A large deviation principle for a natural sequence of point processes on a Riemannian two-dimensional manifold

David García Zelada

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Abstract

We follow the techniques of Paul Dupuis, Vaios Laschos, and Kavita Ramanan in [8] to prove a large deviation principle for a sequence of point processes defined by Gibbs measures on a compact orientable two-dimensional Riemannian manifold. We see that the corresponding sequence of empirical measures converges to the solution of a partial differential equation and, in some cases, to the volume form of a constant curvature metric.

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1 Université Paris-Dauphine, PSL Research University.
1 Model and results

Let \((M, g)\) be a compact oriented two-dimensional Riemannian manifold of genus different from one. Denote by \(\text{vol}\) the normalized volume form associated to \(g\) and the orientation. Define the 2-form

\[
\Lambda = \frac{\text{Ric } g}{2\pi \chi(M)},
\]

(1.1)

where \(\text{Ric } g\) is the Ricci curvature seen as a 2-form and \(\chi(M)\) denotes the Euler characteristic. More precisely, write \(\text{Ric } g = K_g \text{vol}\), where \(K_g\) is the Gaussian curvature of \(g\), while the usual symmetric Ricci curvature would be \(K_g g\). We shall think of \(\Lambda\) as a signed measure.

It is known that there exists a continuous symmetric function \(G : M \times M \to \mathbb{R} \cup \{\infty\}\) such that the function \(G_x : M \to \mathbb{R} \cup \{\infty\}\) defined by \(G_x(y) = G(x, y)\) is integrable and satisfies

\[
\Delta G_x = -\delta_x + \Lambda
\]

(1.2)

for every \(x \in M\). More precisely, for every \(f \in C^\infty(M)\) and \(x \in M\), we have

\[
\int_M G(x, y)\Delta f(y) = -f(x) + \int_M f(y)d\Lambda(y),
\]

(1.3)

here \(\Delta : C^\infty(M) \to \Omega^2(M)\) is the usual Laplacian, i.e. \(\Delta = d \ast d\), where \(\ast\) is the Hodge star operator and \(d\) is the exterior derivative. Moreover, such \(G\) is unique up to an additive constant and we can choose \(G\) such that

\[
\int_M G(x, y)d\Lambda(y) = 0
\]

(1.4)

for every \(x \in M\). See [6] for a proof and more information.

Take an integer \(n \geq 2\). We consider a system of \(n\) indistinguishable particles with total charge 1 interacting via the electrostatic force. In other words, each particle has charge \(1/n\) and the two-particle interaction
potential is $G$. This means that the total energy will be $H_n : M^n \to \mathbb{R} \cup \{\infty\}$, defined by

$$H_n(x_1, ..., x_n) = \frac{1}{n^2} \sum_{i<j} G(x_i, x_j).$$

Choose a sequence of positive numbers $\{\beta_n\}_{n \geq 2}$ and a positive number $\beta > 0$ such that $\beta_n \to \beta$. We define the Gibbs non-normalized measure associated to $H_n$ and $\beta_n$ as the finite measure $\gamma_n$ on $M^n$ given by

$$d\gamma_n = \exp \left( -n \beta_n H_n \right) d\text{vol}^\otimes n.$$ 

The Gibbs probability measure will be the probability measure

$$\mathbb{P}_n = \frac{\gamma_n}{Z_n},$$

where $Z_n = \gamma_n(M^n)$ is called the partition function. The probability measure $\mathbb{P}_n$ describes a system of $n$ particles with Hamiltonian $H_n$ and inverse temperature $n\beta_n$.

For any metrizable compact space $E$ we endow $\mathcal{P}(E)$, the space of probability measures on $E$, with the smallest topology such that for every continuous function $f : E \to \mathbb{R}$ the application $\mu \to \int_E f d\mu$ is continuous. We can see that this is again a metrizable compact space. See Appendix for a short proof, and [5] for extra information. Furthermore, a sequence of probability measures on $E$, say $\{\mu_n\}_{n \in \mathbb{N}}$, converges to $\mu \in \mathcal{P}(E)$ if and only if $\int_E f d\mu_n$ converges to $\int_E f d\mu$ for every continuous function $f : E \to \mathbb{R}$. In fact, it is enough to verify that $\int_E f d\mu_n$ converges to $\int_E f d\mu$ for $f$ belonging to a countable dense family of the space of continuous functions with the uniform topology.

The space $M^n$ is to be ‘injected’ in $\mathcal{P}(M)$ by means of the continuous application

$$i_n : M^n \to \mathcal{P}(M)$$

$$(x_1, ..., x_n) \mapsto \frac{1}{n} \sum_{i=1}^n \delta_{x_i},$$
and we will study the limit of the sequence of probabilities \( i_n(\mathbb{P}_n) \), the \textbf{pushforward laws of} \( \mathbb{P}_n \).

Define the \textbf{macroscopic energy} as

\[
W : \mathcal{P}(M) \to \mathbb{R} \cup \{ \infty \} \\
\mu \mapsto \int_{M \times M} G(x, y)\,d\mu(x)\,d\mu(y),
\]

and the \textbf{free energy} as

\[
F : \mathcal{P}(M) \to \mathbb{R} \cup \{ \infty \} \\
\mu \mapsto \beta \frac{1}{2} W(\mu) + D(\mu \parallel \text{vol}),
\]

where \( D(\mu \parallel \text{vol}) \) denotes the relative entropy of \( \mu \) with respect to \( \text{vol} \), also known as the \textbf{Kullback-Leibler divergence}, defined by

\[
D(\mu \parallel \text{vol}) = \int_M \log \left( \frac{d\mu}{d\text{vol}} \right) d\mu
\]

if \( \mu \) is absolutely continuous with respect to \( \text{vol} \), and \( D(\mu \parallel \text{vol}) = \infty \) otherwise. Now we can state our main result.

\textbf{Theorem 1.1 (Laplace principle).} \textit{For every continuous} \( f : \mathcal{P}(M) \to \mathbb{R} \) \textit{we have the convergence}

\[
\frac{1}{n} \log \int_{M^n} e^{-nf \circ i_n} d\gamma_n \xrightarrow{n \to \infty} - \inf_{\mu \in \mathcal{P}(M)} \{ f(\mu) + F(\mu) \}.
\]

To state a large deviation principle as an easy corollary we need to define first the function

\[
I : \mathcal{P}(M) \to \mathbb{R} \cup \{ \infty \} \\
\mu \mapsto F(\mu) - \inf F.
\]

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Corollary 1.2 (A large deviation principle). For every closed set $C \subset \mathcal{P}(M)$ we have

$$\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}_n(i_n^{-1}(C)) \leq - \inf_{x \in C} I(x),$$

and for every open set $O \subset \mathcal{P}(M)$ we have

$$\liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}_n(i_n^{-1}(O)) \geq - \inf_{x \in O} I(x).$$

This tells us that to understand the limiting behavior of $i_n(\mathcal{P}_n)$ we must study the free energy $F$. The first two main properties will be studied in Section 2.

Proposition 1.3 (Convexity and lower semicontinuity of $F$). The function $F$ is strictly convex and lower semicontinuous.

Thus, $F$ achieves its minimum at only one point. The following theorem characterizes this minimum.

Theorem 1.4 (Minimum of $F$). The function $F$ achieves its minimum at a probability measure $\mu_{eq}$ that is absolutely continuous with respect to $\text{vol}$ and such that $\rho = \frac{d\mu_{eq}}{d\text{vol}}$ is a $C^\infty$ strictly positive everywhere function that satisfies the differential equation

$$\Delta \log \rho = \beta \mu_{eq} - \beta \Lambda. \quad (1.5)$$

Remark 1.5 (Equivalent formulation: equation on the Ricci curvature). If we define the metric $\bar{\omega} = \rho g$, we have that $\mu_{eq}$ is the volume form associated to $\bar{\omega}$, and Equation 1.5 can be written as

$$\text{Ric} \bar{\omega} = \left(2\pi \chi(M) + \frac{\beta}{2}\right) \text{Ric} g - \frac{\beta}{2} \mu_{eq}$$

(because of the identity $\Delta \log \rho = 2\text{Ric} g - 2\text{Ric} \bar{\omega}$).

Finally, by Corollary 1.2 and an application of the Borel-Cantelli lemma we get the following.
Corollary 1.6 (Convergence of the empirical measures). If \( \{X_n\}_{n \geq 2} \) is a sequence of random elements in \( \mathcal{P}(M) \) such that \( X_n \sim i_n(\mathcal{P}_n) \), then
\[
X_n \xrightarrow{a.s.} \mu_{eq},
\]
where \( \mu_{eq} \) is the unique minimizer of \( F \).

2 Lower semicontinuity and convexity of I

In this section we prove Proposition 1.3. It is well known that \( D(\cdot \| \text{vol}) \) is lower semicontinuous and strictly convex (see [9, Lemma 1.4.3]). What we need to establish is lower semicontinuity and convexity for \( W \).

Proof of the lower semicontinuity of \( W \). For positive \( m \) set \( G_m(x, y) = G(x, y) \wedge m = \min\{G(x, y), m\} \). Then
\[
\mu \mapsto \int_{M \times M} G_m(x, y) \, d\mu(x) \, d\mu(y)
\]
is a continuous function of \( \mu \). As \( W \) is the increasing limit of functions as \( m \) tends to infinity, we get that \( W \) is lower semicontinuous.

Proof of the convexity of \( W \). To prove convexity it is enough to show that for every \( \mu, \nu \in \mathcal{P}(M) \) we have
\[
W\left( \frac{1}{2} \mu + \frac{1}{2} \nu \right) \leq \frac{1}{2} W(\mu) + \frac{1}{2} W(\nu) \tag{2.1}
\]
due to the lower semicontinuity of \( W \). Inequality (2.1) is equivalent to
\[
\frac{1}{2} W(\mu) + \frac{1}{2} W(\nu) \geq \int_{M \times M} G(x, y) \, d\mu(x) \, d\nu(y). \tag{2.2}
\]
This inequality is easy to verify if \( \mu \) and \( \nu \) are differentiable, i.e., given by differentiable forms. Indeed, in that case, the functions
\[
f(x) = \int M G(x, y) \, d\mu(y) \quad \text{and} \quad g(x) = \int M G(x, y) \, d\nu(y)
\]
A large deviation principle

satisfy

$$\Delta f = -\mu + \Lambda \quad \text{and} \quad \Delta g = -\nu + \Lambda$$

because of [1,3] are differentiable due to the ellipticity of the Laplacian, and have zero integral with respect to $\Lambda$ because of [1,4]. So, in terms of $f$ and $g$ Inequality 2.2 reads

$$-\frac{1}{2} \int_M f \Delta f - \frac{1}{2} \int_M g \Delta g \geq - \int_M f \Delta g,$$

which is equivalent to

$$\int_M \|\nabla (f - g)\|^2 \, d\operatorname{vol} \geq 0.$$

For the general case we need two lemmas.

**Lemma 2.1.** Let $\mu$ be a continuous probability measure, i.e., given by a continuous 2-form. Then there exists a sequence $\mu_n$ of differentiable probability measures such that

$$\mu_n \to \mu \quad \text{and} \quad W(\mu_n) \to W(\mu).$$

*Proof.* As $\mu$ is continuous, we can write $d\mu = \rho \, d\operatorname{vol}$ with $\rho$ continuous. Take a sequence of differentiable functions $\{\rho_n\}_{n \in \mathbb{N}}$ such that $\rho_n \to \rho$ uniformly. We can assume $\rho_n \geq 0$ (redefine $\rho_n = \rho_n + \|\rho - \rho_n\|_{\infty}$) and $\int_M \rho_n \, d\operatorname{vol} = 1$ (because $\int_M \rho_n \, d\operatorname{vol} \to \int_M \rho \, d\operatorname{vol}$). Define $\mu_n$ by means of $d\mu_n = \rho_n \, d\operatorname{vol}$. We notice that

$$\mu_n \to \mu$$

holds due to the uniform convergence and, as $\rho_n \otimes \rho_n \to \rho \otimes \rho$ uniformly, we obtain

$$\int_{M \times M} G(x, y)\rho_n(x)\rho_n(y) \, d\operatorname{vol}(x) \, d\operatorname{vol}(y)$$

$$\to \int_{M \times M} G(x, y)\rho(x)\rho(y) \, d\operatorname{vol}(x) \, d\operatorname{vol}(y)$$

by the dominated convergence theorem outside the diagonal (because $G$ is $\operatorname{vol} \otimes \operatorname{vol}$ - integrable and the sequence $\rho_n$ is uniformly bounded). \qed

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To approximate arbitrary probability measures with continuous ones we refer to [12, Lemma 6.3.1]. It states the following.

**Lemma 2.2.** Let $\mu$ be any probability measure such that $W(\mu) < \infty$. Then there exists a sequence $\mu_n$ of continuous probability measures such that

$$
\mu_n \to \mu \quad \text{and} \quad W(\mu_n) \to W(\mu).
$$

□

To complete the proof of the convexity, let $\mu, \nu \in \mathcal{P}(M)$. Take two sequences $\{\mu_n\}_{n \in \mathbb{N}}$ and $\{\nu_n\}_{n \in \mathbb{N}}$ of differentiable measures such that $\mu_n \to \mu$, $W(\mu_n) \to W(\mu)$, and $\nu_n \to \nu$, $W(\nu_n) \to W(\nu)$. We want to take the lower limit to the sequence of inequalities

$$
\frac{1}{2} W(\mu_n) + \frac{1}{2} W(\nu_n) \geq \int_{M \times M} G(x,y) \, d\mu_n(x) d\nu_n(y).
$$

For this, notice that $(\mu, \nu) \mapsto \int_{M \times M} G(x,y) \, d\mu(x) d\nu(y)$ is lower semicontinuous. This can be seen as a consequence of the fact that it can be reexpressed as the increasing limit as $m$ goes to infinity of the continuous functions $(\mu, \nu) \mapsto \int_{M \times M} G_m(x,y) \, d\mu(x) d\nu(y)$ where $G_m(x,y) = G(x,y) \land m$. Then, we get

$$
\frac{1}{2} W(\mu) + \frac{1}{2} W(\nu) \geq \liminf_{n \to \infty} \int_{M \times M} G(x,y) \, d\mu_n(x) d\nu_n(y)
\geq \int_{M \times M} G(x,y) \, d\mu(x) d\nu(y),
$$

and the proof is complete. □

### 3 The minimum of $F$

Now we prove Theorem [14] Let $\rho \in C^\infty(M)$ be a differentiable positive solution of the equation (see [7])

$$
\Delta \log \rho = \beta \mu_{eq} - \beta \Lambda,
$$

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where $d\mu_{eq} = \rho \, d\text{vol}$. We will prove that the functional $F$ achieves its minimum at $\mu_{eq}$. For this we shall calculate the derivative of $F$ at $\mu_{eq}$ and prove that it is zero. We start with the following result.

**Lemma 3.1** (Derivative of the energy). Let $\mu_0$ and $\mu_1$ be two probability measures. Define $\mu_t = t\mu_1 + (1-t)\mu_0$, for $t \in [0,1]$. If $W(\mu_0) < \infty$ and $W(\mu_1) < \infty$ then $W(\mu_t)$ is differentiable at $t = 0$, and satisfies

$$
\frac{d}{dt} W(\mu_t) |_{t=0} = 2 \int_{M \times M} G(x,y) \, d\mu_0(x) \, (d\mu_1(y) - d\mu_0(y)).
$$

**Proof.** As $W(\mu_0)$ and $W(\mu_1)$ are finite, due to the convexity of $W$ we have that

$$
W(\mu_t) = t^2 \int_{M \times M} G(x,y) d\mu_1(x) \, d\mu_1(y) +
+ 2t(1-t) \int_{M \times M} G(x,y) d\mu_0(x) \, d\mu_1(y) +
+ (1-t)^2 \int_{M \times M} G(x,y) d\mu_0(x) \, d\mu_0(y)
$$

is finite. The linear term (in the variable $t$) is given by

$$
2 \int_{M \times M} G(x,y) d\mu_0(x) \, (d\mu_1(y) - d\mu_0(y)),
$$

which is the sought derivative. $\square$

**Lemma 3.2** (Derivative of the entropy). Let $\mu_0$ and $\mu_1$ be two probability measures. Define $\mu_t = t\mu_1 + (1-t)\mu_0$, for $t \in [0,1]$. If $D(\mu_0 \| \text{vol}) < \infty$, $D(\mu_1 \| \text{vol}) < \infty$ and $\int_M \left| \log \left( \frac{d\mu_0}{d\text{vol}} \right) \right| d\mu_1 < \infty$, then $D(\mu_t \| \text{vol})$ is differentiable at $t = 0$, and satisfies

$$
\frac{d}{dt} D(\mu_t \| \text{vol}) |_{t=0} = \int_M \log \left( \frac{d\mu_0}{d\text{vol}} \right) (d\mu_1(y) - d\mu_0(y)).
$$

**Proof.** We use the notation

$$
\rho_0 = \frac{d\mu_0}{d\text{vol}} \quad \text{and} \quad \rho_1 = \frac{d\mu_1}{d\text{vol}}.
$$
As $D(\mu_0||vol)$ and $D(\mu_1||vol)$ are finite, by the convexity of the entropy we get that $D(\mu_t||vol)$ is also finite. In particular, we have
\[
\int_M \log \left( t \rho_1(x) + (1-t) \rho_0(x) \right) d\mu_t < \infty
\]
and, as $\mu_t = t \mu_1 + (1-t) \mu_0$, if $0 < t < 1$, we get
\[
\int_M \log \left( t \rho_1(x) + (1-t) \rho_0(x) \right) d\mu_0 < \infty
\]
and
\[
\int_M \log \left( t \rho_1(x) + (1-t) \rho_0(x) \right) d\mu_1 < \infty.
\]
Keeping this in mind it makes sense to write
\[
D(\mu_t||vol) = \int_M \log \left( t \rho_1(x) + (1-t) \rho_0(x) \right) t \rho_1(x) dvol(x) + \\
+ \int_M \log \left( t \rho_1(x) + (1-t) \rho_0(x) \right) (1-t) \rho_0(x) dvol(x) \\
= t \int_M \log \left( t \rho_1(x) + (1-t) \rho_0(x) \right) \left( \rho_1(x) - \rho_0(x) \right) dvol(x) + \\
+ \int_M \log \left( t \rho_1(x) + (1-t) \rho_0(x) \right) \rho_0(x) dvol(x),
\]
which together with
\[
D(\mu_0||vol) = \int_M \log(\rho_0(x)) \rho_0(x) dvol(x)
\]
yields
\[
\frac{1}{t} (D(\mu_t||vol) - D(\mu_0||vol)) = \int_M \log \left( t \rho_1(x) + (1-t) \rho_0(x) \right) d\mu_1(x) + \\
- \int_M \log \left( t \rho_1(x) + (1-t) \rho_0(x) \right) d\mu_0(x) \\
+ \int_M \frac{1}{t} \left[ \log \left( t \rho_1(x) + (1-t) \rho_0(x) \right) \right] \rho_0(x) dvol(x) \\
- \int_M \frac{1}{t} \log \rho_0(x) d\rho_0(x).
\]
For the first two terms we notice that, as $t \to 0$, we get
\[ \log \left( t \rho_1(x) + (1-t) \rho_0(x) \right) \to \log \rho_0(x), \]
for every $x \in M$. We know that $|\log \left( t \rho_1(x) + (1-t) \rho_0(x) \right)|$ is bounded by $|\log \left( \frac{1}{2} \rho_1 + \frac{1}{2} \rho_0 \right)| + |\log \rho_0(x)|$, for $0 < t \leq \frac{1}{2}$, and we can use the dominated convergence theorem to reach
\[ \int_M \log \left( t \rho_1(x) + (1-t) \rho_0(x) \right) d\mu_1(x) \to \int_M \log \rho_0(x) d\mu_1(x) \]
and
\[ \int_M \log \left( t \rho_1(x) + (1-t) \rho_0(x) \right) d\mu_0(x) \to \int_M \log \rho_0(x) d\mu_0(x). \]
Finally, we notice the convergence
\[ \frac{1}{t} \left( t \rho_1(x) + (1-t) \rho_0(x) \right) \to \rho_0(x) \]
as $t \downarrow 0$ for every $x \in M$ and since each term is integrable, we can use the monotone convergence theorem to obtain
\[ \int_M \frac{1}{t} \left( t \rho_1(x) + (1-t) \rho_0(x) \right) d\mu(x) \to 0. \]

Now we are in position to prove Theorem 1.4.

Proof of Theorem 1.4. Let $\mu$ be any probability measure different from $\mu_{eq}$ and such that $F(\mu) < \infty$. Define $\mu_t = t\mu + (1-t)\mu_{eq}$, for $t \in [0,1]$. Multiply the equality
\[ \Delta \log \rho = \beta \rho \operatorname{vol} - \beta \Lambda, \]
by $G(x,y)$ and integrate in one variable to get
\[ -\log \rho(y) + \int_M \log \rho(x) d\Lambda(x) = \beta \int_M G(x,y) \rho(x) d\operatorname{vol}(x) \]
for every $y \in M$. Then, by Lemma 3.1 and 3.2, we have
\[
\frac{d}{dt} F(\mu_t)_{t=0} = \beta \int_{M \times M} G(x,y) d\mu_{eq}(x) (d\mu(y) - d\mu_{eq}(y)) + \\
+ \int_{M \times M} \log \rho(y) (d\mu(y) - d\mu_{eq}(y)) \\
= \int_M \left( \beta \int_M G(x,y) d\text{vol}(x) + \log \rho(y) \right) (d\mu(y) - d\mu_{eq}(y)) \\
= \int_M \left( \int_M \log \rho(x) d\Lambda(x) \right) (d\mu(y) - d\mu_{eq}(y)) \\
= \left( \int_M \log \rho(x) d\Lambda(x) \right) \int_M (d\mu(y) - d\mu_{eq}(y)) = 0.
\]
This implies, due to the strict convexity of $F(\mu_t)$, the inequality
\[
F(\mu_{eq}) > F(\mu).
\]

4 Laplace principle

Theorem 1.1 will be proved in this section. For this, we first understand some limiting properties of the energy. Write
\[
W_n(x_1, \ldots, x_n) = 2H_n(x_1, \ldots, x_n) = \frac{1}{n^2} \sum_{i \neq j} G(x_i, x_j).
\]

The easiest property we need to establish is the following.

**Proposition 4.1.** For $\mu \in \mathcal{P}(M)$, we have
\[
\int_{\mathcal{M}^n} W_n d\mu^{\otimes n} \rightarrow W(\mu).
\]

**Proof.** We integrate to get
\[
\int_{\mathcal{M}^n} W_n d\mu^{\otimes n} = \frac{n(n-1)}{n^2} \int_{M \times M} G(x,y) d\mu(x) d\mu(y).
\]
Then we take limits to complete the proof. □
For more general $\tau_n \in \mathcal{P}(M^n)$ (not necessarily of the form $\mu^\otimes n$) we can obtain a bound for below of the lim inf.

**Proposition 4.2.** For each $n$ choose $\tau_n \in \mathcal{P}(M^n)$. Suppose there exists a probability distribution on $\mathcal{P}(M)$, say $\zeta$ (that is $\zeta \in \mathcal{P}(\mathcal{P}(M))$), such that $i_n(\tau_n) \to \zeta$. Then we have

$$\int_{\mathcal{P}(M)} W d\zeta \leq \liminf_{n \to \infty} \int_{M^n} W d\tau_n.$$ 

**Proof.** As usual, for each $m \geq 0$ define $G_m(x,y) = G(x,y) \wedge m$. For each $n$ take a random element $(X^n_1, ..., X^n_n) \in M^n$ with law $\tau_n$ and $\mu \in \mathcal{P}(M)$ with law $\zeta$. Define $\mu_n = i_n(X^n_1, ..., X^n_n)$. We have then

$$\int_{M \times M} G_m(x,y) d\mu_n(x) d\mu_n(y) = \frac{1}{n^2} \sum_{i \neq j} G_m(X^n_i, X^n_j) + \frac{m}{n} \leq \frac{1}{n^2} \sum_{i \neq j} G(X^n_i, X^n_j) + \frac{m}{n},$$

from which, taking expected values, we obtain

$$\mathbb{E} \left[ \int_{M \times M} G_m(x,y) d\mu_n(x) d\mu_n(y) \right] \leq \mathbb{E} [W_n(X^n_1, ..., X^n_n)] + \frac{m}{n}. \quad (4.1)$$

We have thus

$$\mathbb{E} \left[ \int_{M \times M} G_m(x,y) d\mu_n(x) d\mu_n(y) \right] \to \mathbb{E} \left[ \int_{M \times M} G_m(x,y) d\mu(x) d\mu(y) \right]$$

by the continuity of $G_m$. So, by letting $n \to \infty$ in Inequality 4.1 we reach

$$\mathbb{E} \left[ \int_{M \times M} G_m(x,y) d\mu(x) d\mu(y) \right] \leq \liminf_{n \to \infty} \mathbb{E} [W_n(X^n_1, ..., X^n_n)].$$

By letting $m \to \infty$, we finally conclude

$$\mathbb{E} \left[ \int_{M \times M} G(x,y) d\mu(x) d\mu(y) \right] \leq \liminf_{n \to \infty} \mathbb{E} [W_n(X^n_1, ..., X^n_n)]$$

by the monotone convergence theorem. \hfill \Box

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Remark 4.3. In the previous proposition we may choose a sequence of increasing integers $n_k$ and for each $k$ a measure $	au_k \in \mathcal{P}(M^{n_k})$ such that $i_{n_k}(	au_k) \rightarrow \zeta$, and get the same result:

$$\int_{\mathcal{P}(M)} Wd\zeta \leq \liminf_{k \rightarrow \infty} \int_{M^{n_k}} W_{n_k} d\tau_k.$$ 

Now we can start proving Theorem 1.1.

Proof of Theorem 1.1. Take $f : \mathcal{P}(M) \rightarrow \mathbb{R}$ continuous. Because of the identity

$$\frac{1}{n} \log \int_{M^n} e^{-nf \circ i_n} d\gamma_n = \frac{1}{n} \log \int_{M^n} e^{-n(f \circ i_n + \frac{2}{n} W_n)} d\text{vol}^{\otimes n},$$

we only need to prove

$$\frac{1}{n} \log \int_{M^n} e^{-n(f \circ i_n + \frac{2}{n} W_n)} d\text{vol}^{\otimes n} \rightarrow - \inf_{\mu \in \mathcal{P}(M)} \{f(\mu) + F(\mu)\}.$$ 

For that we use the following result (see [9, Proposition 4.5.1]).

Lemma 4.4 (Variational formulation). Let $E$ be a Polish space, $\mu$ a probability measure on $E$ and $g : E \rightarrow \mathbb{R} \cup \{\infty\}$ a measurable function bounded from below. Under those hypothesis, the relation

$$\log \int_{E} e^{-g} d\mu = - \inf_{\tau \in \mathcal{P}(E)} \left\{ \int_{E} g d\tau + D(\tau\|\mu) \right\}.$$ 

holds.

In our case, we have

$$\frac{1}{n} \log \int_{M^n} e^{-n(f \circ i_n + \frac{2}{n} W_n)} d\text{vol}^{\otimes n} =$$

$$= - \inf_{\tau \in \mathcal{P}(M^n)} \left\{ \int_{M^n} f \circ i_n d\tau + \frac{\beta_n}{2} \int_{M^n} W_n d\tau + \frac{1}{n} D(\tau\|\text{vol}^{\otimes n}) \right\}.$$
Let us start with an upper limit inequality. More precisely, we prove the relation

\[
\limsup_{n \to \infty} \inf_{\tau \in \mathcal{P}(M^n)} \left\{ \int_{M^n} f \circ i_n \, d\tau + \frac{\beta_n}{2} \int_{M^n} W_n \, d\tau + \frac{1}{n} D(\tau \| \text{vol}^{\otimes n}) \right\} \\
\leq \inf_{\mu \in \mathcal{P}(M)} \{ f(\mu) + F(\mu) \}.
\]

(4.2)

For this, we need to see that for every probability measure \( \mu \in \mathcal{P}(M) \) we get

\[
\limsup_{n \to \infty} \inf_{\tau \in \mathcal{P}(M^n)} \left\{ \int_{M^n} f \circ i_n \, d\tau + \frac{\beta_n}{2} \int_{M^n} W_n \, d\tau + \frac{1}{n} D(\tau \| \text{vol}^{\otimes n}) \right\} \\
\leq f(\mu) + F(\mu).
\]

(4.3)

It will be enough to find, for every \( n \geq 2 \), a probability measure \( \tau_n \in \mathcal{P}(M^n) \) such that

\[
\limsup_{n \to \infty} \left\{ \int_{M^n} f \circ i_n \, d\tau_n + \frac{\beta_n}{2} \int_{M^n} W_n \, d\tau_n + \frac{1}{n} D(\tau_n \| \text{vol}^{\otimes n}) \right\} \\
\leq f(\mu) + F(\mu).
\]

We choose the simplest one: \( \tau_n = \mu^{\otimes n} \). If so, by the law of large numbers in the compact space \( M \), we have

\[ i_n(\tau_n) \to \delta_\mu. \]

Indeed, take a sequence \( \{X_k\}_{k \in \mathbb{N}} \) of independent and identically distributed random elements of \( M \) with law \( \mu \) and take any continuous function \( g : M \to \mathbb{R} \). Then, \( \{g(X_k)\}_{k \in \mathbb{N}} \) is a sequence of independent and identically distributed bounded random variables. By the strong law of large numbers we have

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} g(X_k) = \mathbb{E}[g(X_1)]. \]
almost surely. This can be written as

\[
\lim_{n \to \infty} \int_M g \, d[i_n(X_1, \ldots, X_n)] = \int_M g \, d\mu,
\]

and taking a countable dense family of functions we get

\[
\lim_{n \to \infty} i_n(X_1, \ldots, X_n) = \mu
\]

almost surely. By the dominated convergence theorem, the almost sure convergence implies the convergence of their laws, and so, as the law of \(i_n(X_1, \ldots, X_n)\) is \(i_n(\tau_n)\) and \(\mu\) is deterministic (of law \(\delta_\mu\)), we obtain

\[
i_n(\tau_n) \to \delta_\mu.
\]

Hence, we get

\[
\lim_{n \to \infty} \int_{M^n} f \circ i_n \, d\tau_n = f(\mu).
\]

The second term has already been studied in Proposition 4.1: we have

\[
\lim_{n \to \infty} \int_{M^n} W_n \, d\tau_n = W(\mu).
\]

Finally, we use

\[
D(\tau_n \| \text{vol} \otimes^n) = nD(\mu \| \text{vol})
\]

to get

\[
\lim_{n \to \infty} \left\{ \int_{M^n} f \circ i_n \, d\tau_n + \frac{\beta_n}{2} \int_{M^n} W_n \, d\tau_n + \frac{1}{n} D(\tau_n \| \text{vol} \otimes^n) \right\} = f(\mu) + \frac{\beta}{2} W(\mu) + D(\mu \| \text{vol}).
\]

The second and final step is to prove the lower bound

\[
\lim_{n \to \infty} \inf_{\tau \in P(M^n)} \left\{ \int_{M^n} f \circ i_n \, d\tau + \frac{\beta_n}{2} \int_{M^n} W_n \, d\tau + \frac{1}{n} D(\tau \| \text{vol} \otimes^n) \right\} \geq \inf_{\mu \in P(M)} \{ f(\mu) + F(\mu) \}.
\]

(4.4)
We proceed by contradiction. Suppose this is not true, i.e. we have
\[
\lim_{n \to \infty} \inf_{\tau \in \mathcal{P}(M^n)} \left\{ \int_{M^n} f \circ i_n d\tau + \frac{\beta_n}{2} \int_{M^n} W_n d\tau + \frac{1}{n} D(\tau \| \text{vol}^\otimes n) \right\} < \inf_{\mu \in \mathcal{P}(M)} \{ f(\mu) + F(\mu) \}.
\]
Then we can find \( C \in \mathbb{R} \) subject to
\[
\inf_{\tau \in \mathcal{P}(M^n)} \left\{ \int_{M^n} f \circ i_n d\tau + \frac{\beta_n}{2} \int_{M^n} W_n d\tau + \frac{1}{n} D(\tau \| \text{vol}^\otimes n) \right\} < C < \inf_{\mu \in \mathcal{P}(M)} \{ f(\mu) + F(\mu) \}
\]
for every \( n \) along a subsequence. For each of those \( n \) we pick \( \tau_n \in \mathcal{P}(M^n) \) such that
\[
\int_{M^n} f \circ i_n d\tau_n + \frac{\beta_n}{2} \int_{M^n} W_n d\tau_n + \frac{1}{n} D(\tau_n \| \text{vol}^\otimes n) < C
\]
The idea now is to take the limit (or just the limit of a subsequence) and derive a contradiction. To achieve that we use the following lemma.

**Lemma 4.5.** There exists a subsequence of \( \{\tau_n\} \), that we will still call \( \{\tau_n\} \) for ease of notation, and a probability distribution \( \zeta \) (i.e. \( \zeta \in \mathcal{P}(\mathcal{P}(M)) \)) on \( \mathcal{P}(M) \), such that \( i_n(\tau_n) \to \zeta \) and
\[
\int_{\mathcal{P}(M)} D(\cdot \| \text{vol}) d\zeta \leq \liminf_{n \to \infty} \frac{1}{n} D(\tau_n \| \text{vol}^\otimes n).
\]

**Proof.** Given a probability measure \( \tau_n \in \mathcal{P}(M^n) \) we can construct a \( n \)-tuple of random probabilities in \( M \) by means of marginals. More precisely, there exists a random variable \( (T^1_n, T^2_n, ..., T^n_n) \) on \( \mathcal{P}(M)^n \) and a random variable \( (X_1, ..., X_n) \in M^n \) with law \( \tau_n \), such that
\[
\int_M g dT^i_n = \mathbb{E}[g(X_i)|X_1, ..., X_{i-1}],
\]
for every continuous function \( g : M \to \mathbb{R} \).
We can prove (see Proposition 7.2 in the Appendix for an idea of the proof, or see [9, Theorem C.3.1] for a complete proof) that

\[ D(\tau_n\|\text{vol}^{\otimes n}) = \mathbb{E} \left[ \sum_{i=1}^{n} D(T_n^i\|\text{vol}) \right] \]

holds. So, by the convexity of \( D(\cdot\|\text{vol}) \) we get

\[ \mathbb{E} \left[ D \left( \frac{1}{n} \sum_{i=1}^{n} T_n^i\|\text{vol} \right) \right] \leq \frac{1}{n} \mathbb{E} \left[ \sum_{i=1}^{n} D(T_n^i\|\text{vol}) \right] = \frac{1}{n} D(\tau_n\|\text{vol}^{\otimes n}). \]

The compactness of \( \mathcal{P}(\mathcal{P}(M) \times \mathcal{P}(M)) \) allows us to extract a subsequence of \( \left( \frac{1}{n} \sum_{i=1}^{n} T_n^i, \frac{1}{n} \sum_{i=1}^{n} \delta_X^i \right) \in \mathcal{P}(M) \times \mathcal{P}(M) \) such that \( \left( \frac{1}{n} \sum_{i=1}^{n} T_n^i, \frac{1}{n} \sum_{i=1}^{n} \delta_X^i \right) \) converges in law to, say, \((\chi, \tilde{\chi})\). Then, we get \( \chi = \tilde{\chi} \) almost surely (see Proposition 7.4 in the Appendix or [8, Lemma 3.5]). Denote by \( \zeta \) the common law of \( \chi \) and \( \tilde{\chi} \). The fact that \( D(\cdot\|\text{vol}) \) is lower semicontinuous and bounded from below implies that it can be written as an increasing pointwise limit of bounded continuous functions, and then the function \( \alpha \mapsto \int_{\mathcal{P}(M)} D(\cdot\|\text{vol})d\alpha \) is also lower semicontinuous. In particular, we get

\[ \int_{\mathcal{P}(M)} D(\cdot\|\text{vol}) d\alpha \leq \liminf \mathbb{E} \left[ D \left( \frac{1}{n} \sum_{i=1}^{n} T_n^i\|\text{vol} \right) \right]. \]

\( \square \)

We can now complete the proof by noticing that Lemma 4.5 and Proposition 4.2 imply

\[ \int_{\mathcal{P}(M)} \left( f + \frac{\beta}{2} W + D(\cdot\|\text{vol}) \right) d\zeta \leq C < \inf_{\mu \in \mathcal{P}(M)} \{ f(\mu) + F(\mu) \}, \]

or, equivalently,

\[ \int_{\mathcal{P}(M)} (f + F) d\zeta \leq C < \inf_{\mu \in \mathcal{P}(M)} \{ f(\mu) + F(\mu) \}, \]

which is impossible. \( \square \)
5 Convergence of $i_n(\mathbb{P}_n)$

We prove the corollaries in this section: Corollary 1.2 about the large deviation principle, and Corollary 1.6 about the convergence of the empirical measures.

Proof of Corollary 1.2. By [9, Theorem 1.2.3] and the fact that $I$ is lower semicontinuous the following Laplace principle implies the large deviation principle: for every continuous function $f : \mathcal{P}(M) \to \mathbb{R}$ we have

$$\frac{1}{n} \log \int_{M^n} e^{-nf_{\mu_n}} d\mathbb{P}_n \nrightarrow_{n \to \infty} \inf_{\mu \in \mathcal{P}(M)} \{f(\mu) + I(\mu)\}.$$  

Using the measures $\gamma_n$ and the definition of $I$ it is enough to prove that for every continuous function $f : \mathcal{P}(M) \to \mathbb{R}$ we have

$$\frac{1}{n} \log \int_{M^n} e^{-nf_{\mu_n}} d\gamma_n \nrightarrow_{n \to \infty} \inf_{\mu \in \mathcal{P}(M)} \{f(\mu) + F(\mu) - \inf F\}.$$ 

However, by Theorem 1.1 applied to the function $f = 0$, we get

$$\frac{1}{n} \log Z_n \nrightarrow_{n \to \infty} -\inf F,$$

and combining this with the same theorem for general $f$, we get

$$\frac{1}{n} \log \int_{M^n} e^{-nf_{\mu_n}} d\gamma_n \nrightarrow_{n \to \infty} \inf_{\mu \in \mathcal{P}(M)} \{f(\mu) + F(\mu)\},$$

and the proof is finished.

Proof of Corollary 1.6. Take random probabilities $\{X_n\}_{n \geq 2}$ coupled in any way but such that $X_n \sim i_n(\mathbb{P}_n)$. For any closed set $C$ that does not contain $\mu_{eq}$, we have $\inf_{x \in C} I(x) > 0$ due to the semicontinuity of $I$. The property

$$\limsup_{n \to \infty} \frac{1}{n} \log P_n(i_n^{-1}(C)) \leq -\inf_{x \in C} I(x)$$
implies that there exists $A > 0$ and $N \in \mathbb{N}$ such that
\[
\frac{1}{n} \log \mathbb{P}_n(i_n^{-1}(C)) \leq -A
\]
for every $n > N$. Hence we have
\[
\mathbb{P}_n(i_n^{-1}(C)) \leq e^{-nA}
\]
for every $n > N$, which yields
\[
\sum_{n=1}^{\infty} \mathbb{P}_n(i_n^{-1}(C)) < \infty.
\]
By the Borel-Cantelli lemma we get then
\[
\mathbb{P}(\text{there exists } M \in \mathbb{N} \text{ such that } i > M \text{ implies } X_i \notin C) = 1.
\]
Take a countable local base $\{O_i\}_{i \in \mathbb{N}}$ around $\mu_{eq}$ and apply the previous argument for every $C = O_i^c$ to obtain almost sure convergence.

\section{Final comments}

This work has been inspired on the article by Robert Berman \cite{berman} where a slightly different model is treated. Our proof of the large deviation principle is an adaptation of the article by Paul Dupuis, Vaios Laschos, and Kavita Ramanan \cite{dupuis} to the case of compact manifolds.

Here we have studied just one kind of limiting behavior for a sequence of point processes on a surface. There are two main issues that, to our knowledge, are still open: the fluctuations and the local behaviour.

By fluctuations we mean the following. Take $f \in C^\infty(M)$ and $\mu_n$ a sequence with law $i_n(\mathbb{P}_n)$. We have proved, in Corollary \ref{cor:1.6}, the convergence
\[
\int f d\mu_n \to \int f d\mu_{eq},
\]
A large deviation principle

what we could rewrite as

$$\int f \, d\mu_n = \int f \, d\mu_{eq} + o(1).$$

The idea is to find the next order terms (to prove a central limit type theorem). More precisely, to find a sequence $\alpha_n \to \infty$ such that

$$\alpha_n \left( \int f \, d\mu_n - \int f \, d\mu_{eq} \right)$$

converges weakly, and describe such limit.

When we talk about **local behavior** we take $x \in M$ and a chart

$$\phi : U \to T_x M$$

such that $\phi(x) = 0$ and $d\phi_x = id|_{T_x M}$. We fix $n$ points $(X_1, ..., X_n)$ distributed according to $\mathbb{P}_n$. We get a point process in $T_x M$ with points $\phi(X_1), ..., \phi(X_n)$ (when $X_i \in U$). We then scale this point process by $\sqrt{n}$ and find the limit (in some sense) point process. We ask how this point process depends on $x \in M$.

These questions are already answered in the case of some determinantal point processes (see [1] and [3]) and in the one dimensional case (see [10]). Very recent results about fluctuations on $\mathbb{R}^2$ can be found in [2] and [11].

7 Appendix

Here we deal with several tools used along this paper.

**Proposition 7.1.** Let $E$ be a compact metrizable space. Then $\mathcal{P}(E)$, the space of probability measures on $E$, is a compact metrizable space.

**Proof.** By the Stone-Weierstrass theorem we know that the space of continuous functions on $E$ is separable in the topology of uniform convergence. Choose a dense countable set $\{f_m\}_{m \in \mathbb{N}}$. 

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Let \( d \) be a metric in \( E \) that induces its topology. Define \( \bar{d} : \mathcal{P}(E) \times \mathcal{P}(E) \to \mathbb{R} \) by
\[
\bar{d}(\mu, \nu) = \sum_{m \in \mathbb{N}} \frac{1}{2^m} \wedge \left| \int_E f_m d\mu - \int_E f_m d\nu \right|.
\]
We can see that the topology induced by \( \bar{d} \) is the smallest topology such that \( \mu \mapsto \int_E f_m d\mu \) is continuous for every \( m \in \mathbb{N} \). But by density and uniform convergence the functional \( \mu \mapsto \int_E f_m d\mu \) is continuous for every \( m \in \mathbb{N} \) if and only if \( \mu \mapsto \int_E f d\mu \) is continuous for any continuous function \( f : E \to \mathbb{R} \). So, the topology induced by \( \bar{d} \) is the weak topology of \( \mathcal{P}(E) \).

To see that \( \mathcal{P}(E) \) is compact it is enough to show that it is sequentially compact. Take a sequence \( \{\mu_n\}_{n \in \mathbb{N}} \) of probability measures on \( E \). By a diagonal procedure we can choose a subsequence \( \{\mu_{n_i}\}_{i \in \mathbb{N}} \) such that \( \int_E f_m d\mu_{n_i} \) converges as \( i \) goes to infinity for every \( m \in \mathbb{N} \). This implies that \( \int_E f d\mu_{n_i} \) converges as \( i \) goes to infinity for every continuous function \( f : E \to \mathbb{R} \). Indeed, we can prove that \( \{\int_E f d\mu_{n_i}\}_{i \in \mathbb{N}} \) is Cauchy. For this, take \( \epsilon > 0 \) and choose \( m \in \mathbb{N} \) such that \( \|f_m - f\| < \epsilon/3 \). Take a number \( M \) such that if \( i, j > M \) then \( \left| \int_E f_m d\mu_{n_i} - \int_E f_m d\mu_{n_j} \right| < \epsilon/3 \). Then, whenever \( i, j > M \), we have
\[
\left| \int_E f d\mu_{n_i} - \int_E f d\mu_{n_j} \right| \leq \left| \int_E f d\mu_{n_i} - \int_E f_m d\mu_{n_i} \right| + \left| \int_E f_m d\mu_{n_i} - \int_E f_m d\mu_{n_j} \right| + \left| \int_E f_m d\mu_{n_j} - \int_E f d\mu_{n_j} \right| < \epsilon.
\]
Define \( \Lambda : C(E) \to \mathbb{R} \) as \( \Lambda(f) = \lim_{i \to \infty} \int_E f d\mu_{n_i} \). Then \( \Lambda \) is a positive linear functional and so, there exists a positive measure \( \mu \) on \( E \) such that \( \Lambda(f) = \int_E f d\mu \) for every \( f \in C(E) \). As \( \Lambda(1) = \lim_{i \to \infty} \int_E 1 d\mu_{n_i} = 1 \), we obtain \( \mu \in \mathcal{P}(E) \). In this way, we have extracted a subsequence of \( \{\mu_n\}_{n \in \mathbb{N}} \) that converges. \( \Box \)

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In what follows, instead of writing $d\mu(x)$ we write $\mu(dx)$.

As in the proof of Lemma 4.5, given a probability measure $\mu \in \mathcal{P}(M^n)$ we can construct an $n$-tuple of random probabilities $(\mu_1, \mu_2, \ldots, \mu_n)$ in $\mathcal{P}(M)^n$ and a random element $(X_1, \ldots, X_n) \in M^n$ with law $\mu$ such that

$$\int_M f \, d\mu_i = \mathbb{E}[f(X_i) | X_1, \ldots, X_{i-1}]$$

holds.

**Proposition 7.2 (Chain rule).** We have

$$D(\mu \| \text{vol}^\otimes n) = \mathbb{E} \left[ \sum_{i=1}^n D(\mu_i \| \text{vol}) \right].$$

**Sketch of the proof.** We will give an idea of the proof ignoring issues of measurability and finiteness of the entropy. For extra details we refer to [9, Theorem C.3.1].

We consider $M^n$ with a probability measure $\mu$. In this case the random element with law $\mu$ is $(X_1, \ldots, X_n)$ where $X_i : M^n \to M$ is the projection onto the $i$-th coordinate. Suppose that

$$\tilde{\mu}_k : M^{k-1} \to \mathcal{P}(M)$$

is a transition kernel from $(X_1, \ldots, X_{k-1})$ to $X_k$.

If we define

$$\mu_k = \tilde{\mu}_k \circ \pi_{k-1},$$

where $\pi_{k-1} : M^n \to M^{k-1}$ is the projection onto the first $k - 1$ coordinates, we see that $(\mu_1, \ldots, \mu_n)$ satisfies the properties of the definition. If we assume all entropies are finite, we get
\[
\mathbb{E} \left[ \sum_{k=1}^{n} D(\mu_k \| \text{vol}) \right] = \sum_{k=1}^{n} \mathbb{E} \left[ D(\mu_k \| \text{vol}) \right] \\
= \sum_{k=1}^{n} \int_{M^{k-1}} D(\tilde{\mu}_k(x) \| \text{vol}) \left[ \pi_{k-1}(\mu) \right] (dx) \\
= \sum_{k=1}^{n} \int_{M^{k-1}} \left( \int_{M} \log \left( \frac{\tilde{\mu}_k(x, dy)}{\text{vol}(dy)} \right) \mu_k(x, dy) \right) \left[ \pi_{k-1}(\mu) \right] (dx) \\
= \sum_{k=1}^{n} \int_{M^{k-1} \times M} \log \left( \frac{\tilde{\mu}_k(x, dy)}{\text{vol}(dy)} \right) \left[ \pi_k(\mu) \right] (dx, dy) \\
= \sum_{k=1}^{n} \int_{M^n} \log (\rho_k(x)) \mu(dx),
\]

where \( \rho_k : M^n \to [0, \infty) \) is equal to \( \rho_k = \frac{\tilde{\mu}_k(x, dy)}{\text{vol}(dy)} \circ \pi_k \).

Then we just have to notice the equality
\[
\prod_{i=1}^{n} \rho_i(x) = \frac{\mu(dx)}{\text{vol}^\otimes_n (dx)},
\]
that follows from the definition.

\[\square\]

**Lemma 7.3.** Let \((X_1, \ldots, X_n) \in M^n\) and \((\mu_1, \ldots, \mu_n) \in \mathcal{P}(M)^n\) be random elements as before. Consider the random measures
\[
\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} \mu_i, \quad \hat{\nu} = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}.
\]

Then we have
\[
\mathbb{P} \left( \left| \int_{M} f(x) \hat{\mu}(dx) - \int_{M} f(y) \hat{\nu}(dy) \right| > \epsilon \right) \leq 4 \frac{\|f\|_{\infty}^2}{n \epsilon^2}.
\]
Proof. By Chebyshev’s inequality, we need to understand the quantity
\[ \text{Var} \left( \int_M f(x)\hat{\mu}(dx) - \int_M f(y)\hat{\nu}(dy) \right). \]
The first term is
\[ \int_M f(y)\hat{\mu}(dy) = \frac{1}{n} \sum_{k=1}^{n} \int_M f(y)\mu_k(dy) = \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}[f(X_k)|X_1,\ldots,X_{k-1}] \]
while the second is
\[ \int_M f(x)\hat{\nu}(dx) = \frac{1}{n} \sum_{i=1}^{n} f(X_i). \]
We can see that both have the same expected value, and if \( i < j \), we have
\[ \mathbb{E}\left[ \left( f(X_i) - \mathbb{E}[f(X_i)|X_1,\ldots,X_{i-1}] \right) \left( f(X_j) - \mathbb{E}[f(X_j)|X_1,\ldots,X_{j-1}] \right) \right] = \]
\[ = \mathbb{E}\left[ \left( f(X_i) - \mathbb{E}[f(X_i)|X_1,\ldots,X_{i-1}] \right) f(X_j) \right] \]
because \( f(X_i) - \mathbb{E}[f(X_i)|X_1,\ldots,X_{i-1}] \) is \( (X_1,\ldots,X_{j-1}) \) measurable. Then we get
\[ \mathbb{E}\left[ \left( f(X_i) - \mathbb{E}[f(X_i)|X_1,\ldots,X_{i-1}] \right) \left( f(X_j) - \mathbb{E}[f(X_j)|X_1,\ldots,X_{j-1}] \right) \right] = 0. \]
So we have
\[ \text{Var} \left( \int_M f(x)\hat{\mu}(dx) - \int_M f(y)\hat{\nu}(dy) \right) = \]
\[ = \frac{1}{n^2} \sum_{i=1}^{n} \mathbb{E}\left[ \left( f(X_i) - \mathbb{E}[f(X_i)|X_1,\ldots,X_{i-1}] \right)^2 \right] \]
\[ \leq \frac{1}{n^2} \sum_{i=1}^{n} 4\|f\|_{\infty}^2 = \frac{1}{n} 4\|f\|_{\infty}^2, \]
and by Chebyshev’s inequality we conclude our claim. \( \square \)
Proposition 7.4. Using the notation of the proof in Lemma 4.5, if we have
\[(\hat{\mu}_n, \hat{\nu}_n) = \left(\frac{1}{n} \sum_{i=1}^n \tau_i, \frac{1}{n} \sum_{i=1}^n \delta_{X_i}\right) \to (\chi, \tilde{\chi})\]
in law, then we have \(\chi = \tilde{\chi}\) almost surely.

Proof. For any continuous \(f : M \to \mathbb{R}\), the function
\[T_f : \mathcal{P}(M) \times \mathcal{P}(M) \to \mathbb{R}\]
\[(\mu, \nu) \mapsto \int_M f(x) \mu(dx) - \int_M f(y) \nu(dy)\]
is continuous. By Lemma 7.3, for every continuous \(f\), we get
\[\mathbb{P}( |T_f(\hat{\mu}_n, \hat{\nu}_n)| > \epsilon) \leq 4 \frac{\|f\|_\infty^2}{n\epsilon^2},\]
and, by the Portmanteau theorem (taking the lower limit on both sides), we reach
\[\mathbb{P}( |T_f(\chi, \tilde{\chi})| > \epsilon) = 0\]
for every \(\epsilon > 0\). Thus we have
\[\mathbb{P}( |T_f(\chi, \tilde{\chi})| = 0) = 1.\]

Next, choose a dense sequence \(\{f_m\}_{m \in \mathbb{N}}\) in the space of continuous functions on \(M\) endowed with the topology of uniform convergence in order to obtain
\[\mathbb{P}( |T_{f_m}(\chi, \tilde{\chi})| = 0 \text{ for all } m) = 1.\]

But, by density, we have
\[\{|T_{f_m}(\chi, \tilde{\chi})| = 0 \text{ for all } m\} = \{|T_f(\chi, \tilde{\chi})| = 0 \text{ for all continuous } f\},\]
which means \(\chi = \tilde{\chi}\) almost surely. \(\square\)
References


**Resumen**

Siguiendo las técnicas desarrolladas por Paul Dupuis, Vaios Laschos y Kavita Ramanan en [8], se establecerá un principio de grandes desviaciones para una secuencia de procesos puntuales definidos por medidas de Gibbs en una variedad riemanniana bidimensional compacta y orientable. Veremos que la correspondiente secuencia de medidas empíricas converge a la solución de una ecuación diferencial parcial y, en ciertos casos, a la forma de volumen de una métrica de curvatura constante.

**Palabras clave:** Medidas de Gibbs; gas de Coulomb; medida empírica; principio de grandes desvíos; sistemas de partículas interactuantes; potencial singular; variedad de Einstein 2-dimensional; entropía relativa.

David García Zelada  
Université Paris-Dauphine  
PSL Research University, CNRS, CEREMADE  
75016 PARIS, FRANCE.  
garciazelada@ceremade.dauphine.fr