DISCRETE ANALOGUE OF CAUCHY’S INTEGRAL FORMULA

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1. Introduction

In an earlier paper [3], in 1993, a class of functions named (p, q)-analytic functions and defined on a suitable geometric lattice was introduced. In yet another paper [4] we made a study on complex integrals of discrete functions. The present paper is a continuation of [4] and here a discrete analogue of Cauchy’s integral formula has been established.

2. Notations, Definitions and The Lattice

In order to introduce the concept of (p, q)-analytic function the following geometric lattice was considered in [3]:

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\[ K = \{ (p^m x_0, q^n y_0); \; m, n \in \mathbb{Z}, \; the \; set \; of \; integers, \; 0 < p < 1. \} \]

\[ 0 < q < 1, \; (x_0, y_0) \; fixed, \; x_0 > 0, \; y_0 > 0 \}, \quad (2.1) \]

where the complex number \( z \) is used synonymously with its components \((x, y)\). Thus if \( z \in K \) then \( z = (x, y) = (p^m x_0, q^n y_0) \).

We also recall here, for convenience, some of the definitions given in [3].

**Definition 2.1.** The 'discrete plane' \( Q' \) with respect to some fixed point \( z' = (x', y') \) in the first quadrant, is defined by the set of lattice points,

\[ Q' = \{ (p^m x', q^n y'); \; m, n \in \mathbb{Z}, \; the \; set \; of \; integers \} \].

**Definition 2.2.** Two lattice points \( z_i, z_{i+1} \in Q' \) are said to be 'adjacent' if \( z_{i+1} \) is one of \((px_i, y_i), (p^{-1} x_i, y_i), (x_i, q y_i)\) or \((x_i, q^{-1} y_i)\).

**Definition 2.3.** A 'discrete curve' \( C \) in \( Q' \) connecting \( z_0 \) to \( z_n \) is denoted by the sequence

\[ C = < z_0, z_1, \ldots, z_n > , \]

where \( z_i, z_{i+1}; \; i = 0, 1, \ldots, (n - 1) \), are adjacent points of \( Q' \).

If the points are distinct \((z_i \neq z_j; \; i \neq j)\) then the discrete curve \( C \) is said to be 'simple'.

**Definition 2.4.** A 'discrete closed curve' \( C \) \( Q' \) is given by the sequence \(< z_0, z_1, z_2, \ldots, z_{n} > \) where \(< z_0, z_1, \ldots, z_{n-1} > \) is simple and \( z_0 = z_n \).

Denote by \( \overline{C} \) the continuous closed curve formed by joining adjacent points of the discrete closed curve \( C \). Then \( \overline{C} \) encloses certain points of \( Q' \), denoted by \( \text{Int.} \; (C) \).

**Definition 2.5.** A 'finite discrete domain' \( D \) is defined as

\[ D = \{ z \in Q'; \; z \in C \cup \text{Int.} \; (C) \}. \]
In general a 'discrete domain' $D$ is defined as a union (infinite or otherwise) of finite discrete domains.

$\partial (D)$ denotes the discrete closed curve around the finite discrete domain $D$.

i.e. $\partial (D) = D - \text{Int} (D)$.

**Definition 2.6.** A 'basic set' with respect to $z \in Q$ is defined as

$$S(z) = \{(x, y), (px, y), (px, qy), (x, qy)\},$$

and the discrete closed curve around $S(z)$ is denoted by

$$\partial (S) = <(x, y), (px, y), (px, qy), (x, qy), (x, y)> \quad (2.2)$$

The order of points of $\partial (S)$ as in (2.2) is said to be positive direction. The reverse sequence is denoted by $-\partial (S)$.

It is evident from the above definitions that a discrete domain $D$ is composed of a union of basic sets.

**Definition 2.7.** Function defined on the points of a discrete domain $D$ are said to be 'discrete functions'.

**Definition 2.8.** The p-difference and q-difference operator $D_{p,x}$ and $D_{q,y}$ are defined as follows:

$$D_{p,x}[f(z)] = \frac{f(z) - f(px, y)}{(1-p)x} \quad (2.3)$$

$$D_{q,y}[f(z)] = \frac{f(z) - f(x, qy)}{(1-q)iy} \quad (2.4)$$

where $f$ is a discrete function.

**Definition of $(p, q)$ – Analytic Functions.** The two operators (2.3) and (2.4) involve a 'basic triad' of points denoted by

$$T(z) = \{(x, y), (px, y), (x, qy)\} \quad (2.5)$$

Let $D$ be a discrete domain. Then a discrete function $f$ is said to be $(p,q)$-analytic at $z \in D$ if
\[ D_{p,x} [f(z)] = D_{q,y} [f(z)] \]  

(2.6)

If in addition (2.6) holds for every \( z \in D \) such that \( T(z) \subseteq D \) then \( f \) is said to be \((p, q)\) – analytic in \( D \).

(2.7)

For simplicity if (2.6) or (2.7) holds, the common operator \( D \) is used where

\[ D = D_{p,x} \equiv D_{q,y} \]  

(2.8)

Further, the operator \( R_{p,q} \) is defined as

\[ R_{p,q} [f(z)] = \{(1-p)x - i(1-q)y\} f(x,y) - (1-p)x f(x,qy) + i(1-q)y f(px,y), \]  

(2.9)

where \( f : K \rightarrow \mathbb{C} \) (the field of complex numbers).

\( R_{p,q} f(z) \) is called \((p, q)\) – residue of the function at \( z \).

From (2.7) it is easily seen that \( f \) is \((p, q)\)-analytic in a discrete domain \( D \) iff

\[ R_{p,q} [f(z)] = 0, \]  

(2.10)

for every \( z \in D \) with \( T(z) \subseteq D \).

Since a discrete domain \( D \) is the union of basic sets \( S \) so if the discrete domain \( D \) is given by

\[ D = \bigcup_{i=1}^{N} S(z_i), \]

then the ‘subdomain \( D_N \)’ is defined by

\[ D_N = \{ z_i ; i = 1, 2, ..., N \} \]  

(2.11)

3. \((r, s)\) – Analytic Functions

In order to develop a discrete analogue of Cauchy’s integral formula it is also necessary to introduce the concept of \((r, s)\)-analytic function.
If \( r = p^{-1} \) and \( s = q^{-1} \) then \( r > 1 \) and \( s > 1 \). Let the operators \( \theta_{r,x}, \theta_{s,y} \) be defined in a manner similar to the operators \( D_{p,x}, D_{q,y} \).

\[
\theta_{r,x}[f(z)] = \frac{f(z) - f(rx, y)}{(1-r)x},
\]

\[
\theta_{s,y}[f(z)] = \frac{f(z) - f(x, sy)}{(1-s)iy}.
\]

A discrete function \( f \), defined on \( Q \), is said to be \("(r, s)\)-analytic\) at \( z \) if

\[
\theta_{r,x}[f(z)] = \theta_{s,y}[f(z)] \quad \quad (3.1)
\]

and the common operator is denoted by \( \theta \).

Now equation (3.1) is equivalent to \( B_{p,q}[g(z)] = 0 \), where the operator \( B_{p,q} \) is defined as

\[
B_{p,q}[g(z)] = \{(1-r)x - (1-s)iy\} \ g(z) - (1-r)x \ g(x, sy) + (1-s)iy \ g(rx, y)
\]

\[
= \{(1-p^{-1})x - (1-q^{-1})iy\} \ g(z) - (1-p^{-1})xg(x, q^{-1}y)\]

\[
+ (1-q^{-1})iy \ g(p^{-1}x, y) \quad \quad (3.2)
\]

4. Cauchy's integral formula

The following definition of a \("conjoint line integral\) is the \((p, q)\)-function analogue of the one introduced by Isaacs [2].

If \( C = < z_0, z_1, \ldots, z_n > \) is a discrete curve in \( D \), and if \( f \) and \( g \) are two discrete functions, then the conjoint line integral along \( C \) is defined as

\[
\int_{z_0}^{z_n} (f \oplus g)(t) \ d(t ; p, q) = \sum_{j=0}^{n-1} \int_{z_j}^{z_{j+1}} (f \oplus g)(t) \ d(t ; p, q)
\]

where
The following two theorems are \((p, q)\)-analogues of monodiffric results given by Kurowski [5] and Berzseuyi [1]. The proofs are essentially the same and so are omitted.

**Theorem 4.1.** If \(f\) is \((p, q)\)-analytic and \(g\) is \((r, s)\)-analytic in \(D\) then,
\[
\int_C (f \oplus g)(t) \, d(t; p, q) = 0
\]
where \(C\) is any closed curve in \(D\).

**Theorem 4.2.** If \(D\) is a finite discrete domain and if \(f\) and \(g\) are discrete functions defined on \(D\), then
\[
\int_{\partial(D)} (f \oplus g)(t) \, d(t; p, q) = \sum_{t \in D_N} [f(t)B_{p, q} g(p \rho, q \sigma) - g(p \rho, q \sigma)R_{p, q} f(t)]
\]
where \(D_N\) is given by (2.11) and \(t = p + i \sigma\) so that \(f(t) \equiv f(p, \sigma)\).

The latter theorem is the \((p, q)\)–analogue of Green’s Identity.

If \(f\) is \((p, q)\)–analytic, then since \(R_{p, q} f = 0\) the following holds.

**Corollary 4.1.** If \(f\) is \((p, q)\)–analytic on some finite discrete domain \(D\) then,
\[
\int_{\partial(D)} (f \oplus g)(t) \, d(t; p, q) = \sum_{t \in D_N} f(t)B_{p, q} g(p \rho, q \sigma).
\]
A discrete function $G_a$ is called a 'singularity function' if it satisfies

$$B_{p,q} [G_a (t)] = \begin{cases} 1; & t = a, \quad a = a_1 + ia_2 \\ 0; & t \neq a; \quad a, \ t \in Q' \end{cases}$$

If such a function can be found then Corollary (4.1) reduces to

$$\int_{\partial(D)} (f \oplus G_a)(t) \ d(t; p, q) = f(p^{-1} a_1, q^{-1} a_2),$$

an analogue of Cauchy's integral formula.

5. References


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