A NOTE ON
ŠMULIAN'S THEOREM

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Abstract
An extension of Šmulian's theorem in which neither the linearity nor the local convexity are presupposed is obtained.
As a consequence, a non-Archimedean version of Šmulian's theorem is derived.

Definition [1] Let $\mathbb{K}$ be a metrizable topological division ring, $G$ a vector space over $\mathbb{K}$, $H$ a set and $f$ a mapping from $G \times H$ into $\mathbb{K}$ such that for each $h \in H$, the mapping

$$ f_h : g \in G \mapsto f_h(g) = f(g, h) \in \mathbb{K} $$

is linear, and such that $f$ separates points of $G$ (that is, for all $g \in G$, $g \neq 0$, there is $h \in H$ with $f_h(g) \neq 0$). The initial topology on $g$ for the family $\mathcal{F}_H = \{f_h : h \in H\}$ will be denoted by $\sigma(G, H)$; $G$, endowed with $\sigma(G, H)$, is a topological vector space over $\mathbb{K}$.

Similarly, we define $\sigma(H, G)$ as the initial topology on $H$ for the family $\mathcal{F}_G$.

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Example 1 (a) Let $H$ be a set, $G$ a vector subspace of $\mathcal{F}(H; \mathcal{K})$ (the vector space of all mappings from $H$ into $\mathcal{K}$) and

$$f: (g,h) \in G \times H \mapsto g(H) \in \mathcal{K}.$$  

It is clear that $f$ satisfies the conditions of the Definition. In this case, $\sigma(G,H)$ coincides with the topology of simple convergence on $G$.

(b) Let $\mathcal{K}$ be a separated topological field and let $G$ be a topological vector space over $\mathcal{K}$ with the Hahn-Banach property ([6], p.7). Let

$$f: (x,h) \in G \times G' \mapsto h(x) \in \mathcal{K},$$

where $G'$ denotes the topological dual of $G$. Then $f$ satisfies the conditions of the Definition, the fact that $f$ separates points of $G$ being a consequence of Lemma 1.1.3.1. of [6].

As particular cases of Example 1(b) we have the separated locally convex spaces over $\mathcal{R}$ or $\mathcal{C}$ [3] and the polar separated locally convex spaces over a non-Archimedean non-trivially valued complete field [8]. On the other hand, if $0 < p < 1$, then $l^p$ endowed with the metric $d(x,y) = \sum_{n=1}^{\infty} |x_n - y_n|^p$ is a topological vector space with the Hahn-banach property which is not locally convex ([7], Chap.3, Exercise 5(d)).

The next result has Šmulian's theorem [9] and a non-Archimedean version for this theorem (see the Corollary) as particular cases.

Theorem Let $\mathcal{K}$, $G$, $H$ and $f$ be as in the Definition. If $(H, \sigma(H,G))$ is separable, then every $\sigma(G,H)$-(relatively)-countably compact subset of $G$ is $(G,H)$-(relatively)-sequentially compact.

Proof: Since $(H, \sigma(H,G))$ is separable and $\mathcal{K}$ is a separated uniform space in which the countably compact subsets are sequentially compact ($\mathcal{K}$ is metrizable), it follows from the Corollary of Proposition 5 of [2] that every countably compact subset of $C_s((H,\sigma(H,G)), \mathcal{K})$ is sequentially compact (where $C_s((H,\sigma(H,G)), \mathcal{K})$ denotes the space of all continuous functions from $(H,\sigma(H,G))$ into $\mathcal{K}$ endowed with the vector topology of pointwise convergence, $\tau_s$).
Now consider the function

\[ F : g \in G \mapsto f_g \in C((H, \sigma(H,G), \mathcal{K}), \mathcal{K}), \]

where \( f_g(h) = f(g,h) \) for all \( h \in H \). By the fact that \( f \) separates points of \( G \), \( F \) is injective. Therefore \( F \) is a homeomorphism from \( (G, \sigma(G,H)) \) onto \( M = F(G) \), \( M \) endowed with the vector topology induced by \( \tau_s \).

As the countably compact subsets and the sequentially compact subsets of \( M \) coincide, we conclude that every \( \sigma(G,H) \)-(relatively)-countably compact subset of \( G \) is \( \sigma(G,H) \)-(relatively)-sequentially compact, as was to be shown.

**Remark 1** If \( \mathcal{K} \) is an arbitrary topological division ring, then the Theorem holds if and only if every countably compact subset of \( \mathcal{K} \) is sequentially compact.

**Example 2** In this example, we will see that for each locally compact separated topological division ring \( \mathcal{K} \) there is a topological vector space \( G \) over \( \mathcal{K} \) such that the canonical bilinear form

\[ (x, \varphi) \in G \times G' \mapsto \varphi(x) \in \mathcal{K} \]

separates points of \( G \) (so that the Definition is satisfied). Moreover, there is a \( \sigma(G',G) \)-compact subset of \( G' \) which is not \( \sigma(G',G) \)-sequentially compact.

Indeed, by Theorem 1.44 of [5], there is an absolute value \( \| \cdot \| \) on \( \mathcal{K} \) which defines its topology.

Let \( \Gamma' \) be the vector space over \( \mathcal{K} \) of all bounded sequences of elements of \( \mathcal{K} \), endowed with its usual norm. Then \( (\Gamma')' \) endowed with the norm

\[ \| \varphi \| = \sup_{\lambda \neq 0} \| \varphi(\lambda) \| / \| \lambda \| \]

is a Banach space over \( \mathcal{K} \).

For each \( n \geq 1 \), let \( \varphi_n((\lambda_j)_{j \geq 1}) = \lambda_n \); then \( \varphi_n \in (\Gamma')' \) and \( \| \varphi_n \| = 1 \). \( \{ \varphi_n : n \geq 1 \} \) is \( \sigma((\Gamma')', \Gamma') \)-relatively compact, being a subset of the closed unit ball of \( (\Gamma')' \), which is \( \sigma((\Gamma')', \Gamma') \)-compact ([4], Lemma 3.1). We claim that \( \overline{\{ \varphi_n : n \leq 1 \}}^{\sigma((\Gamma')', \Gamma')} \) is not \( \sigma((\Gamma')', \Gamma') \)-sequentially compact.

For otherwise, there would be a subsequence \( (\varphi_{n_j})_{j \geq 1} \) of \( (\varphi_n)_{n \geq 1} \) and \( \varphi \in (\Gamma')' \) such that \( (\varphi_{n_j}(\lambda))_{j \geq 1} \to \varphi(\lambda) \), for all \( \lambda \in \Gamma' \).
But for $\lambda = (\lambda_i)_{i \geq 1} \in l^\infty$ given by

$$
\begin{align*}
\lambda_i &= 0, \text{ if } i \neq n_j \text{ for all } j \geq 1, \\
\lambda_i &= 0, \text{ if } i = n_j \text{ with } j \text{ even}, \\
\lambda_i &= 0, \text{ if } i \neq n_j \text{ with } j \text{ odd},
\end{align*}
$$

we have that $(\phi_{n_j}(\lambda))_{i \geq 1}$ does not converge.

**Corollary** Let $\mathcal{K}$ be a non-Archimedean non-trivially valued complete field and $E$ a strongly polar separated locally convex space over $\mathcal{K}$. If $E'$ is $\sigma(E',E)$- separable, then every $\sigma(E,E')$- (relatively)- countably compact subset of $E$ is (relatively)- sequentially compact.

**Proof:** Let $\tau$ be the given topology of $E$. By Proposition 4.11 of [8], every $\sigma(E,E;\tau)$-convergent sequence is $\tau$- convergent. Consequently, the $\sigma(E,E')$- (relatively)- sequentially compact subsets of $E$ and the $\tau$- (relatively)- sequentially compact subsets of $E$ coincide. Thus it is sufficient to apply the Theorem to finish the proof.

**Remark 2** If we consider in Example 2 $\mathcal{K}$ as a non-Archimedean non-trivially valued locally compact field, then we obtain an example of a locally convex space over $\mathcal{K}$ for which the Corollary is not true.

**References**


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