CONVERGENCE OF SINGULAR INTEGRALS IN WEIGHTED $L^1$ SPACES

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It is shown that for a singular integral operator $K$ and a given function $f$ in $L^1_w$, $w$ belonging to the $A_1$ class of Muckenhoupt, if the image $Kf$ is also in $L^1_w$, then the truncated operator $K_\varepsilon$ applied to $f$ converges in $L^1_w$ to $Kf$. This is a generalization to the weighted case of a result due to A.P. Calderón and O.N. Capri [2]. As an application of the method developed, a new proof of a result on $H^1_w$ of R.L. Wheeden [9] is given.

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1. Introduction

Let us denote by $\mathbb{R}^n$ the $n$-dimensional euclidean space. By $x \cdot y = x_1y_1 + x_2y_2 + \cdots + x_ny_n$ we denote the scalar product of $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

The norm of $x \in \mathbb{R}^n$ is given by $|x| = (x \cdot x)^{1/2}$ and $B(x, r)$ designates the ball $\{y : |y - x| < r\}$, where $x \in \mathbb{R}^n$ and $r > 0$.

All the functions considered here are Lebesgue measurable and $|E|$ stands for the Lebesgue measure of a measurable set $E$. Let $f$ be a locally integrable function. The Hardy-Littlewood maximal function of $f$ is defined as

$$Mf(x) = \sup_{r > 0} |B(x, r)|^{-1} \int_{B(x, r)} |f(y)| \, dy.$$ 

Let $w(x)$ be a non-negative and locally integrable function. The weight $w(x)$ is said to belong to the $A_p$ class of Muckenhoupt, $1 < p < \infty$, if there exists a constant $c$ such that the inequality

$$\left( |B|^{-1} \int_B w \, dx \right) \left( |B|^{-1} \int_B w^{-1/(p-1)} \, dv \right)^{p-1} \leq c$$

holds for every ball $B \subset \mathbb{R}^n$. We say that $w$ belongs to $A_1$ if there exists a constant $c$ such that $Mw(x) \leq cw(x)$ holds for almost every point $x$ in $\mathbb{R}^n$. Results concerning $A_p$ weights can be founded in B. Muckenhoupt [6], R. Coiman and C. Fefferman [3] and R. Hunt, B. Muckenhoupt and R. Wheeden [4]. We define the measure $w(E) = \int_E w(x) \, dx$, where $E$ is any Lebesgue measurable subset of $\mathbb{R}^n$. $\Omega_n$ denotes the volume of the unit ball of $\mathbb{R}^n$ and $\omega_{n-1}$ the surface area of the unit sphere. The letter $c$ designates a constant not necessarily the same at different occurrences.

The type of singular integral kernels $k(x)$ that shall be considered in this paper will satisfy the following conditions:

(i) $|k(x)| \leq c \, |x|^{-n}$, if $x \neq 0$;
(ii) there exists an increasing function $\theta(t)$ such that $\theta(2t) \leq c\theta(t)$,

$$ | k(x - y) - k(x) | \leq c\theta(|y| / |x|) / |x|^n $$

if $|x| \geq 2|y|$, and $\int_0^1 \theta(t)d(t)/t < \infty$;

(iii) for every $0 < a < b$, $\int_{a < x < b} k(x)dx \leq c$;

(iv) the limit $\lim_{n \to 0} \int_{n < x < 1} k(x)dx$ exists.

A kernel $k(x)$ satisfying these properties allows us to define a temperate distribution, which is called the principal value of $k$, by means of

$$ < p.v.k, \varphi > = \lim_{\varepsilon \to 0} \int_{x > \varepsilon} k(x)\varphi(x)dx = $$

$$ \int_{|x| < 1} k(x)[\varphi(x) - \varphi(0)]dx + $$

$$ \varphi(0) \cdot \lim_{\varepsilon \to 0} \int_{\varepsilon < |x| < 1} k(x)dx + \int_{|x| \geq 1} k(x)\varphi(x)dx $$

for every $\varphi \in S$. In order to calculate the Fourier transform of the distribution $p.v.k$, we observe that the conditions assumed on the kernel imply the hypothesis of Theorem 1. From the proof of that theorem it follows that the Fourier transforms of the truncated kernels, $k_\varepsilon(x) = k(x)$ if $|x| \geq \varepsilon$ and $k_\varepsilon(x) = 0$ if $|x| < \varepsilon$, are uniformly bounded, i.e. for every $\varepsilon > 0$.

(1) $$ |\hat{k}_\varepsilon(x)| \leq M. $$

Moreover, we have
\[ \hat{k}_\varepsilon(x) = \int_{|y| < 1} [e^{-2\pi i x \cdot y}] k(y) dy + \hat{k}_1(x) \]

\[ = \int_{|y| < 1} [e^{-2\pi i x \cdot y} - 1] k(y) dy + \int_{|y| < 1} k(y) dy + \hat{k}_1(x). \]

Letting \( \varepsilon \) go to zero and taking into account (i) and (iv) we obtain

\[ \lim_{\varepsilon \to 0} \hat{k}_\varepsilon(x) = \int_{|y| < 1} [e^{-2\pi i x \cdot y} - 1] k(y) dy \]

\[ + \lim_{\varepsilon \to 0} \int_{|y| < 1} k(y) dy + \hat{k}_1(x). \]

This proves the existence of the limit. Let us denote by \( \hat{k}(x) \) the function \( \lim_{\varepsilon \to 0} \hat{k}_\varepsilon(x) \). We shall show that

\( \langle (p.v.k)^\wedge, \varphi \rangle = \langle p.v.k, \hat{\varphi} \rangle = \int \hat{k}(x) \varphi(x) dx. \) (2)

Indeed, from the relation \( \langle \hat{k}_\varepsilon, \varphi \rangle = \langle k_\varepsilon, \hat{\varphi} \rangle \) and (1), by the Lebesgue dominated convergence theorem, it follows (2) for every \( \varphi \in S \). In addition, by (1), we have

\[ | \hat{k}(x) | \leq M. \) (3)

Let \( k(x) \) be a kernel satisfying conditions (i) to (iv). For \( \varepsilon > 0 \), we define the truncated singular integral operator \( K_\varepsilon \) as

\[ K_\varepsilon f(x) = \int k_\varepsilon(x - y) f(y) dy, \]

where \( f \) belongs to \( L^p_w \), \( 1 \leq p < \infty \), and \( w \in A_p \). The maximal singular integral operator \( K^* \) associated to \( k(x) \) is defined as

\[ K^* f(x) = \sup_{\varepsilon > 0} | K_\varepsilon f(x) |. \]
Next, we gather in theorems A and B some known results that will be needed in the sequel.

**Theorem A.** Let $K^*$ be the maximal singular integral operator associated to a kernel $k(x)$ which satisfies conditions (i), (ii), (iii) and (iv).

(i) If $f \in L^p_w$, $1 < p < \infty$, and $w \in A_p$, then $\|K^*f\|_{L^p_w} \leq c\|f\|_{L^p_w}$, where the constant $c$ does not depend on $f$.

(ii) If $f \in L^1_w$ and $w \in A_1$, then there exists a constant $c$ such that $w(\{x : K^*f(x) > \lambda\}) \leq c\lambda^{-1}\|f\|_{L^1_w}$ holds for every $\lambda > 0$.

Theorem A is a consequence of Theorem 5 of [5] since hypothesis (a) of that theorem holds by (3) and the other hypothesis (b) and (c) of the same theorem coincide with our assumptions (i) and (ii) on the kernel.

**Theorem B.** Let $k(x)$ be a kernel satisfying the same assumptions as in Theorem A.

(i) If $f \in L^p_w$, $1 \leq p < \infty$, $w \in A_p$, then $Kf(x) = \lim_{\varepsilon \to 0} K_\varepsilon f(x)$ exists almost everywhere.

(ii) If $f \in L^p_w$, $1 < p < \infty$ and $w \in A_p$, then $\|Kf\|_{L^p_w} \leq c\|f\|_{L^p_w}$, where $c$ does not depend on $f$.

(iii) If $f \in L^1_w$ and $w \in A_1$, then, there exists a constant $c$ such that $w(\{x : |Kf(x)| > \lambda\}) \leq c\lambda^{-1}\|f\|_{L^1_w}$ holds for every $\lambda > 0$.

(iv) If $f \in L^p_w$, $1 < p < \infty$ and $w \in A_p$, then $\lim_{\varepsilon \to 0}\|K_\varepsilon f - Kf\|_{L^p_w} = 0$.

It is easy to verify that (i) holds for $f \in C_0^\infty$. Arguing as in [7], p. 45, and taking into account Theorem A we obtain (i). Parts
(ii) and (iii) follow from part (i) and Theorem A. As for (iv), it follows from (i) and part (i) of Theorem A by applying the Lebesgue dominated convergence theorem.

2. Convergence in $L^1_w$.

Let $\varphi(x)$ be a function defined on $\mathbb{R}^n$. The least decreasing radial majorant function $\psi(x)$ of $\varphi(x)$ is given by

$$\psi(x) = \sup_{|y| \geq |x|} |\varphi(y)|.$$  

With an abuse of notation, we write $\varphi(r) = \psi(x)$ if $|x| = r$. For $\varepsilon > 0$, we set $\varphi_\varepsilon(x) = \varepsilon^{-n} \varphi(x/\varepsilon)$.

Lemma 1. Let $\varphi(x)$ be a function whose least decreasing radial majorant $\psi(x)$ is integrable. Assume that $f(x)$ belongs to $L^p_w$, $1 \leq p < \infty$, $w \in A_p$. If for $\varepsilon > 0$ we define $f_\varepsilon(x) = \int f(x - y) \varphi_\varepsilon(y) dy$ then,

(i) for every $\varepsilon > 0$, $|f_\varepsilon(x)| \leq M f(x) \int \psi(y) dy$,

(ii) for every $\varepsilon > 0$, $\|f_\varepsilon\|_{L^p_w} \leq c \int \psi(y) dy \|f\|_{L^p_w}$,

(iii) for every Lebesgue point $x$ of $f$ and therefore, for almost every point $x \in \mathbb{R}^n$, $\lim_{\varepsilon \to 0} f_\varepsilon(x) = f(x) \int \psi(y) dy$ and

(iv) $\lim_{\varepsilon \to 0} \int |f_\varepsilon(x) - f(x) \int \varphi(y) dy|^p w(x) dx = 0$

To prove Lemma 1 we shall need some results that we state in Lemmas 2 and 3.

Lemma 2. Let $h(x) \geq 0$ be a locally integrable function and let $\psi(x) \geq 0$ be a decreasing radial and integrable function. Then, the following properties hold

(i) $\int h(y) \psi(y) dy \leq M h(0) \int \psi(y) dy$, 

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(ii) \( \int_{|y| \leq \rho} h(y)\psi(y)dy \leq \sup_{0<r \leq \rho} (\Omega_n r^n)^{-1} \int_{|y| \leq r} h(y)dy \int_{|y| \leq \rho} \psi(y)dy, \)

(iii) \( \int_{|y| > \rho} h(y)\psi(y)dy \leq 2^n (2^n - 1)^{-1} \sup_{r > \rho} (\Omega_n r^n)^{-1} \int_{r < |y| \leq \rho} h(y)dy \int_{|y| > \rho} \psi(y)dy. \)

Proof. Part (i), which is the core of the lemma is well known (see [7], Theorem 2, p.62). Part (ii) follows from (i) considering \( h(y) \) and \( \psi(y) \) as equal to zero for \( |y| > \rho \). To get part (iii) we observe that since \( \psi \) is a radial and decreasing function, we have

\[
[1 - 2^{-n}]\Omega_n \rho^n \psi(\rho) \leq \int_{\frac{\rho}{2} < |y| < \rho} \psi(y)dy.
\]

If we apply part (i) to \( h_\rho(y) = h(y) \) if \( |y| > \rho \), \( h_\rho(y) = 0 \) if \( |y| \leq \rho \) and \( \psi_\rho(y) = \psi(y) \) if \( |y| > \rho \) and \( \psi_\rho(y) = \psi(\rho) \) if \( |y| \leq \rho \), then

\[
\int_{|y| > \rho} h(y)\psi(y)dy = \int_{|y| > \rho} h_\rho(y)\psi_\rho(y)dy
\]

\[
\leq M h_\rho(0)[\Omega_n \rho^n \psi(\rho) + \int_{|y| > \rho} \psi(y)dy]
\]

\[
\leq 2^n (2^n - 1)^{-1} M h_\rho(0) \int_{|y| > \frac{\rho}{2}} \psi(y)dy,
\]

which is (iii).

Lemma 3. Let \( w \in A_1 \) and \( f \in L^1_w \). If, for \( \rho > 0 \), we set \( \Delta_f(\rho) = (\Omega_n \rho^n)^{-1} \int_{|y| \leq \rho} \int |f(x - y) - f(x)| w(x)dx dy \), then,

(i) \( \Delta_f(\rho) \leq c\|f\|_{L^2_w}, \)

(ii) \( \lim_{\rho \to 0} \Delta_f(\rho) = 0 \).

Proof. We have,
\[ \Delta_f(p) \leq (\Omega_\rho^n)^{-1} \int_{|y| \leq \rho} \int |f(x)| w(x+y) dx dy \]

\[ + \int |f(x)| w(x) dx \]

\[ = \int |f(x)| \left[ (\Omega_\rho^n)^{-1} \int_{|y| \leq \rho} w(x+y) dy \right] dx + \|f\|_{L^1_w} \]

\[ \leq c \int |f(x)| w(x) dx + \|f\|_{L^1_w} \]

\[ = c\|f\|_{L^1_w} \]

This proves (i). In order to prove part (ii), let \( g \) be a continuous function with compact support such that \( \|f - g\|_{L^1_w} < \epsilon \). Then, there exists \( \rho_0 \) such that

\[ \int |g(x-y) - g(x)| w(x) dx < \epsilon, \]

for every \( |y| < \rho_0 \). Therefore, by part (i), we have

\[ \Delta_f(p) \leq (\Delta_{f-g}(p) + \Delta_g(p)) \leq c\|f - g\|_{L^1_w} + \epsilon \leq (c + 1)\epsilon, \text{ if } \rho < \rho_0. \]

**Proof of Lemma 1.** Part (i) of Lemma 1 is an easy consequence of part (i) of Lemma 2. Part (i) and the assumption \( w \in A_p \) imply (ii) for \( p > 1 \). If \( p = 1 \), by part (i) of Lemma 1, we have

\[ \int |f_\epsilon(x)| w(x) dx \leq \int \left\{ \int |f(y) \psi_\epsilon(x-y) dy \right\} w(x) dx \]

\[ = \int |f(y)| \left\{ \int w(x+y) \psi_\epsilon(x) dx \right\} dy \]

\[ \leq \int |f(y)| Mw(y) dy \]

\[ \leq c \int |f(y)| w(y) dy. \]
since \( w \in A_1 \).
Let us prove part (iii). If \( B \) is a ball, we have

\[
(4) \quad |B|^{-1} \int_B |f(y)| \, dy \leq c\|f\|_{L^p_w}(\int_B w(y)dy)^{-\frac{1}{p}}
\]

which shows that \( f \) is a locally integrable function and therefore, that almost every point \( x \in \mathbb{R}^n \) is a Lebesgue point for \( f \). Then, assuming that \( x \) is a Lebesgue point for \( f \), there exists \( \rho_1 > 0 \) such that

\[
(\Omega_n r^n)^{-1} \int_{|x-y| \leq r} |f(y) - f(x)| \, dy \leq 1
\]
holds for every \( 0 < r \leq \rho_1 \). Thus,

\[
(5) \quad (\Omega_n r^n)^{-1} \int_{|x-y| \leq r} |f(y)| \, dy \leq 1 + |f(x)|,
\]

for \( 0 < r \leq \rho_1 \). Taking into account (4) with \( B = \{y : |x-y| \leq r\}, \ r > \rho_1 \) and (5) it follows that \( Mf(x) < \infty \). Now, we have

\[
|f_\varepsilon(x) - f(x)\int \varphi(y)dy| \\
\leq \{\int_{|y| \leq \rho} + \int_{|y| > \rho}\} |f(x-y) - f(x)| \psi_\varepsilon(y)dy.
\]

By Lemma 2, we obtain

\[
|f_\varepsilon(x) - f(x)\int \varphi(y)dy| \\
\leq \sup_{0<r\leq \rho} (\Omega_n r^n)^{-1} \int_{|y| \leq \rho} |f(x-y) - f(x)| \, dx.
\]

\[
= I_1 + I_2 + I_3.
\]
Choosing \( \rho \) small enough, \( I_1 \) can be made less than any given \( n > 0 \) no matter the value of \( \epsilon \). Once \( \rho \) is fixed, both \( I_2 \) and \( I_3 \) go to zero for \( \epsilon \) going to zero. Therefore, (ii) is proved.

As for part (iv), we observe that the case \( p > 1 \) follows immediately from (i) and (iii) by applying the Lebesgue dominated convergence theorem. If \( p = 1 \), we have

\[
\int \left| f_\epsilon(x) - f(x) \right| \varphi(y)dy \mid w(x)dx
\leq \int \{ \int \left| f(x - y) - f(x) \right| \psi_\epsilon(y)dy \} w(x)dx
\leq \int \{ \int \left| f(x - y) - f(x) \right| w(x)dx \} \psi_\epsilon(y)dy
= \int h(y)\psi_\epsilon(y)dy,
\]

where

\[
h(y) = \int \left| f(x - y) - f(x) \right| w(x)dx.
\]

By Lemma 3, the function \( h(y) \) satisfies \( Mh(0) < \infty \) and \( y = 0 \) is a Lebesgue point for \( h \). The proof of part (iii) of this lemma shows that

\[
\lim_{\epsilon \to 0} \int h(y)\psi_\epsilon(y)dy = h(0) = 0,
\]

which ends the proof of part (iv).

Let \( m \) be a positive integer and \( k = (k_1, \ldots, k_n) \in \mathbb{Z}^n \). Let \( y_k^m \) stand for the point \( 2^{-m}k \in \mathbb{R}^n \) and \( Q_k^m = \{ y \in \mathbb{R}^n : 2^{-m}k_i \leq y_i < 2^{-m}(k_i + 1), 1 \leq i \leq n \} \). We observe that \( y_k^m \in Q_k^m \) and that the length of the sides of \( Q_k^m \) are equal to \( 2^{-m} \). In addition, for any given \( m \), the family \( \{ Q_k^m : k \in \mathbb{Z}^n \} \) is a partition of \( \mathbb{R}^n \). Let us assume that \( g \) is a bounded function with bounded support,
$w \in A_1$ and $f \in L^1_w$. Then, for any given positive integer $m$, we define

$$c_m(f)(x) = \sum_{k \in \mathbb{Z}^n} f(x - y_k^m) \int_{Q_k^m} g(y)dy$$

(6)

The set $I$ of indices $k$ with the property that $Q_k^m$ has a non-empty intersection with the support of $g$ is a finite set. If the support of $g$ is contained in the unit ball, the points $\{y_k^m\}_{k \in I}$ satisfy $|y_k^m| \leq 1 + 2^{-m}\sqrt{n}$. In the sequel we shall assume that $m$ is large enough so that $|y_k^m| \leq 2$.

**Lemma 4.** Let $w \in A_1, f \in L^1_w$ and let $g$ be bounded function with bounded support. Then, for every $R > 0$,

$$\lim_{m \to \infty} \int_{|x| \leq R} |C_m(f)(x) - (f \ast g)(x)| \, dx = 0.$$

**Proof.** Without loss of generality, we may assume that the support of $g$ is contained in the unit ball. If we take $x, |x| \leq R$ then since $|x - y_k^m| \leq R + 2$ and $|x - y| \leq R + 1$ for $|y| \leq 1$, it follows that

$$C_m(f)(x) - (f \ast g)(x) = C_m(f \chi_{R+2})(x) - ((f \chi_{R+2}) \ast g)(x),$$

where $\chi_{R+2}(y)$ stands for the characteristic function of $|y| \leq R+2$. Therefore, it is enough to prove the lemma for functions which are Lebesgue integrable. Then, assuming that $f \in L^1(R^n)$, we have

$$\int |C_m(f)(x) - (f \ast g)(x)| \, dx$$

$$\leq \sum_{k \in \mathbb{Z}^n} \int_{Q_k^m} \int |f(x - y_k^m) - f(x - y)| \, dx \, |g(y)| \, dy$$

which tends to zero as $m$ tends to infinity.
Lemma 5. Let \( w \in A_1 \) and let \( g \) be a bounded function with bounded support. If \( f \) and \( \tilde{f} = Kf \) belong to \( L^1_w \), then

\[
(f \ast g)\sim(x) = (\tilde{f} \ast g)(x)
\]

holds for almost every \( x \) in \( \mathbb{R}^n \).

Proof. Without loss of generality we may assume that the support of \( g \) is contained in the unit ball. From Theorem B, we know that the operator \( k \) is defined on \( L^1_w \) and since \( f \ast g \) belongs to \( L^1_w \), then it follows that \( (f \ast g)\sim(x) \) is defined almost everywhere. The assumption that \( \tilde{f} \in L^1_w \) implies the almost everywhere existence of \( (\tilde{f} \ast g)(x) \). Let \( T > 0 \) and \( \varepsilon > 0 \) Choose \( R \) such that \( R > T + 6 \) and

\[
(7) \quad \int_{|y| > R-2} |f(y)| \, w(y) dy < \varepsilon^2.
\]

By Lemma 4 applied to \( f \) and \( \tilde{f} \) there exists \( m \) such that

\[
(8) \quad \int_{|y| \leq R} \left| C_m(f)(y) - (f \ast g)(y) \right| dy < \varepsilon^2 \text{ and}
\]

\[
(9) \quad \int_{|y| \leq R} \left| C_m(\tilde{f})(y) - (\tilde{f} \ast g)(y) \right| dy < \varepsilon^2.
\]

Then, denoting \( B_T = \{ x : |x| > T \} \) we have

\[
(10) \quad | \{ x : (f \ast g)\sim(x) - (\tilde{f} \ast g)(x) |> \varepsilon \} \cap B_T |
\]

\[
\leq \{ x : (f \ast g)\sim(x) - (\chi_R(f \ast g))\sim(x) |> \frac{\varepsilon}{4} \} \cap B_T |
\]

\[
+ \{ x : (\chi_R(f \ast g))\sim(x) - (\chi_R C_m(f))\sim(x) |> \frac{\varepsilon}{4} \} \cap B_T |
\]

\[
+ \{ x : (\chi_R C_m(f))\sim(x) - C_m(\tilde{f})(x) |> \frac{\varepsilon}{4} \} \cap B_T |
\]

\[
+ \{ x : C_m(\tilde{f})(x) - (\tilde{f} \ast g)(x) |> \frac{\varepsilon}{4} \} |
\]

\[= \alpha + \beta + \gamma + \delta.\]
Let us estimate $\alpha$. By the $A_1$ condition on $w$ and the $(1,1)$-weak type of the operator $K$, we have

$$\alpha \leq c(\|B_T \| / w(B_T)) w(\{x : |(f \ast g)^+(x) - (\chi_R(f \ast g))^+(x)| > \frac{\varepsilon}{4}\})$$

(11) \hspace{1cm} \leq c(\|B_T \| / w(B_T)) w(\{x : |(1 - \chi_R)(f \ast g)^+(x)| > \frac{\varepsilon}{4}\})$$

$$\leq c'\varepsilon^{-1}(\|B_T \| / w(B_T))\|\|1 - \chi_R)(f \ast g)\|_{L^1_w}.$$  

For this norm, we have

$$\|(1 - \chi_R)(f \ast g)\|_{L^1_w} \leq \int_{|x| \geq R} \{ \int |f(y) \| g(x - y) \| dy\} w(x) dx.$$ 

Since the support of $g$ is contained in the unit ball, the relevant values of $x$ and $y$ in the integral satisfy $|x| \geq R$ and $|x - y| \leq 1$. Therefore, $R - |y| \leq |x| - |y| \leq 1$, which implies $R - 1 \leq |y|$. Then, by condition $A_1$ and (7), we obtain

$$\|(1 - \chi_R)(f \ast g)\|_{L^1_w} \leq \int_{|y| \geq R-1} |f(y)| \{ \int_{|x - y| \leq 1} |g(x - y)| \| w(x) dx \} dy \leq \Omega_n\|g\|_\infty \int_{|y| \geq R-1} |f(y)| \{ \Omega_n^{-1} \int_{|x - y| \leq 1} w(x) dx \} dy \leq \Omega_n c\|g\|_\infty \int_{y \geq R-1} |f(y)| \| w(y) dy \leq c\|g\|_{\infty^{\infty}}^2$$

which together with (11) gives

$$\alpha \leq c(\|B_T \| / w(B_T))\|g\|_{\infty^\infty}. \quad (12)$$
Estimation of $\beta$. By (8) and the $(1,1)$-weak type of the operator $k$, we get

\begin{align*}
(13) \quad \beta \leq \{ x : | (x_R C_m(f))^\sim(x) - (x_R (f \ast g))^\sim(x) | > \varepsilon/4 \} \\
&\leq 4\varepsilon^{-1} \int_{|x| \leq R} | C_m(f)(x) - (f \ast g)(x) | \, dx \\
&\leq 4\varepsilon. 
\end{align*}

Estimation of $\gamma$. We have

\begin{align*}
(x_R C_m(f))^\sim(x) - C_m(\tilde{f})(x) \\
= \sum_k pv \int k(x-y) x_R(y) f(y-y_k^m) dy \int_{Q_k^m} g(z) dz \\
- \sum_k pv \cdot \int k(x-y_k^m-y) f(y) dy \int_{Q_k^m} g(z) dz \\
= \sum_k pv \int k(x-y-y_k^m) [x_R(y+y_k^m) - 1] f(y) dy \int_{Q_k^m} g(z) dz.
\end{align*}

Thus

\begin{align*}
(14) \quad | (x_R C_m(f))^\sim(x) - C_m(\tilde{f})(x) | \leq \\
\sum_k \int \left| k(x-y-y_k^m) - k(x-y) \right| \left| x_R(y+y_k^m) - 1 \right| f(y) \mid dy \cdot \\
\quad \int_{Q_k^m} g(z) \mid dz \\
+ | \int k(x-y) f(y) \sum_k [x_R(y+y_k^m) - 1] dy \int_{Q_k^m} g(z) dz | \\
= I(x) + J(x).
\end{align*}
If $|y| \leq R - 2$, since $|y_k^m| \leq 2$ for $m$ large enough, then $|y + y_k^m| \leq R$ and thus $[\chi_R(y + y_k^m) - 1] = 0$. If $x \in B_T, T < R - 6$ and $|y| > R - 2$, we have

$$|x - y| \geq |y| - |x| \geq (R - 2) - (R - 6) = 4 \geq 2 |y_k^m|.$$  

Therefore, by condition (ii) on the kernel $k(x)$ we get

$$|k(x - y - y_k^m) - k(x - y)| \leq c |x - y|^{-n} \Theta(|y_k^m|/|x - y|).$$

Then, taking into account that $|x - y| \geq 4$, we can write

$$\sum_k \{ \int_{|y| \geq R - 1} [\int_{|x - y| \geq 4} c |x - y|^{-n} \Theta(2/|x - y|) w(x) dx] |f(y)| dy \cdot \int_{Q_x} g(z) dz \} \leq (15) \int_{B_T} I(x) w(x) dx \leq$$

$$\sum_k \{ \int_{|y| \geq R - 1} [\int_{|x - y| \geq 4} c |x - y|^{-n} \Theta(2/|x - y|) w(x) dx] |f(y)| dy \cdot \int_{Q_x} g(z) dz \}$$

To estimate the integral with respect to $dx$, we apply part (iii) of Lemma 2, obtaining

$$\int_{|x - y| \geq 4} |x - y|^{-n} \Theta(2/|x - y|) w(x) dx \leq M w(y) \int_{|x| \geq 2} |x|^{-n} \Theta(2/|x|) dx \leq c w(y) \int_0^1 \Theta(t) \frac{dt}{t}, \text{ since } w \in A_1.$$  

Returning to (15), the estimate obtained together with (7) gives

$$\int_{B_T} I(x) w(x) dx \leq c \|g\|_{L^1} \int_{|y| > R - 2} |f(y)| w(y) dy \leq c \|g\|_{L^1}.$$
Then, since \( w \in A_1 \) and by Tchebichef’s inequality it follows that

\[
| \{ x : I(x) > \frac{\epsilon}{\delta} \} \cap B_T | \\
\leq c(| B_T | / w(B_T)) w(\{ x : I(x) > \frac{\epsilon}{\delta} \} \cap B_T) \\
\leq c(| B_T | / w(B_T))(\frac{8}{\epsilon} \int_{B_T} I(x)wdx) \\
\leq c(| B_T | / w(B_T))\epsilon.
\]

By the \((1, 1)\)-weak type of the operator \( K \) and (7), we obtain

\[
| \{ x : J(x) > \frac{\epsilon}{\delta} \} \cap B_T | \\
\leq c(| B_T | / w(B_T)) w(\{ x : J(x) > \frac{\epsilon}{\delta} \}) \\
\leq c \epsilon^{-1}(| B_T | / w(B_T)) (\int_{|y| > R^{-2}} |f(y)| w(y)dy)\|g\|_{L^1} \\
\leq c(| B_T | / w(B_T))\epsilon.
\]

Then, from (14), (16) and (17), it follows that

\[
\gamma \leq | \{ x : I(x) > \frac{\epsilon}{\delta} \} \cap B_T | + | \{ x : J(x) > \frac{\epsilon}{\delta} \} \cap B_T | \\
\leq c(| B_T | / w(B_T))\epsilon.
\]

Let us estimate \( \delta \). By Tchebichef’s inequality and (9), we have

\[
\delta = | \{ x : C_m(\tilde{f})(x) - (\tilde{f} * g)(x) | > \frac{\epsilon}{4} \} \cap B_T | \\
\leq (\frac{4}{\epsilon}) \int_{|x| \leq R} | C_m(\tilde{f})(x) - (\tilde{f} * g)(x) | dx \\
\leq 4\epsilon.
\]
The estimates (12), (13), (18) and (19) for \( \alpha, \beta, \gamma \) and \( \delta \) respectively show that (10) is smaller than a constant \( c_T \) times \( \varepsilon \). This ends the proof of the lemma.

**Lemma 6.** Let \( w \in A_1 \) and \( f \in L^1_w \). If \( g \) is a function with a decreasing radial majorant \( \psi \) such that \( \psi \in L^1 \cap L^{p_0}, 1 < p_0 \leq \infty \) then

\[
(f * g) = (f \ast \tilde{g})(x)
\]

holds almost everywhere.

**Proof.** If \( w \in A_1 \), it follows from the reverse Hölder inequality that there exists \( \delta > 1 \) such that for every \( p, 1 < p \leq \delta, w^p \in A_1 \). We choose \( p \) satisfying \( 1 < p \leq \min(p_0, \delta) \). By Lemma 2, part (i), and the assumption that \( w \in A_1 \), we have

\[
\int |g(x-y)|^p w(x)^p dx \leq M w^p(y) \int \psi(y)^p dy 
\]

\[
\leq c \omega(y)^p \|\psi\|_{L^p}^p.
\]

Then, by Minkowski's integral inequality and (20), we obtain

\[
\|f \ast g\|_{L^p_w} \leq \int |f(y)| \left( \int |g(x-y)|^p w(x)^p dx \right)^{1/p} dy 
\]

\[
\leq c \|\psi\|_{L^p} \int |f(y)| w(y) dy 
\]

\[
= c \|f\|_{L^1_w} \|\psi\|_{L^p}.
\]

On the other hand, since \( w^p \in A_1 \subseteq A_p \) and recalling that the operator \( K \) is of strong type \((p, p)\) for weights belonging to \( A_p \), we have

\[
\int |\tilde{g}(x-y)|^p w(x)^p dx \leq c \int |g(x-y)|^p w(x)^p dx.
\]
Then, applying Minkowski's integral inequality and (20), we obtain

$$
\|f \ast \tilde{g}\|_{L^p_w} \leq \int \left| f(y) \right| \left( \int \left| \tilde{g}(x-y) \right|^p \ w(x)\,dx \right)^{1/p} \,dy
\leq c\|f\|_{L^1_w} \|\psi\|_{L^p}.
$$

Applying Fubini's theorem, it is easy to show that for every $\varepsilon > 0$

$$(f \ast g) \ast k_\varepsilon = f \ast (g \ast k_\varepsilon).$$

Since, as we have shown in (21), $f \ast g \in L^p_w$, then by Theorem B, part (iv), we have that $f \ast (g \ast k_\varepsilon)$ converges to $(f \ast g')$ in $L^p_w$ as $\varepsilon$ goes to zero. The proof of the lemma will be completed once we show that

$$(22) \quad \lim_{\varepsilon \to 0} \|f \ast (g \ast k_\varepsilon) - f \ast \tilde{g}\|_{L^p_w} = 0.$$ 

We observe that (20) says that $g(x-y)$ belongs to $L^p_w$ for almost every $y$. Therefore, in virtue of Theorem B, part (iv), if $\tilde{g}_\varepsilon = K_\varepsilon g$, we have that

$$(23) \quad \int \left| \tilde{g}_\varepsilon(x-y) - \tilde{g}(x-y) \right|^p \ w^p(x)\,dx$$

tends to zero as $\varepsilon \to 0$ for almost every $y \in \mathbb{R}^n$. Moreover, since $K^*g(x) = \sup_{\varepsilon > 0} \left| \tilde{g}_\varepsilon(x) \right|$, then, by Theorem A, part (i), we have

$$
\int \left| \tilde{g}_\varepsilon(x-y) - \tilde{g}(x-y) \right|^p \ w(x)^p\,dx
\leq 2^p \int (K^*g(x-y))^p \ w(x)^p\,dx
\leq c \int \left| g(x-y) \right|^p \ w(x)^p\,dx
\leq c w(y)^p \|\psi\|_{L^p}. $$
Then, by Minkowski's integral inequality, we get

\[(24) \quad \| f \ast (g * k_\varepsilon) - f \ast \tilde{g} \|_{L^p_{\text{w,p}}} \leq \int | f(y) | \left( | \tilde{g}_\varepsilon(x - y) - \tilde{g}(x - y) |^p \omega^p(x) \, dx \right) \frac{1}{r} \, dy, \]

and, by the considerations just made in (23) and the Lebesgue dominated convergence theorem, we conclude that the right hand side of (24) tends to zero with \( \varepsilon \). This ends the proof of the lemma.

**Lemma 7.** Let \( k(x) \) be a singular integral kernel and \( w \in A_1 \). Then,

\[ \int_{|x| \geq 2|y|} | k(x) - k(x - y) | \, w(x) \, dx \leq cw(y) \]

holds for almost every \( y \in \mathbb{R}^n \).

**Proof.** By condition (ii) on the kernel and Lemma 2, part (iii), we have

\[
\int_{|x| \geq 2|y|} | k(x - y) - k(x) | \, w(x) \, dx \\
\leq c \int_{|x| \geq 2|y|} | x |^{-n} \Theta(|y|/|x|)w(x) \, dx \\
\leq c \sup_{r > 2|y|} \left( \frac{1}{\Omega_n r^n} \int_{2|y| < |z| < r} w(x) \, dx \right) \cdot \left( \int_{|x| > |y|} | x |^{-n} \Theta(|y|/|x|) \, dx \right). 
\]

Taking polar coordinates, we get

\[
\int_{|x| \geq |y|} | x |^{-n} \Theta(|y|/|x|) \, dx = w_{n-1} \int_{|y|}^\infty r^{-1} \Theta(|y|/r) \, dr \\
= w_{n-1} \int_0^1 \Theta(t) \, dt / t < \infty.
\]
Now, since \( \{ z : 2 \mid y \mid < \mid z \mid < r \} \subset B(y, 3r/2) \), we have

\[
\frac{1}{\Omega_n r^n} \int_{|y|<|z|} w(z)dz \leq c \mid B(y, 3r/2) \mid^{-1} \int_{B(y,3r/2)} w(z)dz
\]

\[
\leq cw(y),
\]

for almost every \( y \), ending the proof of the lemma.

**Theorem 1.** Let \( Kf = f \) be a singular integral operator with a kernel \( k(x) \) and let \( w \in A_1 \). If \( f \) and \( \tilde{f} \) belong to \( L^1_w \), then

1) for every \( \epsilon > 0 \), \( k_\epsilon * f \) belongs to \( L^1_w \) and

2) \( \|k_\epsilon * f - \tilde{f}\|_{L^1_w} \to 0 \) as \( \epsilon \to 0 \).

**Proof.** Let \( \varphi \geq 0 \), \( \varphi \in C_0^\infty \), \( \text{supp} \varphi \subset \{ x : | x | \leq 1 \} \) and \( \int \varphi(x)dx = 1 \). Let \( \varphi_\epsilon(x) = \epsilon^{-n} \varphi(x/\epsilon) \) and set \( \delta_\epsilon(x) = \varphi_\epsilon(x) - K_\epsilon(x) \). If \( g \) is a continuous function with bounded support we shall show that \( \|g * \delta_\epsilon\|_{L^2_w} \) tends to zero as \( \epsilon \) tends to zero. In fact, we have \( g * \varphi_\epsilon = (g * \varphi_\epsilon)\sim = \tilde{g} * \varphi_\epsilon \) (see [2], Lemma 3). Therefore, \( (g * \delta_\epsilon)(x) = (\tilde{g} * \varphi_\epsilon)(x) - (k_\epsilon * g)(x) \), a.e. Adding and subtracting \( \tilde{g} \), and taking into account Lemma 1 and Theorem B, part (iv) we have

\[
\lim_{\epsilon \to 0} \|g * \delta_\epsilon\|_{L^2_w} \
\leq \lim_{\epsilon \to 0} \|\tilde{g} * \varphi_\epsilon - \tilde{g}\|_{L^2_w} + \lim_{\epsilon \to 0} \|\tilde{g} - k_\epsilon * g\|_{L^2_w} = 0.
\]

Now, arguing as in the proof of Lemma 5 of [2], we shall prove that \( g * \delta_\epsilon \) also tends to zero in \( L^1_w \) when \( \epsilon \) goes to zero. In fact, if the support of \( g \) is contained in the ball \( | x | \leq N \), then support of \( g * \varphi_\epsilon \) is contained in \( | x | \leq 2N \) provided \( 0 < \epsilon < N \). By Lemma 3 of [2], if \( | x | \geq 4N \) and \( 0 < \epsilon < N \), we have

\[
(\varphi_\epsilon * g)(x) = (\varphi_\epsilon * g)\sim(x)
\]

\[
= \int_{|y| \leq 2N} k(x - y)(\varphi_\epsilon * g)(y)dy.
\]
Moreover, since
\[(k_\varepsilon * g)(x) = \int_{|y| \leq 2N} k(x - y)g(y)dy,\]
it follows that
\[(g * \delta_\varepsilon)(x) = \int_{|y| \leq 2N} k(x - y)[(\varphi_\varepsilon * g)(y) - g(y)]dy,\]
for \(|x| \geq 4N\) and \(0 < \varepsilon < N\).

Observing that
\[\int_{|y| \leq 2N} [(\varphi_\varepsilon * g)(y) - g(y)]dy = 0,\]
We write,
\[(g * \delta_\varepsilon)(x) = \int_{|y| \leq 2N} [k(x - y) - k(x)][(\varphi_\varepsilon * g)(y) - g(y)]dy,\]
for \(|x| \geq 4N\) and \(0 < \varepsilon < N\). Then, multiplying by \(w(x)\) and integrating on \(|x| \geq 4N\), we get
\[\int_{|x| \geq 4N} |(g * \delta_\varepsilon)(x)| w(x)dx \leq \int_{|y| \leq 2N} \left\{ \int_{|x| \geq 2y} |k(x - y) - k(x)| w(x)dx \right\} |(\varphi_\varepsilon * g)(y) - g(y)| dy.

By Lemma 7, the right hand side of the inequality above is bounded by a constant times \(\|\varphi_\varepsilon * g - g\|_{L^1_w}\) which in turn tends to zero when \(\varepsilon \to 0\), by Lemma 1. If \(|x| \leq 4N\), we have
\[\int_{|x| \leq 4N} |(g * \delta_\varepsilon)(x)| w(x)dx \leq \left( \int_{|x| \leq 4N} w(x)dx \right)^\frac{1}{2} \|g * \delta_\varepsilon\|_{L^2_w},\]

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Which, by (25), tends to zero with \( \varepsilon \). Next, we shall find a decreasing, radial and integrable function \( \Delta(x) \) such that \( |\delta_\varepsilon(x)| \leq \Delta_\varepsilon(x) = \varepsilon^{-n} \Delta(x/\varepsilon) \). In fact, if \( |x| \leq 2\varepsilon \), we have

\[
\delta_\varepsilon(x) = \varphi_\varepsilon(x) - k_\varepsilon(x)
= \lim_{n \to 0} \int_{n < |y| < 4\varepsilon} k(y)[\varphi_\varepsilon(x - y) - \varphi_\varepsilon(x)]dy
+ \varphi_\varepsilon(x) \lim_{n \to 0} \int_{n < |y| < 4\varepsilon} k(y)dy - k_\varepsilon(x).
\]

Hence, by conditions (i), (iii) and (iv) on \( k(x) \), we obtain

\[
(26) \quad |\delta_\varepsilon(x)| \leq c\varepsilon^{-n}, \text{ for } |x| \leq 2\varepsilon.
\]

Let \( |x| \geq 2\varepsilon \). Then,

\[
\delta_\varepsilon(x) = \int_{|y| \leq \varepsilon} [k(x - y) - k(x)]\varphi_\varepsilon(y)dy.
\]

Thus, by condition (ii) on \( k(x) \), we have

\[
(27) \quad |\delta_\varepsilon(x)| \leq c\int_{|y| \leq \varepsilon} |x|^{-n} \Theta(|y|/|x|)\varphi_\varepsilon(y)d(y).
\]

Defining \( \Delta(x) = c \) for \( |x| \leq 2 \) and

\[
\Delta(x) = c \int_{|y| \leq 1} |x|^{-n} \Theta(|y|/|x|)\varphi_\varepsilon(y)d(y) \quad \text{for } |x| \geq 2.
\]

We have that by (26) and (27), \( |\delta_\varepsilon(x)| \leq \Delta_\varepsilon(x) = \varepsilon^{-n} \Delta(x/\varepsilon) \).

As for the integrability of \( \Delta(x) \).

\[
\int_{|y| \geq 2} \Delta(x)dx = \int_{|y| \leq 1} \varphi(y)(\int_{|x| \geq 2} |x|^{-n} \Theta(|y|/|x|)dx)dy
= n^{-1} \int \varphi(y)dy \int_0^{1/2} \Theta(t)dt/t < \infty,
\]

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shows that \( \Delta(x) \) is integrable.

By Lemma 5 and 6, we know that \((\tilde{f} * \varphi_\varepsilon)(x) = (f * \tilde{\varphi}_\varepsilon)(x)\) almost everywhere. Hence, \((f * k_\varepsilon)(x) = (\tilde{f} * \varphi_\varepsilon)(x) - (f * \delta_\varepsilon)(x)\), a.e. Therefore,

\[
\|f * k_\varepsilon\|_{L^1_w} \leq \|\tilde{f} * \varphi_\varepsilon\|_{L^1_w} + \|f * \delta_\varepsilon\|_{L^1_w}.
\]

Since \(|\delta_\varepsilon(x)| \leq \Delta_\varepsilon(x)\), by Lemma 1, part (ii), we obtain

\[
\|f * k_\varepsilon\|_{L^1_w} \leq \|f\|_{L^1_w} + \|\tilde{f}\|_{L^1_w}.
\]

For \(n > 0\), let \(g\) be a continuous function with bounded support such that \(\|f - g\|_{L^1_w} < n\). Then,

\[
\|f * \delta_\varepsilon - \tilde{f}\|_{L^1_w} \leq \|\tilde{f} * \varphi_\varepsilon - \tilde{f}\|_{L^1_w} + \|(f - g) * \delta_\varepsilon\|_{L^1_w} + \|g * \delta_\varepsilon\|_{L^1_w}.
\]

Since \(\|(f - g) * \delta_\varepsilon\|_{L^1_w} \leq \|f - g\|_{L^1_w} * \Delta_\varepsilon\), by Lemma 1, part (ii), we get \(\|(f - g) * \delta_\varepsilon\|_{L^1_w} \leq cn\). On the other hand, since we have already shown that \(\|g * \delta_\varepsilon\|_{L^1_w}\) tends to zero and by Lemma 1 applied to \(\|\tilde{f} * \varphi_\varepsilon - \tilde{f}\|_{L^1_w}\) we get

\[
limsup_{\varepsilon \to 0} \|f * k_\varepsilon - \tilde{f}\|_{L^1_w} \leq cn.
\]

Then arbitrariness of \(n > 0\) proves that the limit exists and that it is equal to zero.

Lemmas 5 and 6 show that for a function \(f\) such that \(f\) and \(\tilde{f}\) belong to \(L^1_w\), \(w \in A_1\),

\[
(\tilde{f} * g)(x) = (f * \tilde{g})(x)
\]

holds almost everywhere provided that \(g\) is a bounded function with bounded support. This result can be generalized as follows.

**Theorem 2.** Let \(w \in A_1\) and let \(f\) be a function such that \(f\) and \(\tilde{f}\) belong to \(L^1_w\). Let us assume that \(g\) is a function with a decreasing and radial majorant \(\psi\) such that \(\psi \in L^1 \cap L^{p_0}\), \(1 < p_0 \leq \infty\). Then
holds almost everywhere on $x$.

Proof. By Fubini's theorem, $(f * K_\varepsilon) * g = f * (k_\varepsilon * g)$.

Since $w \in A_1$, we have

$$
\|(f * k_\varepsilon) * g - \tilde{f} * g\|_{L^1_w} \leq c\|f * k_\varepsilon - \tilde{f}\|_{L^1_w} \|\psi\|_{L^1}
$$

By Theorem 1, $f * k_\varepsilon - \tilde{f}$ converges to zero in $L^1_w$ as $\varepsilon \to 0$. Therefore, $(f * k_\varepsilon) * g$ converges to $\tilde{f} * g$ in $L^1_w$. On the other hand, observing that the hypothesis of the theorem imply those of Lemma 6, by (22) we have that $f * (k_\varepsilon * g)$ converges in $L^p_{w,p}$, $1 < p \leq p_0$ to $f * \tilde{g}$ when $\varepsilon \to 0$. This completes the proof of the theorem.

3. Application $H^1_w$.

Let $F(x,t) = (u(x,t), v_1(x,t), \ldots, v_n(x,t))$, $x \in \mathbb{R}^n$, $t > 0$ be a vector function satisfying the Cauchy-Riemann equations in the sense of Stein and Weiss [8]. The vector $F(x,t)$ is said to belong to $H^1_w$ if

$$
\|\|F\|\|_{H^1_w} = \sup_{t > 0} \int_{\mathbb{R}^n} |F(x,t)| w(x) dx < \infty.
$$

The Poisson integral of a function $f$ is defined as

$$
Pf(x,t) = \int f(x-y) P(y,t) dy,
$$

where $P(y,t) = c_n t (t^2 + |y|^2)^{(n+1)/2}$.

The $j$-conjugate Poisson integral, $1 \leq j \leq n$, is given by

$$
Q_j f(x,t) = \int f(x-y) Q_j(y,t) dy,
$$

where $Q_j(y,t) = c_n y_j (t^2 + |y|^2)^{(n+1)/2}$.
Let $R_j$ denote the $j$-Riesz transform, i.e.

$$R_jf(x) = p.v. \int c_n y_j |y|^{-n-1} f(x - y)dy.$$ 

It is well known that $Q_j(x, t) = R_j(P(\cdot, t))(x)$.

We shall apply Theorem 2 to give a proof of a result due to R. L. Wheeden in [9].

**Theorem 3.** (see [9], Theorem 1, part (ii)). Let $w \in A_1$ and $f \in L^1_w$. If each $R_j f \in L^1_w$, $1 \leq j \leq n$, then, the vector

$$F = (Pf, Q_1f, \ldots, Q_nf)$$

belongs to $H^1_w$. Moreover, for $1 \leq j \leq n$,

(28) \hspace{1cm} Q_jf = P(R_jf), \text{ and}

$$c_1\|F\|_{H^1_w} \leq \|f\|_{L^1_w} + \sum_{j=1}^{n} \|R_j f\|_{L^1_w} \leq c_2\|F\|_{H^1_w},$$

where the constants $c_1$ and $c_2$ do not dependent on $f$.

**Proof.** Since $R_j(P) = Q_j$ and observing that $P(x, t)$ is a radial decreasing function in $L^1 \cap L^\infty$, from Theorem 2 we have that if $f$ and $R_j f$, $1 \leq j \leq n$, belong to $L^1_w$, then

(29) \hspace{1cm} \int f(y)Q_j(x - y, t)dy = \int R_j f(y)P(x - y, t)dy

holds for almost every point $x \in \mathbb{R}^n$ for each given $t > 0$. It is easy to show that both sides of (29) are continuous function of $x$ and $t > 0$. Therefore (29) holds for every $x \in \mathbb{R}^n$ and $t > 0$. This proves (28). Now, since
\[ F = (Pf, Q_1f, \ldots, Q_nf) = (Pf, P(R_1f), \ldots, P(R_nf)), \]
we have

\[
\int |f(x,t)| w(x)dx \leq \int |Pf(x,t)| w(x)dx + \sum_{j=1}^{n} \int |P(R_jf)(x,t)| w(x)dx.
\]

By Lemma 1, the right hand side of this inequality is bounded by

\[
c \int |f(x)| w(x)dx + \sum_{j=1}^{n} \int |R_jf(x)| w(x)dx.
\]

Conversely, since by Lemma 1, \( \|f\|_{L^1_w} = \lim_{t \to 0} \|Pf(x,t)\|_{L^1_w} \) and \( \|R_jf\|_{L^1_w} = \lim_{t \to 0} \|P(R_jf)(x,t)\|_{L^1_w} \), we obtain

\[
\|f\|_{L^1_w} + \sum_{j=1}^{n} \|R_jf\|_{L^1_w} \leq \sqrt{n} \|F\|_{H^1_w},
\]

which ends the proof of the theorem.

References


