ON THE CONSTANT OF HOMOTHEITY FOR COVERING A CONVEX SET WITH ITS SMALLER COPIES

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Abstract

Let $H_d$ denote the smallest integer $n$ such that for every convex body $K$ in $\mathbb{R}^d$ there is a $0 < \lambda < 1$ such that $K$ is covered by $n$ translates of $\lambda K$. In [2] the following problem was posed: Is there a $0 < \lambda_d < 1$ depending on $d$ only with the property that every convex body $K$ in $\mathbb{R}^d$ is covered by $H_d$ translates of $\lambda_d K$? We prove the affirmative answer to the question and hence show that the Gohberg-Markus-Boltyanski-Hadwiger Conjecture (according to which $H_d \leq 2^d$) holds if, and only if, a formally stronger version of it holds.

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1 Definitions and Results

A convex body in $\mathbb{R}^d$ is a compact convex set $K$ with non-empty interior. Its volume is denoted by $\text{vol}(K)$.

**Definition 1.1.** For $d \geq 1$ let $H_d$ denote the smallest integer $n$ such that for every convex body $K$ in $\mathbb{R}^d$ there is a $0 < \lambda < 1$ such that $K$ is covered by $n$ translates of $\lambda K$. Furthermore, let $\overline{H}_d$ denote the smallest integer $m$ such that there is a $0 < \lambda_d < 1$ with the property that every convex body $K$ in $\mathbb{R}^d$ is covered by $m$ translates of $\lambda_d K$.

Clearly, $H_d \leq \overline{H}_d$. The following question was raised in [2] (Problem 6 in Section 3.2): Is it true that $H_d = \overline{H}_d$?

We answer the question in the affirmative using a simple topological argument.

**Theorem 1.2.** $H_d = \overline{H}_d$.

The famous conjecture of Gohberg, Markus, Boltyanski and Hadwiger states that $H_d \leq 2^d$ (and only the cube requires $2^d$ smaller positive homothetic copies to be covered). For more information on the conjecture, refer to [1], [7] and [11]. In view of Theorem 1.2, the conjecture is true if, and only if, the following, formally stronger conjecture holds:

**Conjecture 1.3.** (Strong Gohberg-Markus-Boltyanski-Hadwiger Conjecture). For every $d \geq 1$ there is a $0 < \lambda_d < 1$ such that every convex body $K$ in $\mathbb{R}^d$ is covered by $2^d$ translates of $\lambda_d K$.

In Section 2 we prove the Theorem. We note that the proof provides no upper bound on $\lambda_d$ in terms of $d$. In Section 3 we show an upper bound on the number of translates of $\lambda K$ required to cover $K$, improving a result of Januszewski and Lassak [5].
2 Proof of Theorem 1.2

We define the following function on the set of convex bodies:

\[ \lambda(K) := \inf \{ \lambda > 0 : K \text{ is covered by } H_d \text{ translates of } \lambda K \} . \]

By [8], \( H_d \) is finite for every \( d \), so \( \lambda(.) \) is well defined.

**Remark 2.1.** Clearly, \( \lambda(.) \) is affine invariant; that is, if \( T \) is an invertible affine transformation of \( \mathbb{R}^d \) then \( \lambda(K) = \lambda(TK) \). Moreover, \( 0 < \lambda(K) < 1 \).

We recall the definition of the (multiplicative) Banach-Mazur distance of two convex bodies \( L \) and \( K \) in \( \mathbb{R}^d \):

\[
d_{BM}(L,K) = \inf \{ \lambda > 0 : L - a \subseteq T(K - b) \subseteq \lambda(L - a) \}
\quad \text{for some } a, b \in \mathbb{R}, T \in GL(\mathbb{R}^d) \} \tag{2.1}
\]

The following proposition states that \( \lambda(.) \) is upper semi-continuous. Similar statements have been proved before, cf. Lemma 2, in [3].

**Proposition 2.2.** For every convex body \( K \) and \( \varepsilon > 0 \) there is a \( \delta > 0 \) with the property that for any convex body \( L \), if \( d_{BM}(L,K) < 1 + \delta \) then \( \lambda(L) < \lambda(K) + \varepsilon \).

**Proof.** Let \( \lambda := \lambda(K) + \frac{\varepsilon}{2} \). Then there is a set \( \Lambda \subset \mathbb{R}^d \) with card \( \Lambda \leq H_d \) such that \( K \subseteq \Lambda + \lambda K \). Now, let \( \delta > 0 \) be such that

\[
1 + \delta < \frac{\lambda + \frac{\varepsilon}{2}}{\lambda} \tag{2.2}
\]

Assume that \( d_{BM}(L,K) < 1 + \delta \); that is,

\[
L - a \subseteq K \subseteq (1 + \delta)(L - a), \tag{2.3}
\]
where \( \overline{K} \) is an affine image (under an invertible affine transformation) of \( K \). Clearly, we may assume that \( \overline{K} = K \).

It follows that \( L - a \subseteq \Lambda + (1 + \delta)\lambda(L - a) \), and hence, \( \lambda(L) \leq (1 + \delta)\lambda \leq \lambda(k) + \varepsilon \).

Let \( \mathcal{K}_d^a \) denote the set of affine equivalence classes of convex bodies in \( \mathbb{R}^d \) equipped with the topology induced by the metric \( d_{BM} \). In [6] it is shown that \( \mathcal{K}_d^a \) is a compact space. (Note that Macbeath uses a different metric on \( \mathcal{K}_d^a \) however, that metric induces the same topology as \( d_{BM} \), cf. [4].)

It follows from Remark 2.1 and Proposition 2.2 that \( \lambda(.) \) is an upper semicontinuous function on a compact space. Hence, it attains its maximum, which (by Remark 2.1) is less than one. This proves Theorem 1.2.

3 Quantitative Results

Januszewski and Lassak [5] proved that for every \( k + l > d^d \), any convex body \( K \subset \mathbb{R}^d \) is covered by \( k \) translates of \( \lambda K \) and \( l \) translates of \( -\lambda K \), where \( \lambda = 1 - \frac{1}{(d+1)d^d} \). The following argument shows that one may obtain a better estimate on the number of translates of \( \lambda K \) required to cover \( K \), using results of Rogers [8], Rogers and Shephard [9], and Rogers and Zhong [10].

Let \( K, L \) be convex bodies in \( \mathbb{R}^d \). Let \( N(K, L) \) denote the covering number of \( K \) and \( L \); that is, the smallest number of translates of \( L \) required to cover \( K \). In [10] it is shown that

\[
N(K, L) \leq \frac{\text{vol}(K - L)}{\text{vol}(L)} \Theta(L),
\]

where \( \Theta(L) \) is the covering density of \( L \). By [8], \( \Theta(L) \leq d \log d + \)
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\[ \log \log d + 5d \] for every convex body \( L \) in \( \mathbb{R}^d \). It follows that for any \( 0 < \lambda < 1 \) we have

\[
N(K, \lambda K) \leq \lambda^{-d} \frac{\text{vol}(K - K)}{\text{vol} K} (d \log d + \log \log d + 5d)
\]

\[
\leq \lambda^{-d} \left( \frac{2d}{d} \right) (d \log d + \log \log d + 5d) \tag{3.1}
\]

The last inequality follows from the Rogers-Shephard Inequality [9]. Similarly,

\[
N(K, -\lambda K) \leq \lambda^{-d} \frac{\text{vol}(K + K)}{\text{vol} K} (d \log d + \log \log d + 5d)
\]

\[
= \lambda^{-d} 2^d (d \log d + \log \log d + 5d) \tag{3.2}
\]

By substituting \( \lambda = \frac{1}{2} \) into (3.1) and (3.2), we obtain the following:

**Remark 3.1.** The number of translates of \( \frac{1}{2} K \) that cover \( K \) is of order not greater than \( 8d \sqrt{d} \log d \); and the number of translates of \( -\frac{1}{2} K \) that cover \( K \) is of order not greater than \( 4d^2 \log d \).

**Definition 3.2.** Let \( 0 < \lambda < 1 \), and \( d \geq 1 \). We denote by \( H_d(\lambda) \) the smallest integer \( n \) such that every convex body \( K \) in \( \mathbb{R}^d \) is covered by \( n \) translates of \( \lambda K \).

It follows from Remark 3.1 that \( \overline{H}_d \left( \frac{1}{2} \right) \) is finite for every \( d \). A natural strengthening of the question we discussed in this note is the following:

**Question 3.3.** Is there a universal constant \( 0 < \lambda < 1 \) such that for every dimension \( d \), \( H_d \) is equal to \( \overline{H}_d(\lambda) \)?

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References


Resumen

Llamemos \( H_d \) al menor entero positivo \( n \) con la propiedad de que para todo cuerpo convexo \( K \) en \( \mathbb{R}^d \) hay una constante \( 0 < \lambda < 1 \) tal que \( K \) se cubre por medio de \( n \) traslaciones de \( \lambda K \). En el libro *Research problems in discrete geometry*, de Brass, Moser y Pach, el siguiente problema fue propuesto: ¿Es posible encontrar una constante \( 0 < \lambda_d < 1 \), que dependa solo de la dimensión \( d \), tal que todo cuerpo convexo \( K \) en \( \mathbb{R}^d \) es cubierto por \( H_d \) traslaciones de \( \lambda_d K \)? Demostraremos que la respuesta a esta pregunta es afirmativa, y por tanto que la conjetura de Gohberg-Markus-Boltyanski-Hadwiger (la cual postula que \( H_d \leq 2^d \)) se cumple si, y solo si, se satisface una versión formalmente más fuerte de la misma.

**Palabras Clave:** Iluminación, Conjetura de Boltyanski-Hadwiger, Conjuntos convexos, Cubrimiento de conjuntos convexos.

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