1} = 0\,.$$

After some computations we see that $\mathcal{C}$ is non-singular. Next we show that $\mathcal{C}$ is Frobenius non-classical.

For $P = (a : b : c) \in \mathcal{C}$, the tangent line $T_P\mathcal{C}$ is given by

$$a^m X + b^m Y + c^m Z = 0\,. \tag{3.1}$$

Then we have $\phi_q(P) = (a^{m^2} : b^{m^2} : c^{m^2})$ and hence

$$a^m a^{m^2} + b^m b^{m^2} + c^m c^{m^2} = (a^{m+1} + b^{m+1} + c^{m+1})^m = 0\,,$$

which implies $\phi_q(P) \in T_P\mathcal{C}$.

Next let us compute the set $\mathcal{C}(\mathbb{F})$.

• If $Z = 0$ then $(1 : \alpha : 0) \in \mathcal{C}$ with

$$\alpha^{m+1} = -1\,,$$

hence $\alpha \in \mathbb{F}$. Thus there are $m + 1$ $\mathbb{F}$-rational points over $Z = 0$.
• Let $Z \neq 0$ and consider the affine equation $y^{m+1} = -x^{m+1} - 1$. There are $m + 1$ elements of $\mathcal{C}(F)$ of type $(\alpha : 0 : 1)$ with α subject to $\alpha^{m+1} = -1$.
• Let $\alpha \in \mathbb{F}$ so that $\alpha^{m+1} \neq -1$. There are $(m^2 - (m+1))(m+1)$ points of $\mathcal{C}$ of type $(\alpha : \beta : 1)$ with $\beta^{m+1} = -\alpha^{m+1} - 1$.

Summing up, we have $\#\mathcal{C}(\mathbb{F}) = (m+1)+(m+1)+(m^3+m^2-m^2-2m-1) = m^3+1$ (this result also follows from Proposition 2.4 above). We do observe that the set of $\mathbb{F}$-rational points of $\mathcal{C}$ attains the Hasse-Weil bound, namely $\#\mathcal{C}(\mathbb{F}) = m^2 + 1 + 2g_0 m$, where $g_0 = m(m-1)/2$. In this case, we say that $\mathcal{C}$ is $\mathbb{F}$**-maximal**. As a matter of fact, $\mathcal{C}$ is, up to isomorphism, the unique $\mathbb{F}$-maximal curve of genus g_0, compare [11].

Finally we show that $\mathcal{C}(\mathbb{F})$ is a complete $(m^3 + 1, m + 1)$-arc.

We shall apply Corollary 3.4. From (3.1) we can identify $T_P\mathcal{C}$ with the point $(a^m : b^m : c^m)$, where $P = (a : b : c) \in \mathcal{C}$. Thus we get

$$(a^m)^{m+1} + (b^m)^{m+1} + (c^m)^{m+1} = 0$$

and hence the degree of the dual curve of $\mathcal{C}$ is $r^* = m+1$ and so $r^* < m^2+1$; the result now follows immediately.

Remark 3.7. Let $\mathcal{C}$ be the Hermitian curve of degree $r = m+1$ in Example 3.6. Let ϵ be the generic order of contact of $\mathcal{C}$ with lines. Since the degree of the dual curve of $\mathcal{C}$ is $r^* = m+1$, by Kaji [10] we have $\epsilon = m$ as $r^* = n(n-1)/\epsilon$. Thus the intersection divisor of $\mathcal{C}$ and $T_P\mathcal{C}$ at a point P with $j_2(P) = \epsilon$ is of type

$$\mathcal{C} \cdot T_P\mathcal{C} = mP + \phi_q(P)\,. \tag{3.2}$$

We observe that in fact $j_2(P) = \epsilon$ holds for any $P \notin \mathcal{C}(\mathbb{F})$. If $P \in \mathcal{C}(\mathbb{F})$ we have $j_2(P) \geq \epsilon + 1$ and hence we get

$$\mathcal{C} \cdot T_P\mathcal{C} = (m+1)P\,. \tag{3.3}$$

Finally, the arc $\mathcal{C}(\mathbb{F})$ has the following incident property:

- For $P \in \mathcal{C}(\mathbb{F}) \cap \ell$, we get either $\#\ell \cap \mathcal{C}(\mathbb{F}) = 1$, or $\#\ell \cap \mathcal{C}(\mathbb{F}) = m+1$.

Indeed, if $\ell = T_P\mathcal{C}$, from (3.3) we have $\#\ell \cap \mathcal{C}(\mathbb{F}) = 1$. On the contrary if $\ell \neq T_P\mathcal{C}$, then $I(P; \ell \cap \mathcal{C}) = 1$. If there is $Q \in \ell \cap \mathcal{C}$, with $\phi_q(Q) \neq Q$, then we have $\phi_q(Q) \in \ell \cap \mathcal{C}$, and so we conclude $\ell = T_Q\mathcal{C}$, which is not possible by (3.2). Now for $Q \in \ell \cap \mathcal{C} \subseteq \mathcal{C}(\mathbb{F})$, we have $I(Q; \ell \cap \mathcal{C}(\mathbb{F})) = 1$ by (3.1), and the result follows from Bezout's theorem.

Remark 3.8. For the Hermitian curve $\mathcal{C}$ in Example 3.6 we just observed the equality $\epsilon = \epsilon(\mathcal{C}) = m$. As a matter of fact this curve is the unique non-singular Frobenius non-classical curve $\mathcal{C}$ of degree at most $m+1$ that verifies $\epsilon = \epsilon(\mathcal{C})m > 2$ (see [6, Proposition 6]).

Example 3.9. (Related to a Serre's question; compare [15, Proposition 1.1]) We look for a projective non-singular quartic plane curve $\mathcal{C}$ over $\mathbb{F}$ subject to $\#\mathcal{C}(\mathbb{F}) > 4(4+q-1)/2 = 2(3+q)$. Such a curve must be Frobenius non-classical by Proposition 2.1; therefore, we should get $\#\mathcal{C}(\mathbb{F}) = 4q-8$ by Proposition 2.4. Let ϵ be the generic order of contact of $\mathcal{C}$ with lines. Then $\epsilon \in \{2,3\}$ by Lemma 3.1(2).

Suppose $\epsilon = 3$. Then $q = 9$ by Lemma 3.1(3) and so $\#\mathcal{C}(\mathbb{F}) = 28 = 9+1+2g\cdot 3$, with $g = 3$. This means that $\mathcal{C}$ must be a $\mathbb{F}$-maximal curve

of genus $g_0 = 3$; i.e., $\mathcal{C}$ is the Hermitian curve $X^4 + Y^4 + Z^4 = 0$ by [11] (see also [6, Proposition 6]).

Otherwise $\epsilon = 2$. Since ϵ has to be a power of the characteristic, we have that q is a power of two, see [6, Section 3] (or [4, Corollary 3]). Moreover, there exists a $\mathbb{F}$-divisor $S = \sum_{P \in S} v_P(S)P$ on $\mathcal{C}$ (see [12, p. 9]) such that

$$\deg(S) = \epsilon \cdot 4 + (q+2) \cdot 4 \geq 2(4q-8),$$

and hence $q \in \{2, 4, 8\}$. As a matter of fact, Top [15] concluded that q must be 8 and $\mathcal{C}$ must be $\mathbb{F}$-isomorphic to the plane curve $\mathcal{C}$ defined by

$$X^4 + Y^4 + Z^4 + X^2Y^2 + Y^2Z^2 + Z^2X^2 + X^2YZ + XY^2Z + XYZ^2 = 0\,.$$

In the explicit case $\mathcal{A} = \mathcal{C}(\mathbb{F}_8)$, we have $n(n-1) = 12 < 2(8+1)$ and so, by Corollary 3.4, $\mathcal{A}$ is a complete $(24, 4)$-arc in $PG(\mathbb{F}_8)$. We do remark that the largest complete $(n, 4)$-arc in $PG(\mathbb{F}_8)$ is found for $n = 28$; see [8, Table 12.3].

Next we present examples of complete arcs obtained from non-singular, Frobenius non-classical plane curves which do not satisfy the numerical hypothesis in Corollary 3.4.

Example 3.10. ([3], [7, Theorem 8.81]) Let $p > 2$ be a prime, $\alpha \geq 2$ be an integer, $q = p^\alpha$ and set $r = (p^\alpha - 1)/(p-1)$. Let $\mathcal{C}$ be the plane curve in $PG(\overline{\mathbb{F}})$ defined by the afin equation

$$y^r = f(x) = xg^p(x) + h^p(x),$$

where

$$g(x) = x^{\sum_{i=0}^{\alpha-2} p^i} + 1 \quad \text{and} \quad h(x) = \sum_{i=0}^{\alpha-2} x^{p^i}\,.$$

Observe that for $x \in \mathbb{F}$, we have $f(x) = \mathrm{N}(x) + \mathrm{T}(x)$ being T and N respectively the trace and norm functions from $\mathbb{F}$ to $\mathbb{F}_p$.

This curve is non-singular, Frobenius non-classical with $\epsilon(\mathcal{C}) = p$ and degree r. In particular, the condition $r(r-1) < \epsilon(q+1)$ holds only for $\alpha = 2$. In this case we have $r = p+1$ and we obtain a complete $(n, p+1)$-arc with $n = p^3 + 1$.

Let $\alpha \geq 3$. We claim that $\mathcal{C}(\mathbb{F})$ is a complete (n,r)-arc with $n = r(q-r+2)$ (this follows from Proposition 2.4). We have to show property (A_2). Let $P_0 = (a:b:c) \in PG(q) \setminus \mathcal{C}(\mathbb{F})$.

If $c = 0$, the line $\ell_0 : Z = 0$ intersects $\mathcal{C}$ in r points in $PG(\mathbb{F})$, namely those $(\alpha:\beta:0)$ subject to $\alpha^r = \beta^r$ (notice that any r-th root of unity belongs to $\mathbb{F}$).

Let $P_0 = (a:b:1)$. Suppose $f(a) \neq 0$ and consider the line $\ell_0 : X = aZ$. Since $f(a) \in \mathbb{F}_p$, then the points $(a:\beta:1)$ with $\beta^n = f(a)$ belong to $PG(\mathbb{F})$ and we get $\#\mathcal{C}(\mathbb{F}) \cap \ell_0 = r$. For $f(a) = 0$, we have $a \neq 1, -1$ since $p > 2$ and $b \neq 0$ as $P_0 \notin \mathcal{A}$. Let $\ell_1 : Y = m_1(X - aZ) + bZ$ (respectively $\ell_2 : Y = m_2(X - aZ) + bZ$) be the line with $m_1 = b/(a+1)$ (respectively $m_2 = b/(a-1)$). Let

$$(m_1(X-a)+b)^n - f(X) = 0\,, \quad (m_2(X-a)+b)^n - f(X) = 0\,.$$

Whenever $m_1^n = 1$ and $m_2^n = 1$ we would have $(a+1)^n = (a-1)^n$, which together with $f(a) = 0$ forces a contradiction. Hence one of the lines ℓ_i makes (A_2) work and thus $\mathcal{C}(\mathbb{F})$ should be a (n,r)-arc.

Remark 3.11. (On the uniqueness of arcs) In $PG(p^3)$, with $p > 2$ a prime, there are at least two non-isomorphic complete (n,r)-arcs with $r = (p^3-1)/(p-1) = p^2+p+1$ and $n = r(q-r+2)$. Indeed Example 3.10 above defines one such arc, say $\mathcal{A}_1$. Consider also the curve $\mathcal{D}$ given by

$$y^{p^2+p+1} = x^{p^2+p+1} + 1\,.$$

After some computation one concludes that $\mathcal{D}$ is also non-singular and Frobenius non-classical with $\mathcal{D}(\mathbb{F})$ a complete (n,r)-arc. Suppose that there exists a projective bijective map $T : PG(p^3) \to PG(p^3)$ such that $T(\mathcal{A}_1) = \mathcal{A}_2$. By [3] we have $T(\mathcal{C}) \neq \mathcal{D}$ (recall the characterization $\mathcal{A}_1 = \mathcal{C}(\mathbb{F})$) so that by Bezout's theorem we have

$$k = (p^2+p+1)(p^3-p^2-p+1) \leq (p^2+p+1)^2,$$

which is impossible.

Example 3.12. As for a numerical example we let $p = 3$ in Remark 3.11 and so $r = 3^2+3+1 = 13$, $n = 13(27-13+2) = 208$. This yields at least two complete non-isomorphic $(208, 13)$-arcs in $PG(27)$.

Let us recall that the **deficiency** of an (n, r) arc in $PG(q)$ is $D = (r-1)q+r-n$ (cf. Remark 1.2); in our case, $D = 129$. Arcs with "large D" (say $D > n$) can be constructed in general via several combinatorial methods [8, Section 12.4], [9]. Our examples, on the other hand, depict arcs of small deficiency which can be constructed via non-Frobenius plane curves.

Finally, for m the biggest integer for which there is a complete $(m, 13)$-arc in $PG(27)$, we have $208 \leq m \leq 337$. We ask if these bounds can be improved.

We end this paper with a remark on linear codes (cf. [8, Section 2.14], [1]). First of all we notice that an (n, r)-arc in $PG(q)$ can be raised to a linear code over $\mathbb{F}$ with length n, dimension 3 and minimum distance $d = n - r$. We are concerned with the so-called Griesmer bound on n [16, Theorem 5.2.6], namely with the inequality

$$n \geq g_q(3, d) = \sum_{i=0}^{2} \lceil d/q^i \rceil \, .$$

Proposition 3.13. *For a code $[n, 3, d]$ associated to an (n, r)-arc on a projective, non-singular, Frobenius non-classical curve over $\mathbb{F}$ of degree r we have $n = g_q(3, d)$ provided that $r(q - r + 1) \leq q^2$ holds.*

Proof. Here we have $d = n - r = r(q - r + 1)$ by Proposition 2.4 and so we get

$$n \geq n - r + \lceil (n - r)/q \rceil + 1 \, .$$

The result follows from Remark 1.2. □

Example 3.14. The arcs obtained from the Hermitian curve (Example 3.6) and those from the quartics in Example 3.9 satisfy Proposition 3.13. For further considerations see Storme [14].

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Resumen

Investigamos arcos planos completos que emergen del conjunto de puntos racionales de ciertas curvas Frobenius no clásicas planas sobre cuerpos finitos. También apuntamos consecuencias directas de la cota de Griesmer para algunos códigos lineales.

Keywords: Cuerpos finitos, arcos planos, curvas Frobenius no clásicas.

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