ON D-K-MACKey Locally K-Convex Spaces

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Abstract:

D-K-Mackey locally K-convex spaces are introduced and a description of their topologies is obtained.

Introduction.

The non-Archimedean analogues of Mackey, d-barrelled and d-infrabarrelled locally convex spaces over $\mathbb{R}$ or $\mathbb{C}$ were introduced by J. Van Tiel [8] and the author ([2], [3]) respectively. In the present article we define the larger class of D-K-Mackey non-Archimedean locally convex spaces over a spherically complete field $K$, which is an extension of the classical definition of J. Rojo [6]. The main goal of this paper is to give several characterizations of such space by means of topologies. Their relation with other significant class of non-Archimedean locally convex spaces over $K$ (briefly locally K-convex) are established.

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Terminology and Notation.

We shall adopt the notation and terminology of [8], [9] and [3]. Some of the notations and terminology used in the sequel are as follows: $K$ will denote a non-trivial spherically complete non-Archimedean valued field and $(E,\tau)$ a locally $K$-convex space endowed with the locally $K$-convex topology $\tau$. As in [9], if $A$ is a subset of $E$ the pseudo-polar $A^p$ (respectively pseudo-bipolar $A^{pp}$) of $A$ is defined as $A^p = \{ g \in E'; |g(A)| < 1 \}$ (respectively $A^{pp} = \{ x \in E; |A^p(x)| < 1 \}$). We have $A = A^{pp}$ if and only $A$ is $K$-convex and Closed ([9] Proposition 2).

In this paper $E$ will always stand for a separated locally $K$-convex space over a spherically complete field $K$.

Definition 1. Let $E$ be a locally $K$-convex space and $E'$ its dual.

(i) $(E,\tau)$ is said to be $K$-Mackey if the topology $\tau$ coincides with $\tau_c(E,E')$, where $\tau_c(E,E')$ be the locally $K$-convex topology in $E$ of uniform convergence on the collection of all $K$-convex bounded and $c$-compact subset of $(E', \sigma(E',E))$ and is the strongest $(E,E')$-compatible locally $K$-convex topology on $E$.

(ii) $(E,\tau)$ is said to be $d$-$K$-Mackey, if each $\sigma(E',E)$-bounded $H$ of $E'$ which is the countably union of equicontinuous subsets of $E'$ and such that the $K$-convex hull of $H$ is relatively $c$-compact for the topology $\sigma(E',E)$, is itself equicontinuous.

(See [7], for the concept and property of an $c$-compact subset).

Lemma 1. Let $E$ be a locally $K$-convex space. Then the $K$-convex hull of a $K$-convex $c$-compact subset $A$ of $E$ is $c$-compact.

Proof. Since $K$ is spherically complete, the set $B = \{ \lambda \in K; \lambda \leq 1 \}$ is $c$-compact ([8] Theorem 2.6). Therefore it is enough to see that the $K$-convex hull of $A$ is the image of $B \times A$ under the mapping $(\lambda, x) \rightarrow \lambda x$.

Proposition 1. Let $E$ be a locally $K$-convex space and $E'$ its dual. Every $K$-convex subset of $E'$ which is bounded and relatively $c$-compact for the topology $\sigma(E',E)$ is bounded for the topology $b(E',E)$.

Proof. Let $M$ be a $k$-convex bounded and relatively $c$-compact of $(E',\sigma(E',E))$. By ([8] Theorem 2.5 and 2.7) and Lemma 1 the $k$-convex closed hull $N = \overline{C(M)}$ of the closure of $M$ is a $k$-convex bounded and $c$-compact subset of $(E',\sigma(E',E))$. It's pseudo-polar $N^p$ is a neighborhood of
zero in $E$ for the topology $\tau_\mathcal{C} (E,E')$. Let $B$ be an arbitrary bounded subset of $E$. Then $B$ is also bounded for the topology $\tau_\mathcal{C} (E,E')$ ([1] p.70) and thus there exists $\lambda \in K^*$ such that $B \subseteq \lambda N^p$. But then $M \subseteq N = N^p \subseteq \lambda B^p$. Hence by the definition of the topology $b(E',E)$, the set $M$ is $b(E',E)$-bounded.

**Proposition 2.** Let $(E,\tau)$ be a locally $k$-convex space with topology $\tau$. Then $\tau$ coincides with the topology of uniform convergence on the equicontinuous subsets of $E'$.

**Proof.** Let $\Theta$ the collection of all equicontinuous subsets of $E'$ and $\tau_\Theta$ be the locally $k$-convex topology on $E$ of uniform convergence on $\Theta$. If $U$ is a $k$-convex $\tau$-neighborhood of zero in $E$, then $U^p$ is equicontinuous in $E'$. Hence $U = U^{pp}$ is a $\tau_\Theta$-neighborhood of zero in $E$. Thus $\tau_\Theta$ is finer than the topology $\tau$. Conversely, let $H$ be a equicontinuous set in $E'$, we can find a $k$-convex $\tau$-neighborhood $U$ of zero in $E$ such that $|H(U)| < 1$. Then $H \subseteq U^p$. It follows that $H^p \supseteq U^{pp} = U$; i.e., $H^p$ is a $\tau$-neighborhood of zero in $E$. Thus $\tau$ is finer than the topology $\tau_\Theta$ and the desired equality $\tau = \tau_\Theta$ is established.

Our next goal is to prove certain characterizations of $d$-$k$-Mackey spaces. In order to do so we shall the following.

**Definition 2.** Let $E$ be a locally $k$-convex space and let $\Gamma$ be the collection of all $k$-convex bounded relatively $c$-compact subset of $(E',\sigma(E',E))$, which is the countably union of equicontinuous subset of $E'$. Then the corresponding $\Gamma$-topology on $E$ of uniform convergence on $\Gamma$ will be denoted by $\Gamma_d(E,E')$.

Clearly $\sigma(E,E') \subseteq \tau_d(E,E') \subseteq \tau_\mathcal{C} (E,E') \subseteq b(E,E')$. Therefore, the topology $\tau_d(E,E')$ is $(E,E')$-compatible.

The following proposition prove that the given topology of a $d$-$k$-infrabarrelled space $E$ ([3]), over a spherically complete field $k$ is the $\tau_d(E,E')$ locally $k$-convex topology on $E$.

**Proposition 3.** If $(E,\tau)$ is $d$-$k$-infrabarrelled, then the topology $\tau$ coincides with the topology $\tau_d(E,E')$.

**Proof.** If is enough to apply Proposition 1.

**Theorem 1.** For a locally $k$-convex space $(E,\tau)$, the following conditions are equivalent:

(i) $(E,\tau)$ is $d$-$k$-Mackey.
(ii) $\tau = \tau_d(E,E')$.  

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Proof. (i)→(ii): \( \tau = \tau_\Theta \) (Proposition 2) where \( \tau_\Theta \) be the locally \( k \)-convex topology on \( E \) defined by the family \( \Theta = \{ H CE' \; \text{;} \; k \text{-convex equicontinuous} \} \). Since, every \( k \)-convex equicontinuous subset of \( E' \) is relatively \( c \)-compact for the topology \( \sigma(E,E') \) ([8] Theorem 4.4 (b)). Then \( \Theta \subseteq \Gamma \) (\( \Gamma \) as in definition 2). Hence \( \tau_\Theta \) is weaker than \( \tau_d(E,E') \). Let now \( H \in \Gamma \). By Lemma 1 and hypothesis, \( H \) is equicontinuous. Thus \( H \in \Theta \). Hence \( \tau_d(E,E') \) is weaker than \( \tau_\Theta \) and the desired equality \( \tau = \tau_d(E,E') \) is established.

(ii)→(i): Let \( H \in \sigma(E',E) \)-bounded of \( E' \) which is the countably union of equicontinuous subsets of \( E' \) and such that the \( k \)-convex hull \( C(H) \) of \( H \) is relatively \( c \)-compact for the topology \( \sigma(E',E) \). Then \( C(H) \) is a \( k \)-convex, bounded \( e \) relatively \( c \)-compact subset of \( (E',\sigma(E',E)) \). Therefore its pseudo-polar \( (C(H))^p \subseteq (H)^p \) is a neighborhood of zero in \( E \) for the topology \( \tau_d(E,E') \); i.e., by hipothesis a \( \tau \)-neighborhood of zero. Hence \( H \) is equicontinuous. This proves (i).

As a direct consequence of Theorem 1. We have:

**Corollary 1.** A \( k \)-Mackey space is always \( d-k \)-Mackey.

**Proof.** Let \( (E,\tau) \) be a \( k \)-Mackey space. We shall show that \( \tau \) is the topology \( \tau_d(E,E') \). Indeed. By definition 1(i) and remark of definition 2 implies that \( \tau_d(E,E') \leq \tau_C(E,E') = \tau \). On the other hand \( \tau = \tau_\Theta \leq \tau_d(E,E') \). Therefore \( \tau = \tau_d(E,E') \). This prove that \( (E,\tau) \) is a \( d-k \)-Mackey space (Theorem 1).

**Remark 1.**

(i) It follows from Proposition 3 and Theorem 1 that every \( d-k \)-infrabarrelled space is a \( d-k \)-Mackey.

(ii) The following diagram helps to remember some of the relations proved in this and their relation with other classes:

\[
\begin{align*}
\text{k-barrelled} & \quad \xrightarrow{[2]} \quad \text{d-k-barrelled} \\
\text{k-infrabarrelled} & \quad \xrightarrow{[3]} \quad \text{d-k-infrabarrelled} \\
\text{k-Mackey} & \quad \xrightarrow{[\text{Corollary 1}]} \quad \text{d-k-Mackey}
\end{align*}
\]
Theorem 2. Let $E$ and $F$ be two separated locally $k$-convex spaces. Then every linear mapping $f: E \rightarrow F$ which is continuous for the topologies $\sigma(E,E')$ and $\tau_d(F,F')$, is also continuous for the topologies $\tau_d(E,E')$ and $\tau_d(F,F')$.

Proof. Let $V=H^p$ be a neighborhood of zero in $F$ for the topology $\tau_d(F,F')$, where $H = \bigcup_{n \geq 1} H_n$ is a $k$-convex, bounded, relatively $c$-compact subset of $(F', \sigma(F', F))$ and $H_n$ equicontinuous ($n \geq 1$). Since $f: F' \rightarrow E$ ($f$ transpose of $f$) is continuous for the topology $\sigma(F', F)$ and $\sigma(E', E)$ ([1] p.101), the set $X=\{f(H)\}$ is a $k$-convex bounded relatively $c$-compact of $(E', \sigma(E', E))$ which is the countably union of $f(H_n)$ equicontinuous subsets and thus $U=X^p$ is a neighborhood of zero in $E$ for the topology $\tau_d(E,E')$. Since $X=f(H)$, we have $f(U) \subseteq V$, that is $f(U) \subseteq V$, which proves that $f$ is continuous for the topologies $\tau_d(E,E')$ and $\tau_d(F,F')$.

Corollary 2. Let $(E, \tau_E)$ and $(F, \tau_F)$ be locally $k$-convex spaces, $(E, \tau_E)$ $d$-$k$-

Mackey. Then every linear mapping $f: E \rightarrow F$ which is continuous for the topologies $\sigma(E,E')$ and $\sigma(F,F')$ is also continuous for the topologies $\tau_E$ and $\tau_F$.

Proof. By the assumption and by Theorem 2 the mapping $f$ is continuous for the topologies $\tau_E = \tau_d(E,E')$ and $\tau_d(F,F')$. But $\tau_d(F,F')$ is finer than $\tau_F$. Then $f$ is continuous for the topologies $\tau_E$ and $\tau_F$.

Let us recall that the hypothesis of this corollary is satisfied if $E$ is an $d$-$k$-infrabarrelled space (Proposition 3).

Theorem 3. For a locally $k$-convex space $(E, \tau_E)$ the following conditions are equivalent:

(i) $(E, \tau_E)$ is $d$-$k$-Mackey.

(ii) For every locally $k$-convex space $(F, \tau_F)$, each linear mapping $f: E \rightarrow F$ which is continuous for the topologies $\tau_E$ and $\sigma(F,F')$ is also continuous for the topologies $\tau_E$ and $\tau_d(F,F')$.

Proof.:

(i) $\rightarrow$ (ii): By ([1] p.103) the mapping $f$ is continuous for the topology $\sigma(E,E')$ and $\sigma(F,F')$. By the Theorem 2 it is also continuous for $\tau_d(E,E')$ and $\tau_d(F,F')$. Finally since $\tau_E = \tau_d(E,E')$ (Theorem 1), $f$ is continuous for $\tau_E$ and $\tau_d(F,F')$ (also, since $\tau_F \leq \tau_d(F,F')$ the mapping $f$ is continuous for $\tau_E$ and $\tau_F$).
(ii)→(i): Since $\sigma(E,E') \leq \tau_E$, the mapping canonical imbedding $j:E \to E$ is continuous for the topologies $\tau_E$ and $\sigma(E,E')$. By the assumption (ii) it is also continuous for the topologies $\tau_E$ and $\tau_d(E,E')$. Hence $\tau_d(E,E') \leq \tau_E$. Therefore $\tau_d(E,E') = \tau_E$.

d-k-Mackey spaces have remarkable stability properties which we list in the following Proposition and that reasoning as in [3] can be proved.

**Proposition 4.** Let $(E,\tau_E)$ and $(F,\tau_F)$ be two locally $k$-convex spaces.

(i) Let $D$ a dense $k$-subspace of $E$. Then $(E,\tau_E)$ is d-k-Mackey if $(D,\tau_D)$ is d-k-Mackey.

(ii) Let $f$ be a linear continuous, almost open (a fortiori, open) and surjective mapping from $E$ into $F$. Then $(F,\tau_F)$ is d-k-Mackey if $(E,\tau_E)$ is d-k-Mackey.

(iii) If $(E,\tau_E)$ is a d-k-Mackey space and $M$ a closed $k$-subspace of $E$. Then the quotient space $E/M$ is a d-k-Mackey space.

(iv) Let $\mathcal{E}$ be the family of all d-k-Mackey. Then $\mathcal{E}$ is stable under the formation of arbitrary direct sums, inductive limits, and arbitrary products.

Finally we apply these notion of d-k-Mackey to the space of the continuous mappings.

We suppose that $X$ is an ultraregular space, that is a separated topological space where every point has a filterbase of clopen neighborhoods. $C(X,E)$ the space of all continuous $E$-valued mappings on $X$, endowed with the compact-open topology. We call a topological space $w$-compact if every countable union of compact set is relatively compact.

**Theorem 4.** If $C(X,E)$ is a d-k-Mackey space, then $C(X,K)$ and $E$ are d-k-Mackey spaces.

**Proof.** In ([4], Proposition 2.1 and 2.2) it has been show that $C(X,K)$ and $E$ are closed complemented $k$-subspaces of $C(X,E)$. Therefore, there exist two separated quotients spaces of $C(X,E)$ which are isomorphous to $C(X,K)$ and $E$, respectively. Since by Proposition 4(iii) the property of being d-k-Mackey is invariable under separated quotient formation, $C(X,K)$ and $E$ are d-k-Mackey.
Theorem 5. Let $X$ be an ultraregular $w$-compact space and $(E_n, \tau_n)$ be a crescent sequence of locally $k$-convex spaces. If $(E, \tau) = \lim (E_n, \tau_n)$ then the inductive limit $\lim C(X, E_n)$ is a dense topological $k$-subspace of $C(X, E)$.

Proof. See ([4] Theorem 2.5).

Corollary 3. Let $X$ be an ultraregular $w$-compact space and $E$ be the inductive limit of $E_n$ where $(E_n)_{n \in \mathbb{N}}$ is a crescent sequence of non-Archimedean normed spaces. Then $C(X, E)$ is an $d$-$k$-Mackey space.

Proof. By Theorem 5, the inductive limit of spaces $C(X, E_n)$ is a dense topological $k$-subspace in $C(X, E)$. Since $E_n$ is non-Archimedean normed and by ([4] Theorem 4.8) can be proved that the space $C(X, E_n)$ is $d$-$k$-infrabarrelled. Hence and Remark 1(i) $C(X, E_n)$ is $d$-$k$-Mackey. By Proposition 4(iv), the inductive limit of spaces $C(X, E_n)$ is $d$-$k$-Mackey and by the same Proposition 4(i) it results that $C(X, E)$ is $d$-$k$-Mackey.

References


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