GENERALIZED CLOSED SETS VIA IDEALS AND OPERATORS

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Abstract

Given a topological space $(X, \tau)$, three operators $\alpha, \beta, \gamma$ associated to a topology $\tau$ and $I$ an ideal on $X$. The concepts of: $\alpha$-closed set, $\alpha$-semi closed set, $(\alpha, \beta)$-semi closed set and $(I, \gamma)$ g-closed set are generalized. Also new separation axioms are introduced and characterized and new spaces are obtained in such way that the spaces $\alpha - T_{\frac{1}{2}}$, $\alpha$-semi-$T_{\frac{1}{2}}$, $(\alpha, \beta)$-semi-$T_{\frac{1}{2}}$ and $\gamma - T_1$ respectively are generalized.


Keywords: $(\alpha, \beta)$-semi closed, $(I, \gamma)$ g-closed, $(\alpha, \beta)$-semi-$T_{\frac{1}{2}}$, $(\alpha, \beta, \gamma)$ – semi – $T_1$.

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1 Introduction

The study of the concepts of the notions of generalized closed sets goes back to the classic paper of N. Levine [9], where using the basic definition of closed set, he introduces the notion of generalized closed set (abbreviated g-closed) and using this concept in order to define the $T_{\frac{1}{2}}$ spaces. Later on Dunham [5] $T_{\frac{1}{2}}$ spaces are characterized proving that a topological space is $T_{\frac{1}{2}}$ if and only if the unitary sets are open or closed. Khalimsky et al. [7] shown that the digital line is typical example of $T_{\frac{1}{2}}$ space. Using the same idea, many authors have defined and studied many types of generalized closed sets in order to introduce new separations axioms and new spaces.

Recently J Donchev et al. [4], using the theory of topological ideals introduced by R. L. Newcomb [13], the local function defined by D. Jankovic et al. [6] and the operator theory introduced by Kasahara [8], provide the definition of $(I, \gamma)$ generalized closed sets and they introduced a class of spaces denominated $\gamma - T_I$ spaces, that are a generalization of the $T_{\frac{1}{2}}$ spaces given by Levine [9].

In this paper a new variant of a local function given by Jancovik et al. [6],is introduced, in order to define new concepts of g-closed sets that generalize the notions of $(I, \gamma)$ g-closed sets [4], sg-closed set [1], $g\alpha$-closed [11], $\alpha$-sg-closed set [16], $(\alpha, \beta)$-sg-closed set [17], etc. It can be used in order to introduce new spaces and new separation axioms that generalize the well known results given in [2], [4], [9], [15], [16] and [17].

2 Preliminary

Let $X$ be a nonempty set, we say that $\alpha : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is an expansive operator on a family $\Gamma$ of subsets of $X$ if $U \subset \alpha(U)$ for all $U \in \Gamma$. If $(X, \tau)$ is a topological space and $\alpha$ is expansive on the topology $\tau$, then, we say that $\alpha$ is an associated operator on the topology $\tau$ [2].
We denote by \((X, \tau, \alpha)\) the topological space \((X, \tau)\) with the operator \(\alpha\) associated to the topology, also, if \(\alpha(A) \subseteq \alpha(B)\) whenever \(A \subseteq B\), we say that the operator \(\alpha\) is monotone.

Let \((X, \tau, \alpha)\) and \(A\) be a subset of \(X\), we say that \(A\) is \(\alpha\)-open [8] if for each \(x \in A\) there exists an open set \(U\) containing \(x\) such that \(\alpha(U) \subseteq A\). The complement of an \(\alpha\)-open set is called \(\alpha\)-closed set. It is easy to prove that every \(\alpha\)-closed set is closed and the intersection of an arbitrary family of \(\alpha\)-closed sets is an \(\alpha\)-closed set, in this way, we can define the \(\alpha\)-closure of a subset \(A\) of \(X\), denoted by \(\alpha - cl(A)\), as the intersection of all \(\alpha\)-closed sets that contain \(A\). In this case, we can see that \(x \in \alpha - cl(A)\) if and only if for all \(\alpha\)-open set \(U\) containing \(x\), \(U \cap A \neq \emptyset\). We say that the set \(A\) is an \(\alpha\)-generalized closed, denoted by \(\alpha\text{-g-closed}\), if \(\alpha - cl(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(\alpha\)-open. Every \(\alpha\)-closed set is \(\alpha\text{-g-closed}\). We say that \(X\) is an \(\alpha\text{-}\text{T}^\frac{1}{2}\) space if all \(\alpha\text{-g-closed}\) set is \(\alpha\)-closed.

We say that \(A\) subset of \(X\) is an \(\alpha\)-semi-open ([2]) if there exists an open set \(U \in \tau\) such that \(U \subseteq A \subseteq \alpha(U)\). The complement of an \(\alpha\)-semi-open set is called \(\alpha\)-semi-closed set. All closed set is an \(\alpha\)-semi-closed set, in general, the intersection of an arbitrary family of \(\alpha\)-semi closed sets is not an \(\alpha\)-semi closed set; but, if we consider that \(\alpha\) is a monotone operator, then the intersection of an arbitrary family of \(\alpha\)-semi-closed sets is an \(\alpha\)-semi-closed set, in this case, we can define the \(\alpha\)-semi-closure of \(A\), denoted by \(\alpha - scl(A)\), as the intersection of all \(\alpha\)-semi-closed sets containing \(A\); it verifies that \(x \in \alpha - scl(A)\) if and only if all \(\alpha\)-semi-open set \(U\) containing \(x\), \(U \cap A \neq \emptyset\). We say that \(A\) is an \(\alpha\)-semi-generalized closed, denoted by \(\alpha\text{-sg-closed}\), if \(\alpha - scl(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is an \(\alpha\)-semi-open set. If \(\alpha\) is a monotone operator, all \(\alpha\)-semi-closed set is an \(\alpha\)-sg-closed set. We say that \(X\) is an \(\alpha - \text{semi}\text{T}^\frac{1}{2}\) space if all \(\alpha\)-sg-closed set is an \(\alpha\)-semi-closed set.

If \(\beta\) is another associated operator with \(\tau\), then we say that \(A\) subset of \(X\) is an \((\alpha, \beta)\)-semi-open set [15] if for each \(x \in A\) there exists a \(\beta\)-semi-open set \(V\) such that \(x \in V\) and \(\alpha(V) \subseteq A\). The complement of
an \((\alpha, \beta)\)-semi-open set is called \((\alpha, \beta)\)-semi-closed set. We can see the following [15]:

1. If \(A\) is an open set, then \(A\) is an \((id, \beta)\)-semi-open.

2. If \(\alpha = \beta = id\), \(A\) is an \((\alpha, \beta)\)-semi-open set if and only if \(A\) is an open set.

3. If \(\beta = id\) and \(\alpha\) is an arbitrary operator then \(A\) is an \((\alpha, \beta)\)-semi-open set if and only if \(A\) is an \(\alpha\)-open set [8].

4. If \(\alpha = id\), \(\beta\) is a monotone operator then the collection of all \((\alpha, \beta)\)-semi-open sets agree with the collection of all \(\beta\)-semi-open sets.

The intersection of an arbitrary family of \((\alpha, \beta)\)-semi-closed sets is an \((\alpha, \beta)\)-semi-closed set, and we define the \((\alpha, \beta)\)-semi-closure of \(A\), denoted by \((\alpha, \beta) - scl(A)\), as the intersection of all \((\alpha, \beta)\)-semi-closed sets containing \(A\); we can see that \(x \in (\alpha, \beta) - scl(A)\) if and only if for all \((\alpha, \beta)\)-semi-open set \(U\) containing \(x\), \(U \cap A \neq \emptyset\). We say that \(A\) is an \((\alpha, \beta)\)-semi-generalized closed set, abbreviated by \((\alpha, \beta)\)-sg-closed, if \((\alpha, \beta) - scl(A) \subset U\) whenever \(A \subset U\) and \(U\) is an \((\alpha, \beta)\)-semi-open set. All \((\alpha, \beta)\)-semi-closed set is an \((\alpha, \beta)\)-sg-closed set. We say that \(X\) is an \((\alpha, \beta) - semiT_{\frac{1}{2}}\) space if all \((\alpha, \beta)\)-sg-closed set is an \((\alpha, \beta)\)-semi-closed set.

3 Generalized Local Function

In this section, we generalize the concept of local function given in [6]. Also, we study some of its properties.

**Definition 3.1** A non-empty collection \(I\) of subsets of a set \(X\) is said to be an ideal on \(X\) if it satisfies the following two conditions.

1. If \(A_1 \in I\) and \(A_2 \in I\), then \(A_1 \cup A_2 \in I\).
2. If $A_1 \in I$ and $A_2 \subseteq A_1$, then $A_2 \in I$.

**Definition 3.2** Let $X$ be a set, $\mathcal{F}$ be a collection of subsets of $X$ and $I$ be an ideal on $X$. The generalized local function with respect to $I$, is a map that assign each subset $A$ of $X$, the set $A^*(I, \mathcal{F})$, defined as follows:

$$A^*(I, \mathcal{F}) = \{ x \in X : A \cap U_x \notin I \text{ for all } U_x \in \mathcal{F} \text{ such that } x \in U_x \}.$$  

In the above definition $A^*(I, \mathcal{F})$ can be empty and in general can not contain $A$

**Remark 3.1** Observe that

1. When $\mathcal{F} = \tau$ a topology on $X$, the concept of generalized local function agree with the concept of local function given in [6].

2. When $\mathcal{F} = \tau$ a topology on $X$ and the ideal is $\{\emptyset\}$, then $A^*(I, \mathcal{F}) = \text{cl}(A)$.

We describe some properties that satisfies $A^*(I, \mathcal{F})$.

**Theorem 3.1** Let $X$ be a set, $\mathcal{F}$ be a collection of subsets of $X$, $A$ and $B$ subsets of $X$, then

1. If $I = \mathcal{P}(X)$, then $A^*(I, \mathcal{F}) = \emptyset$.

2. If $A \subseteq B$, then $A^*(I, \mathcal{F}) \subseteq B^*(I, \mathcal{F})$.

3. $\emptyset^*(I, \mathcal{F}) = \emptyset$.

4. $A^*(I, \mathcal{F}) \cup B^*(I, \mathcal{F}) \subseteq (A \cup B)^*(I, \mathcal{F})$.

5. $(A^*(I, \mathcal{F}))^*(I, \mathcal{F}) \subseteq A^*(I, \mathcal{F})$.

6. If $J$ is an ideal on $X$ such that $I \subseteq J$, then $A^*(J, \mathcal{F}) \subseteq A^*(I, \mathcal{F})$.

**Definition 3.3** Let $X$ be a set, $\mathcal{F}$ be a family of subsets $X$, $I$ be an ideal on $X$ and $A$ a subset of $X$. We define the $\mathcal{F}$ closure of $A$, denoted by $\mathcal{F} - \text{cl}^*(A)$ as: $\mathcal{F} - \text{cl}^*(A) = A \cup A^*(I, \mathcal{F})$. 

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Remark 3.2 If $\tau$ is a topology on $X$ and $\mathfrak{F} = \tau$, then $\mathfrak{F} - \text{cl}^*$ is a Kuratowski operator and therefore it induce a topology on $X$ denoted by $\tau^*(I)$.

It is easy to prove that $\mathfrak{F} - \text{cl}^*$ satisfies the following properties:

Theorem 3.2 Let $X$ be a set, $\mathfrak{F}$ be a family of subsets of $X$, $I$ be an ideal on $X$, $A$ and $B$ subsets of $X$, then:

1. $A \subset \mathfrak{F} - \text{cl}^*(A)$.
2. If $A \subset B$, then $\mathfrak{F} - \text{cl}^*(A) \subset \mathfrak{F} - \text{cl}^*(B)$.
3. $\mathfrak{F} - \text{cl}^*(\emptyset) = \emptyset$.
4. $\mathfrak{F} - \text{cl}^*(\mathfrak{F} - \text{cl}^*(A)) \subset \mathfrak{F} - \text{cl}^*(A)$.
5. $\mathfrak{F} - \text{cl}^*(A) \cup \mathfrak{F} - \text{cl}^*(B) \subset \mathfrak{F} - \text{cl}^*(A \cup B)$.

4 \hspace{1cm} (I, \gamma)_{\alpha-g-closed \ Sets \ and \ (\alpha, \gamma) - T_I \ Spaces}

In this section, we show that using the concept of generalized local function, we can obtain immediately the notion of $\alpha$-g-closed set given in [9].

Let $(X, \tau, \alpha)$, $I$ an ideal on $X$ and $\tau_\alpha$ the collection of all $\alpha$-open sets in $X$, then the generalized local function taking $\mathfrak{F} = \tau_\alpha$ satisfies the following properties:

Theorem 4.1 Let $(X, \tau, \alpha)$, $I$ be an ideal on $X$ and $A$ be a subset of $X$, then

1. $A^*(I, \tau_\alpha) \subset \alpha - \text{cl}(A)$.
2. If $I = \{\emptyset\}$ and $\alpha$ is any operator, then $A^*(I, \tau_\alpha) = \alpha - \text{cl}(A)$. 

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3. If $\alpha = id$, $I$ any ideal, then $A^*(I, \tau_\alpha) = A^*(I, \tau)$.

4. $A^*(I, \tau_\alpha) = \alpha - cl(A^*(I, \tau_\alpha)) \subset \alpha - cl(A)$.

Proof.

1. Let $x \in A^*(I, \tau_\alpha)$, then for all $\alpha$-open set $U_x$ such that $x \in U_x$ we have $U_x \cap A \notin I$, then $U_x \cap A \neq \emptyset$ therefore $x \in \alpha - cl(A)$.

2. Let $x \in \alpha - cl(A)$, then for all $\alpha$-open set $U_x$ such that $x \in U_x$ we have $U_x \cap A \neq \emptyset$, that is, $U_x \cap A \notin \{\emptyset\} = I$.

3. If $\alpha$ is the identity operator, the collection of all $\alpha$-open sets agree with the collection of all open sets.

4. Let $x \in A^*(I, \tau_\alpha)$, then for all $\alpha$-open set $U_x$ such that $x \in U_x$ we have $U_x \cap A^*(I, \tau_\alpha) \neq \emptyset$, that is $x \in \alpha - cl(A^*(I, \tau_\alpha))$. If $x \in \alpha - cl(A^*(I, \tau_\alpha))$, then for all $\alpha$-open set $U_x$ such that $x \in U_x$, $U_x \cap A^*(I, \tau_\alpha) \neq \emptyset$, that is, $U_x \cap A \notin I$ and therefore $x \in A^*(I, \tau_\alpha)$.

Finally, the last part of 4, follows from 1.  

\begin{remark}
If in Definition 3.3, we take $\mathcal{I} = \tau_\alpha$, and denote by $\tau_\alpha - cl^*(A)$ for $\alpha - cl^*(A)$. We observe that,

1. If $\alpha = id$, then $\alpha - cl^*(A) = cl^*(A)$.

2. If $I = \{\emptyset\}$, then $\alpha - cl^*(A) = \alpha - cl(A)$.

\end{remark}

Now we introduce our generalization of the concepts of $\alpha$-g-closed sets and $(I, \gamma)$-g-closed sets given in [9] and [4] respectively.

Consider now, the triple $(X, \tau, I)$ where $I$ is an ideal defined on $X$, as the topological space.

**Definition 4.1** Let $(X, \tau, I)$, and consider two operators $\alpha, \gamma$ associated with $\tau$. A subset $A$ of $X$ is said to be $(I, \gamma)_\alpha$-generalized closed set, abbreviated $(I, \gamma)_\alpha$-g-closed, if $\alpha - cl^*(A) \subset \gamma(U)$ whenever $A \subset U$ and $U \in \tau_\alpha$.
Remark 4.2 When $\alpha$ is the identity operator, the $(I, \gamma)$-g-closed sets and the $(I, \gamma)_\alpha$-g-closed sets are the same. In the case that $I = \{0\}$, the $(I, id)$-g-closed sets are $\alpha$-g-closed sets.

We can resume the above in the following:

Theorem 4.2 Let $(X, \tau, I)$, $A \subset X$, $\alpha$ and $\gamma$ be two operators associated with $\tau$.

1. If $\alpha = id$, $A$ is an $(I, \gamma)$-g-closed set [4] if and only if $A$ is an $(I, \gamma)_\alpha$-g-closed set.
2. If $I = \{0\}$ and $A$ is an $(I, id)_\alpha$-g-closed set, then $A$ is an $\alpha$-g-closed set [16].
3. If $\alpha = \gamma = id$ and $I = \{0\}$, $A$ is a g-closed set [9] if and only if $A$ is an $(I, \gamma)_\alpha$-g-closed set.

Proof.

1. Suppose that $\alpha = id$ and $A$ is an $(I, \gamma)_\alpha$-g-closed set, let $U$ an open set such that $A \subset U$, then $cl^*(A) \subset \gamma(U)$, since $A^*(I, \tau) \subset cl^*(A)$, we conclude that $A$ is an $(I, \gamma)$-g-closed set.

Reciprocally, suppose that $A$ is an $(I, \gamma)$-g-closed set, let $U$ an $\alpha$-open set such that $A \subset U$. Since $\alpha = id$, then $U$ is open and therefore $A^*(I, \tau) \subset \gamma(U)$, using Theorem 4.1, $A^*(I, \tau) = A^*(I, \tau_\alpha)$, in consequence $\alpha - cl^*(A) \subset \gamma(U)$.

2. Suppose that $A$ is an $(I, id)_\alpha$-g-closed set of $X$ and let $U$ an $\alpha$-open set such that $A \subset U$, then $\alpha - cl^*(A) \subset id(U)$, since $I = \{0\}$; it follows that $\alpha - cl^*(A) = \alpha - cl(A)$, in consequence $A$ is an $\alpha$-g-closed set.

3. Is an immediate consequence of parts 1., 2. \hfill \square

Theorem 4.3 Let $(X, \tau, I)$, $\alpha$ and $\gamma$ two operators associated with $\tau$. Then all $\alpha$-g-closed set is an $(I, \gamma)_\alpha$-g-closed set.
The following example shows the existence of a set that is \((I, \gamma)_{\alpha}\)-g-closed but is not \(\alpha\)-g-closed.

**Example 4.1** Consider \(\mathbb{R}\), the set of real numbers with the finite complement topology \(\tau_f = \{U \subset \mathbb{R} : \mathbb{R} \setminus U \text{ is finite or } \mathbb{R}\}\), and the operator \(\alpha : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})\) associated with the topology defined as \(\alpha(U) = \text{int}(U)\). The set of the rational numbers \(\mathbb{Q}\), is not an \(\alpha\)-g-closed set, because \(\mathbb{R}\setminus\{\sqrt{2}\}\) is an \(\alpha\)-open set that contains \(\mathbb{Q}\); but \(\alpha - \text{cl}(\mathbb{Q}) = \mathbb{R} \nsubseteq \mathbb{R}\setminus\{\sqrt{2}\}\). We prove that, \(\mathbb{Q}\) is an \((I, \gamma)_{\alpha}\)-g-closed set, if we consider the ideal \(I = \mathcal{P}(\mathbb{R})\) and \(\gamma\) any operator associated with the topology.

Using the fact that there exists \((I, \gamma)_{\alpha}\)-g-closed sets that are not \(\alpha\)-g-closed. We introduce a new class of spaces.

**Definition 4.2** Let \((X, \tau, I)\), \(\alpha\) and \(\gamma\) be two operators associated with \(\tau\). We say that \(X\) is an \((\alpha, \gamma) - T_I\) space if all \((I, \gamma)_{\alpha}\)-g-closed set is an \(\alpha\)-closed set.

If we analyze the above definition, we can see that it gives us a general context in comparison with the one described in [4]. The following theorem indicates that taking adequate operators and ideals, we can obtain as particular cases the following well known results in the literature.

**Theorem 4.4** Let \((X, \tau, I)\), \(\alpha\) and \(\gamma\) be two operators associated with \(\tau\).

1. If \(\alpha = \text{id}\) and \(X\) is an \((\alpha, \gamma) - T_I\), then \(X\) is \(\gamma - T_I\) [4].

2. If \(X\) is an \((\alpha, \gamma) - T_I\) then \(X\) is \(\alpha - T_{1/2}\) [16].

3. If \(I = \{\emptyset\}\), \(\gamma = \text{id}\) and \(X\) is an \(\alpha - T_{1/2}\), then \(X\) is \((\alpha, \gamma) - T_I\).

4. If \(I = \{\emptyset\}\), \(\gamma = \alpha = \text{id}\). \(X\) is \((\alpha, \gamma) - T_I\) if and only if \(X\) is \(T_{1/2}\) [9].
5 (I, γ)\textsubscript{α}-sg-closed Sets and (α, γ) - Semi-\textsubscript{T_I} Spaces

In the same way as in the above section, we use the generalized local function in order to obtain the concept of α-sg-closed set given in [16].

Let \((X, τ, α)\) where α a monotone operator, \(I\) an ideal on \(X\) and \(α - \text{SO}(X)\) the collection of all \(α\)-semi-open sets in \(X\), then the generalized local function taking \(J = α - \text{SO}(X)\) satisfies the following properties:

**Theorem 5.1** Let \(A\) be a subset of \(X\) an \(α\) be a monotone operator then:

1. \(A^*(I, α - \text{SO}(X)) \subset α - \text{scl}(A)\).

2. If \(I = \{\emptyset\}\), then
   \[A^*(I, α - \text{SO}(X)) = α - \text{scl}(A)\].

3. If \(α = \text{id}\) and \(I\) is any ideal, then \(A^*(I, α - \text{SO}(X)) = A^*(I, τ)\).

4. \(A^*(I, α - \text{SO}(X)) = α - \text{scl}(A^{*}(I, α - \text{SO}(X))) \subset α - \text{scl}(A)\), that is, \(A^*(I, α - \text{SO}(X))\) is an \(α\)-semi-closed set.

**Proof.**

1. Let \(x \in A^*(I, α - \text{SO}(X))\), then for all \(α\)-semi-open set \(U_x\) such that \(x \in U_x\), we have \(U_x \cap A \notin I\), then \(U_x \cap A \neq \emptyset\). Therefore \(x \in α - \text{scl}(A)\).

2. Let \(x \in α - \text{scl}(A)\), then for all \(α\)-semi-open set \(U_x\) such that \(x \in U_x\) we have \(U_x \cap A \neq \emptyset\), that is, \(U_x \cap A \notin \{\emptyset\} = I\).

3. If \(α\) is the identity operator, the \(α\)-semi-open sets are the same as the open sets.
4. Let \( x \in A^*(I, \alpha - SO(X)) \) then for all \( \alpha \)-semi-open set \( U_x \) such that \( x \in U_x \) we have \( U_x \cap A^*(I, \alpha - SO(X)) \neq \emptyset \); that is \( x \in \alpha - scl(A^*(I, \alpha - SO(X))) \).

If \( x \in \alpha - scl(A^*(I, \alpha - SO(X))) \), then for all \( \alpha \)-semi-open set \( U_x \) such that \( x \in U_x \) we have \( U_x \cap A^*(I, \alpha - SO(X)) \neq \emptyset \), then there exists \( y \in U_x \cap A^*(I, \alpha - SO(X)) \), that is, \( y \in A^*(I, \alpha - SO(X)) \) and \( U_x \) is an \( \alpha \)-semi-open set containing \( y \), it follows that \( U_x \cap A \notin I \), in consequence \( x \in A^*(I, \alpha - SO(X)) \) and therefore \( \alpha - scl(A^*(I, \alpha - SO(X))) \subset A^*(I, \alpha - SO(X)) \).

Finally, the last part of 4, follows from 1. \( \square \)

**Remark 5.1** If \( \mathcal{F} = \alpha - SO(X) \). In Definition 3.3, we denote \( \mathcal{F} - cl^*(A) \) by \( \alpha - scl^*(A) \) that is, \( \alpha - scl^*(A) = A \cup A^*(I, \alpha - SO(X)) \). Also satisfies the following properties:

1. \( \alpha - scl^*(A) \subset \alpha - scl(A) \), for all monotone operator \( \alpha \).
2. If \( \alpha = id \), then \( \alpha - scl^*(A) = cl^*(A) \).
3. If \( I = \{0\} \) and \( \alpha \) is a monotone operator, then \( \alpha - scl^*(A) = \alpha - scl(A) \).

**Definition 5.1** Let \((X, \tau, I)\), and consider two operators \( \alpha, \gamma \) associated with \( \tau \). A subset \( A \) of \( X \) is said to be \((I, \gamma)_{\alpha}\)-semi-generalized closed set, abbreviated \((I, \gamma)_{\alpha}\)-sg-closed, if \( \alpha - scl^*(A) \subset \gamma(U) \) whenever \( A \subset U \) and \( U \in \alpha - SO(X) \).

**Remark 5.2** Observe that when \( \alpha \) is the identity operator, the \((I, \gamma)\)-g-closed sets and \((I, \gamma)_{\alpha}\)-sg-closed sets agree. If we choose \( I = \{0\} \), then all \((I, id)_{\alpha}\)-sg-closed set is a g-closed set.

We can resume the above in the following theorem:

**Theorem 5.2** Let \((X, \tau, I)\), \( A \subset X \), \( \alpha \) and \( \gamma \) be two operators associated with \( \tau \) an \( \alpha \) monotone.
1. If \( \alpha = id \), the set \( A \) is an \((I, \gamma)\)-g-closed set ([4]) if and only if \( A \) is an \((I, \gamma)\)\(_{\alpha}\)-sg-closed set.

2. If \( I = \{\emptyset\} \) and the set \( A \) is an \((I, id)\)\(_{\alpha}\)-sg-closed, then \( A \) is an \(\alpha\)-sg-closed ([16]).

3. If \( \alpha = id \), \( I = \{\emptyset\} \), and \( A \) is an \((I, id)\)\(_{\alpha}\)-sg-closed, then \( A \) is a \(g\)-closed set ([9]).

**Proof.**

1. Suppose that \( \alpha = id \) and \( A \) is an \((I, \gamma)\)\(_{\alpha}\)-sg-closed set. Let \( U \) an open set such that \( A \subset U \), since all open set is an \(\alpha\)-semi-open set, then \( \alpha - \text{scl}^*(A) \subset \gamma(U) \). Since \( \alpha = id \), we have that \( \text{cl}^*(A) \subset \gamma(U) \) and therefore \( A^*(I, \tau) \subset \text{cl}^*(A) \subset \gamma(U) \), it follows that \( A \) is an \((I, \gamma)\)-g-closed set.

The converse follows in the same way.

2. Suppose that \( A \) is an \((I, id)\)\(_{\alpha}\)-sg-closed set of \( X \) and \( U \) an \(\alpha\)-semi-open set such that \( A \subset U \), then \( \alpha - \text{scl}^*(A) \subset id(U) \), since \( I = \{\emptyset\} \); it follows \( \alpha - \text{scl}^*(A) = \alpha - \text{scl}(A) \) in consequence, \( A \) is an \(\alpha\)-sg-closed set.

3. It is an immediate consequence of parts 1., 2. \(\square\)

**Theorem 5.3** Let \((X, \tau, I)\), \( \alpha \) be a monotone operator associated with \( \tau \) and \( \gamma \) an expansive operator on \( \alpha - SO(X) \). Then all \(\alpha\)-sg-closed set is an \((I, \gamma)\)\(_{\alpha}\)-sg-closed set.

**Proof.**

Let \( A \) an \(\alpha\)-sg-closed subset of \( X \) and \( U \) an \(\alpha\)-semi-open set such that \( A \subset U \), then \( \alpha - \text{scl}(A) \subset U \); therefore \( \alpha - \text{scl}^*(A) \subset U \), since \( \gamma \) is expansive on \( \alpha - SO(X) \), then \( U \subset \gamma(U) \), it follows that \( A \) is an \((I, \gamma)\)\(_{\alpha}\)-sg-closed set. \(\square\)

The following example shows the existence of an \((I, \gamma)\)\(_{\alpha}\)-sg-closed set that is not an \(\alpha\)-sg-closed set.
Example 5.1 Consider $\mathbb{R}$, the set of the real numbers, with the finite complement topology $\tau_f = \{U \subseteq \mathbb{R} : \mathbb{R} \setminus U \text{ is finite or } \mathbb{R}\}$, the operator $\alpha : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ associated with this topology is defined as $\alpha(U) = \text{cl}(U)$.

The open interval $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ is not an $\alpha$-sg-closed set because $\mathbb{R} \setminus \{b\}$ is an $\alpha$-semi-open set containing $(a, b)$ and the $\alpha - \text{scl}(A) \notin \mathbb{R} \setminus \{b\}$. $(a, b)$ is an $(I, \gamma)_{\alpha}$-sg-closed set when we consider the ideal $I = \mathcal{P}(X)$ and $\gamma$ the identity operator.

We have shown the existence of an $(I, \gamma)_{\alpha}$-sg-closed set that is not an $\alpha$-semi-closed. Now we introduce a new class of spaces in the following definition.

Definition 5.2 Let $(X, \tau, I)$, $\alpha$ and $\gamma$ be two operators associated with $\tau$. We say that $X$ is an $(\alpha, \gamma)$-semi-$T_1$ space if all $(I, \gamma)_{\alpha}$-sg-closed set is an $\alpha$-semi-closed set.

Theorem 5.4 Let $(X, \tau, I)$, and consider two operators $\alpha$, $\gamma$ associated with $\tau$ and $\alpha$ monotone.

1. If $\alpha = \text{id}$ and $X$ is an $(\alpha, \gamma)$-semi-$T_1$ space, then $X$ is a $\gamma - T_1$ space ([4]).

2. If $X$ is an $(\alpha, \gamma)$-semi-$T_1$ space and $\gamma$ is expansive on $\alpha - \text{SO}(X)$, then $X$ is an $\alpha$-semi-$T_{\frac{1}{2}}$ space ([16]).

3. If $I = \{\emptyset\}$ and $X$ is an $\alpha$-semi-$T_{\frac{1}{2}}$ space, then $X$ is an $(\alpha, \text{id})$-semi-$T_1$ space.

4. If $I = \{\emptyset\}$ and $\gamma = \alpha = \text{id}$. $X$ is an $(\alpha, \gamma)$-semi-$T_1$ space if and only if $X$ is an $T_{\frac{1}{2}}$ space ([9]).
6 (I, \gamma)_{(\alpha,\beta)}-sg$-closed Sets and $(\alpha, \beta, \gamma)-semi-\ T_s$-spaces

Consider $(X, \tau)$, $I$ an ideal on $X$, $\alpha$ and $\beta$ operators on $\tau$ and $(\alpha, \beta) - SO(X)$ the collection of all $(\alpha, \beta)$-semi-open sets in $X$, then the generalized local function taking $\mathfrak{F} = (\alpha, \beta) - SO(X)$ satisfies the following properties:

**Theorem 6.1** Let $A$ be a subset of $X$, then:

1. $A^{*}(I, (\alpha, \beta) - SO(X)) \subseteq (\alpha, \beta) - scl(A)$.

2. If $I = \{\emptyset\}$, $\alpha$ and $\beta$ any two operators, then $A^{*}(I, (\alpha, \beta) - SO(X)) = (\alpha, \beta) - scl(A)$.

3. If $\alpha = id$, $\beta$ a monotone operator and $I$ any ideal, then $A^{*}(I, (\alpha, \beta) - SO(X)) = A^{*}(I, \beta - SO(X))$.

4. If $\beta = id$, $\alpha$ any operator and $I$ any ideal, then $A^{*}(I, (\alpha, \beta) - SO(X)) = A^{*}(I, \tau_{\alpha})$.

5. If $\alpha = \beta = id$, and $I$ any ideal, then $A^{*}(I, (\alpha, \beta) - SO(X)) = A^{*}(I, \tau)$.

6. $A^{*}(I, (\alpha, \beta) - SO(X)) = (\alpha, \beta) - scl(A^{*}(I, (\alpha, \beta) - SO(X))) \subseteq (\alpha, \beta) - scl(A)$, that is, $A^{*}(I, (\alpha, \beta) - SO(X))$ is an $(\alpha, \beta)$-semi-closed set.

**Proof.**

1. Let $x \in A^{*}(I, (\alpha, \beta) - SO(X))$, then for all $(\alpha, \beta)$-semi-open set $U_x$ such that $x \in U_x$ we have $U_x \cap A \notin I$; but $\emptyset \in I$ for any ideal $I$, then $U_x \cap A \neq \emptyset$; it follows that $x \in (\alpha, \beta) - scl(A)$.

2. Let $x \in (\alpha, \beta) - scl(A)$, then for all $(\alpha, \beta)$-semi-open set $U_x$ such that $x \in U_x$, $U_x \cap A \neq \emptyset$, that is, $U_x \cap A \notin \{\emptyset\} = I$, it follows that $x \in A^{*}(I, (\alpha, \beta) - SO(X))$.

Parts 3, 4 and 5 follow directly from the definition 2.5.
6. Let \( x \in A^*(I,(\alpha,\beta) - SO(X)) \) then for all \( (\alpha,\beta) \)-semi-open set \( U_x \) such that \( U_x \cap A^*(I,(\alpha,\beta) - SO(X)) \neq \emptyset \); it follows that \( x \in (\alpha,\beta) - scl(A^*(I,(\alpha,\beta) - SO(X))). \)

Suppose that \( x \in (\alpha,\beta) - scl(A^*(I,(\alpha,\beta) - SO(X))) \), then for all \( (\alpha,\beta) \)-semi-open set \( U_x \) such that \( x \in U_x \) we have \( U_x \cap A^*(I,(\alpha,\beta) - SO(X)) \neq \emptyset \), then there exists \( y \in U_x \cap A^*(I,(\alpha,\beta) - SO(X)) \), that is, \( y \in A^*(I,(\alpha,\beta) - SO(X)) \) and \( U_x \) is an \( (\alpha,\beta) \)-semi-open set containing \( y \), then \( U_x \cap A \notin I \), therefore \( x \in A^*(I,(\alpha,\beta) - SO(X)) \) then \( (\alpha,\beta) - scl(A^*(I,(\alpha,\beta) - SO(X))) \subset A^*(I,(\alpha,\beta) - SO(X)). \)

Finally, the last part of 6, follows from 1. \( \square \)

We denote by \( (\alpha,\beta) - scl^*(A) = A \cup A^*(I,(\alpha,\beta) - SO(X)) \), when \( \mathcal{I} = (\alpha,\beta) - SO(X) \) in the Definition 3.3. It is clear that, \( (\alpha,\beta) - scl^*(A) \subset (\alpha,\beta) - scl(A) \), satisfies the following properties:

1. If \( \alpha = id \) and \( \beta \) is a monotone operator, then \( (\alpha,\beta) - scl^*(A) = \beta - cl^*(A) \).

2. If \( \alpha \) is any operator and \( \beta = id \), then \( (\alpha,\beta) - scl^*(A) = \alpha - cl^*(A) \).

3. If \( \alpha = \beta = id \), then \( (\alpha,\beta) - scl^*(A) = cl^*(A) \).

4. If \( I = \{\emptyset\} \), then \( (\alpha,\beta) - scl^*(A) = (\alpha,\beta) - scl(A) \).

We now introduce the concept of \( (I,\gamma)_{(\alpha,\beta)} \)-sg-closed set as a generalization of the concepts of \( (I,\gamma) \)-g-closed set ([4]) and \( (\alpha,\beta) \)-sg-closed set ([17]).

**Definition 6.1** Let \( (X,\tau,I) \), and \( \alpha, \beta, \gamma \) operators associated with \( \tau \). A subset \( A \) of \( X \) is called \( (I,\gamma)_{(\alpha,\beta)} \)-semi-generalized-closed set, abbreviated \( (I,\gamma)_{(\alpha,\beta)} \)-sg-closed, if \( (\alpha,\beta) - scl^*(A) \subset \gamma(U) \) whenever \( A \subset U \) and \( U \in (\alpha,\beta) - SO(X) \).

**Remark 6.1** 1. Observe that when \( \alpha = \beta = id \), the \( (I,\gamma) \)-g-closed sets and the \( (I,\gamma)_{(\alpha,\beta)} \)-sg-closed sets agree.
2. If $\alpha$ is any operator and $\beta$ is a monotone, the $(I, \gamma)_{(\alpha, \beta)}$-closed sets and the $(I, \gamma)_{(\alpha, \beta)}$-sg-closed sets agree.

3. If $\alpha = id$ and $\beta$ is a monotone, the $(I, \gamma)_{\beta}$-closed sets and the $(I, \gamma)_{(\alpha, \beta)}$-sg-closed sets agree.

4. When $I = \{\emptyset\}$, the $(I, \gamma)_{(\alpha, \beta)}$-sg-closed sets and the $(\alpha, \beta)$-sg-closed sets agree.

We can resume the above in the following theorem.

**Theorem 6.2** Let $(X, \tau, I)$, $\alpha$, $\beta$ and $\gamma$ operators associated with $\tau$ and $A \subset X$, then

1. If $\alpha = id$ and $\beta$ is a monotone operator, $A$ is an $(I, \gamma)_{(\alpha, \beta)}$-sg-closed set if and only if $A$ is an $(I, \gamma)_{\beta}$-sg-closed set.

2. If $\alpha$ is any operator and $\beta = id$, $A$ is an $(I, \gamma)_{(\alpha, \beta)}$-sg-closed set if and only if $A$ is an $(I, \gamma)_{\alpha}$-g-closed set.

3. If $\alpha = \beta = id$, $A$ is an $(I, \gamma)_{(\alpha, \beta)}$-sg-closed set if and only if $A$ is an $(I, \gamma)$-g-closed set (\cite{4}).

4. If $I = \{\emptyset\}$, $\alpha$, $\beta$ any operators, $\gamma = id$ and $A$ is an $(I, \gamma)_{(\alpha, \beta)}$-sg-closed set, then $A$ is an $(\alpha, \beta)$-sg-closed set (\cite{17}).

**Theorem 6.3** Let $(X, \tau, I)$, $\alpha$, $\beta$ be operators on $\tau$ and $\gamma$ an expansive operator on $(\alpha, \beta) - SO(X)$. All $(\alpha, \beta)$-sg-closed set is an $(I, \gamma)_{(\alpha, \beta)}$-sg-closed set.

**Proof.**

Let $A$ an $(\alpha, \beta)$-sg-closed set of $X$ and $U$ an $(\alpha, \beta)$-semi-open set such that $A \subset U$, then $(\alpha, \beta) - scl(A) \subset U$; that is $(\alpha, \beta) - scl(A) \subset U$, since $\gamma$ is expansive on $(\alpha, \beta) - SO(X)$, then $U \subset \gamma(U)$, therefore, $(\alpha, \beta) - scl(A) \subset \gamma(U)$; it follows that $A$ is an $(I, \gamma)_{(\alpha, \beta)}$-sg-closed set. \qed

The following example shows the existence of an $(I, \gamma)_{(\alpha, \beta)}$-sg-closed set that is not an $(\alpha, \beta)$-sg-closed set.
Example 6.1 Consider $\mathbb{R}$, the set of the real numbers, with the finite complement topology $\tau_f = \{ U \subseteq \mathbb{R} : \mathbb{R} \setminus U \text{ is finite or } \mathbb{R} \}$, the operators $\alpha, \beta : \mathcal{P}(\mathbb{R}) \rightarrow \mathcal{P}(\mathbb{R})$ associated with this topology are defined as follow: $\alpha(U) = \text{int}(U)$, $\beta(U) = \text{cl}(U)$.

The set $\mathbb{Q}$ of the rational numbers is not an $(\alpha, \beta)$-sg-closed set because $\mathbb{R} \setminus \{ \sqrt{2} \}$ is an $(\alpha, \beta)$-semi-open set containing $\mathbb{Q}$; but $(\alpha, \beta) - \text{ scl}(\mathbb{Q}) = \mathbb{R} \not\subseteq \mathbb{R} \setminus \{ \sqrt{2} \}$. $\mathbb{Q}$ is an $(I, \gamma)_{(\alpha, \beta)}$-sg-closed set if we consider the ideal $I = \mathcal{P}(\mathbb{X})$ and $\gamma$ the identity operator.

Using the fact that there exist $(I, \gamma)_{(\alpha, \beta)}$-sg-closed sets that are not $(\alpha, \beta)$-semi-closed. We introduce a new class of spaces in the following definition.

Definition 6.2 Let $(X, \tau, I)$, $\alpha, \beta$ and $\gamma$ be operators associated with $\tau$. $X$ is called an $(\alpha, \beta, \gamma)$ – semi – $T_1$ space if all $(I, \gamma)_{(\alpha, \beta)}$-sg-closed set is an $(\alpha, \beta)$-semi-closed set.

Theorem 6.4 Let $(X, \tau, I)$, $\alpha, \beta, \gamma$ be operators associated with $\tau$ and $A$ a subset of $X$, then

1. If $\alpha = \text{id}$ and $\beta$ is monotone, $X$ is an $(\alpha, \beta, \gamma)$ – semi – $T_1$ space if and only if $X$ is an $(\beta, \gamma)$ – semi – $T_1$ space.

2. If $\alpha$ is any operator and $\beta = \text{id}$, $X$ is an $(\alpha, \beta, \gamma)$ – semi – $T_1$ space if and only if $X$ is an $(\alpha, \gamma)$ – semi – $T_1$ space.

3. If $\alpha = \beta = \text{id}$ and $X$ is $(\alpha, \beta, \gamma)$ – semi – $T_1$ space, then $X$ is an $\gamma$ – $T_1$ space ([4]).

4. If $I = \{ \emptyset \}$, $\gamma = \text{id}$ and $X$ is an $(\alpha, \beta)$ – semi – $T_{\frac{1}{2}}$ space, then $X$ is an $(\alpha, \beta, \gamma)$ – semi – $T_1$ space.

5. If $\alpha, \beta$ are any operators, $\gamma$ expansive on $(\alpha, \beta)$ – SO$(X)$ and $X$ is an $(\alpha, \beta, \gamma)$ – semi – $T_1$ space, then $X$ is an $(\alpha, \beta)$ – semi – $T_{\frac{1}{2}}$ space ([17]).
References


Resumen
Son dados un espacio topológico \((X, \tau)\), tres operadores \(\alpha, \beta, \gamma\) asociados a una topología \(\tau\), es un ideal \(I\) en \(X\). Los conceptos de conjunto \(\alpha\)-cerrado, conjunto \(\alpha\)-semicerrado, conjunto \((\alpha, \beta)\)-semicerrado y conjunto \((I, \gamma)\) g-cerrado son generalizados. También nuevos axiomas de separación son introducidos y caracterizados, y nuevos espacios son obtenidos de tal manera que los espacios \(\alpha - T_\frac{1}{2}, \alpha\)-semi \(T_\frac{1}{2}, (\alpha, \beta)\)-semi \(T_\frac{1}{2}\) y \(\gamma - T_1\), respectivamente, son generalizados.
Palabras clave: \((\alpha, \beta)\)-semicerrado, \((I, \gamma)\) g-cerrado, \((\alpha, \beta)\)-semi-\(T_{\frac{1}{2}}\), \((\alpha, \beta, \gamma)\) semi-\(T_i\).

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