WEAK AND PERIODICAL SOLUTIONS OF THE NAVIER-STOKES EQUATION IN NONCYLINDRICAL DOMAINS

Cruz S. Q. de Caldas, Juan Limaco, Pedro Gamboa* and Rioco K. Barreto

Abstract

We consider the Navier Stokes equation in noncylindrical domain and prove the existence of weak and periodical solutions.

Key Words and Phrases: Navier-Stokes, Noncylindrical domains, Weak and periodical solutions.
1 Introduction

Let $T > 0$ be a real number and $\{\Omega_t\}_{0 \leq t \leq T}$ a family of bounded open sets of $\mathbb{R}^n$ with boundary $\Gamma_t$. Let us consider the noncylindrical domain of $\mathbb{R}^{n+1}$, given by $\hat{Q} = \bigcup_{0 < t < T} \Omega_t \times \{t\}$, with lateral boundary $\hat{S} = \bigcup_{0 < t < T} \Gamma_t \times \{t\}$ regular.

We have the Navier-Stokes problem:

\[
\begin{align*}
    u' - \Delta u + \sum_{i=1}^{n} u_i \frac{\partial u}{\partial x_i} &= f - \nabla p \quad \text{in } \hat{Q} \\
    \text{div} u &= 0 \quad \text{in } \hat{Q} \\
    u &= 0 \quad \text{on } \hat{S} \\
    u(x,0) &= u_0(x) \quad x \in \Omega_0
\end{align*}
\]

where $u(x,t) = (u_1(x,t), \ldots, u_n(x,t)), (x,t) \in \hat{Q}$,

$\Delta u = (\Delta u_1, \ldots, \Delta u_n), \nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}).$

In the last thirty years a lot of papers concerning the existence of solutions to the Navier-Stokes equation in noncylindrical domain have been written. Among these papers, it is worth mentioning the articles of J. L. Lions [17], H. Fujita and N. Sauer [7], [8], H. Morimoto [23], in which the penalty method is used, the results of R. Salvi [25], [26], [27], L.A. Medeiros and J. Limaco Ferrel [15] based on elliptic regularization, the paper of O.A. Ladyzhenskaya [13], M. Otani and Y. Yamada [24] obtained with the Rothe’s method and with the subdifferential operator theory respectively, and the works of J.O. Sather [28], D.N. Bock [2], A. Inoue and M. Wakimoto [11], T. Miyakawa and Y. Teramoto [22] derived with tools of Differential Geometry and the paper of M. Milla Miranda and J. Limaco Ferrel [16] used change of variable for a cilyndrical domain and Galerkin’s method.

Let $K : [0,T] \rightarrow \mathbb{R}^{n^2}$, a function, where $K(t)$ is a $n \times n$ matrix. Let $\Omega$ be an open bounded set of $\mathbb{R}^n$, which, without loss of generality, can
be consider containing the origin of $\mathbb{R}^n$. We assume that the boundary $\Gamma$ of $\Omega$ is smooth, and consider the sets

$$\Omega_t = \{x = K(t)y; \ y \in \Omega\}$$

(1)

In this work we study the existence of weak solutions to Problem (I), and also study the existence of periodical solution to (I). For that, by a suitable change of variable which is more general than that one used in [16], we transform the non cylindrical problem (I) in another problem defined in the cylinder $Q = \Omega \times ]0, T[$.

2 Notation and Hypotheses

We make following hypothesis on $K(t)$.

(H1) $K \in C^1$, where $K(t)$ is an invertible matrix.

(H2) $K^{-1} \in C^1$, where $K^{-1}(t)$ is an invertible matrix of $K(t)$.

Consider the notation

$$K(t) = (\alpha_{ij}(t)), \text{ and } K^{-1}(t) = (\beta_{ij}(t))$$

(2)

as well as the convention of summation of repeated indices; that is

$$\alpha_i \beta_i = \sum_{i=1}^{n} \alpha_i \beta_i.$$

By $\langle \ldots \rangle$ we will represent the duality pairing between $X'$ and $X$, $X'$ being the dual of the space $X$.

In order to state the main results we introduce some spaces. Let $^\sqrt{t}$ be the space

$$^\sqrt{t} = \{ \varphi \in (D(\Omega_t))^n; \ div \varphi = 0 \}$$
and \( V_s(\Omega_t) \) be the closure of \( \sqrt{t} \) in the space \( (H^s(\Omega_t))^n \), where \( s \) is a non-negative real number.

We use the particular notation

\[
V_1(\Omega_t) = V(\Omega_t) \quad \text{and} \quad V_0(\Omega_t) = H(\Omega_t).
\]

The inner product of these spaces is denoted, respectively by \( (u, z)_{H(\Omega_t)} \) and \( ((u, z))_{V(\Omega_t)} \). Then for \( u = (u_1, \ldots, u_n) \) and \( z = (z_1, \ldots, z_n) \), we have

\[
(u, z)_{H(\Omega_t)} = \int_{\Omega_t} u_i(x)z_i(x)dx, \quad ((u, z))_{V(\Omega_t)} = \int_{\Omega_t} \frac{\partial u_i(x)}{\partial x_j} \frac{\partial z_i(x)}{\partial x_j} dx.
\]

Note that \( V_s(\Omega_t) \) is continuously embedded in \( (H_0^1(\Omega))^n \) for \( s \geq \frac{n}{2} \) and \( V(\Omega_t) = \{ u \in (H_0^1(\Omega))^n ; \text{div} \ u = 0 \} \).

In a similar way we introduce the spaces \( V_s(\Omega) \), where \( \sqrt{.} \) has a form

\[
\sqrt{} = \{ \psi \in (D(\Omega))^n ; \text{div} \ \psi = 0 \}.
\]

We consider, the particular notations

\( V_1(\Omega) = V, \ V_0 = H \) and \( (v, w)_H = (v, w), \ ((v, w))_V = ((v, w)), \ |v|_H = |v|, \ |v|_V = |v| \).

In order to state the variational formulation of Problem (I) we introduce the following bilinear and trilinear forms respectively:

\[
\hat{a}(t; u, z) = \int_{\Omega_t} \frac{\partial u_i(x)}{\partial x_j} \frac{\partial z_i(x)}{\partial x_j} dx, \quad (3)
\]

\[
\hat{b}(t; u, z, \xi) = \int_{\Omega_t} u_i(x) \frac{\partial z_j(x)}{\partial x_i} \xi_j(x) dx, \quad (4)
\]
We define the weak solution for the problem (I) in the following form

\[ u \in L^2(0, T; V(\Omega_t)) \cap L^\infty(0, T; H(\Omega_t)) \]

\[ \begin{align*}
    &- \int_0^T (u, \varepsilon'(t))_{(\Omega_t)} \, dt + \int_0^T \hat{a}(t; u, \varepsilon) \, dt + \\
    &+ \int_0^T \hat{b}(t; u, u, \varepsilon) \, dt = \int_0^T (f, \varepsilon)_{H(\Omega_t)} \, dt \\
    u(0) &= u_0 \\
    \forall \varepsilon \in L^2(0, T; V(\Omega_t) \cap (L^n(\Omega_t)^n), \varepsilon' \in L^2(0, T; H(\Omega_t)), \\
    \varepsilon(0) &= \varepsilon(T) = 0
\end{align*} \]

In order to transform the noncylindrical problem (II) in a problem defined in the cylinder \( Q \), we introduce the functions:

\[ u(x, t) = K(t)v(K^{-1}(t)x, t) \quad , \quad f(x, t) = K(t)g(K^{-1}(t)x, t) \]  
(5)

\[ p(x, t) = K(t)q(K^{-1}(t)x, t) \quad , \quad u_0(x) = K(0)v_0(K^{-1}(0)x). \]  
(6)

\[ a(t; v, w) = \int_\Omega \delta(t)a_{jk}(t) \frac{\partial v_i(y)}{\partial y_j} \frac{\partial w_k(y)}{\partial y_j} \, dy, \]  
(7)

\[ b(t; v, w, \psi) = \int_\Omega \delta(t)v_i(y) \frac{\partial w_j(y)}{\partial y_i} \psi_j(y) \, dy, \]  
(8)

\[ c(t; v, w) = \int_\Omega \delta(t)b_{ir}^j(t) \alpha_{rj}(t) y_j \frac{\partial v_i(y)}{\partial y_l} \, w_k(y) \, dy, \]  
(9)

\[ d(t; v, w) = \int_\Omega \delta(t) [\alpha_{rl}(t) b_{ir}^j(t) v_k(y) w_k(y) \] 
\[ - \alpha_{ir}(t) b_{ki}^j(t) v_r(y) w_k(y)] \, dy, \]  
(10)

with \( a_{jk}(t) = \beta_{jik}(t) \beta_{ki}(t) \) and \( \delta(t) = |\det K(t)| \), where \( \det M \) means the determinant of a \( n \times n \) matrix \( M \).
Then from (5) to (9) and the problem (II) we obtain the definition of weak solution for the cylinder problem:

\[
\begin{align*}
\nu & \in L^2(0,T; V) \cap L^\infty(0,T; H) \\
- \int_0^T (\delta(t)v, \psi') dt + \int_0^T a(t; v, \psi) dt \int_0^T b(t; v, v, \psi) dt \\
+ \int_0^T c(t; v, \psi) dt + \int_0^T d(t; v, \psi) dt = \int_0^T (\delta(t)g, \psi) dt \\
v(0) = v_0 \\
\forall \psi \in L^2(0,T; V \cap (L^\infty(\Omega_t))^n), \psi' \in L^2(0,T; H), \nu(0) = \psi(T) = 0.
\end{align*}
\]

### 3 Main Results

**Theorem 3.1.** Assume that the hypothesis (H1) and (H2) are satisfied. If \( f \in L^2(0,T; H(\Omega_t)) \) and \( u_0 \in H(\Omega_0) \), then there exists \( u : \bar{Q} \rightarrow \mathbb{R}^n \), solution of Problem (II).

**Theorem 3.2.** If \( g \in L^2(0,T; H) \) and \( v_0 \in H \), then there exists \( v : Q \rightarrow \mathbb{R} \), solution of Problem (III).

The following lemmas will be utilized to prove the theorems given.

**Lemma 3.3.** Consider the bilinear form \( a(t; v, w) \) defined by (7) and the operator \( A(t) \) defined by \( A(t)v = -\frac{\partial}{\partial y_l}(a_{l t}(t)\frac{\partial v}{\partial y_l}), v \in (H^1_0(\Omega))^n \). Then there exist positive constants \( a_0, a_1, a_2 \), such that

\[
\begin{align*}
(i) & \quad < A(t)v, w > = a(t; v, v)\forall v, w \in V \\
(ii) & \quad a(t; v, v) \geq a_0 \delta(t)\|v\|^2, a(t; v, v) \geq a_1 \delta(t)|v|^2, v \in V \\
(iii) & \quad |a(t; v, w)| \leq a_2 \delta(t)\|v\|\|w\|, \forall v, w \in V.
\end{align*}
\]

The proof is given in [16].

**Lemma 3.4.** Let \( b(t; v, w, \psi) \), be the trilinear form, \( c(t; v, w) \), \( d(t; v, w) \) the bilinear forms defined, respectively by (8), (9), (10). Then there exist positive constants \( b_i, c_i, d_i, i = 1, 2 \), such that
(i) \( b(t, v, v, w) = -b(t; v, w, v)\forall v \in V, w \in V_s(\Omega), s = \frac{n}{2} \)

(ii) \( |b(t; v, w, \psi)| \leq b_0 \delta(t)\|v\|\|w\|\|\psi\|_{(L^n(\Omega))^n} \forall v, w \in V, \psi \in V \cap (L^n(\Omega))^n \)

(iii) For each \( v \in V \), the linear form \( w \to b(t; v, v, w) \) is continuous in \( V_s(\Omega), s = \frac{n}{2} \) and \( b(t; v, v, w) = < B(t)v, w >_{V_s}, \) where \( B(t)v \in V_s'(\Omega) \) and \( \|B(t)v\|_{V_s'} \leq b_1\|v\|_{(L^n(\Omega))^n} \) with,

\[
\frac{1}{p} = \frac{1}{2} - \frac{1}{2n} \quad (p < q, \quad \frac{1}{q} = \frac{1}{2} - \frac{1}{n})
\]

(iv) \( |c(t; v, w)| \leq c_0 \delta(t)\|v\|\|w\|, \forall v \in V, w \in H \)

(v) For each \( v \in V \), the linear form \( w \to c(t; v, w) \) is continuous in \( H \) and \( c(t; v, w) = (C(t)v, w), \) where \( C(t)v \in H \) and \( |C(t)v| \leq c_1\|v\| \)

(vi) \( |d(t; v, w)| \leq d_0 \delta(t)\|v\|\|w\|, \forall v, w \in H \)

(vii) For each \( v \in V \), the linear form \( w \to d(t; v, w) \) is continuous in \( H \) and \( d(t; v, w) = (D(t)v, w), \) where \( D(t)v \in H \) and \( |D(t)v| \leq d_1\|v\|. \)

The proof with some modifications is analogues to the proof given in [16].

Lemma 3.5. Let \( v \in L^2(0, T; V) \cap L^\infty(0, T; H). \) Then \( v \in L^4(0, T; (L^p(\Omega))^n), \) where \( p \) is given by (11) and

\[
\|v_i(t)\|_{L^n(\Omega)} \leq c\|v_i(t)\|_{H^0_0(\Omega)}^{1/2}\|v_i(t)\|_{L^2(\Omega)}^{1/2}, \quad v = (v_1, \ldots, v_n)
\]

for some positive constant \( c. \)

The proof of Lemma 3.5 appear in [17].

Now, we consider another hypotheses over \( K(t) \), in order to state results about the existence of periodical solutions

(H3) \( K(0) = K(T) \)

(H4) Assume that there exists a positive constant \( \alpha \), such that
where \( \delta(t) = |\det K(t)| \) and \( a_1, c_0, d_0 \) are constants given in the Lemma 3.4.

**Theorem 3.6.** Assume that hypothesis \( (H1) - (H4) \) are satisfied. If \( f \in L^2(0,T; H(\Omega_t)) \) then there exists \( u : \hat{Q} \to \mathbb{R}^n \) solution of Problem (II) such that \( u(0) = u(T) \).

**Theorem 3.7.** If \( g \in L^2(0,T; H) \), then there exists \( v : Q \to \mathbb{R}^n \), solution of Problem (III) such that \( v(0) = v(T) \).

4 **Proofs**

**Proof of Theorem 2.** Let \( (w_j) \) be a special basis of \( V_s(\Omega) \), with \( s = \frac{n}{2} \). We consider the approximated problem:

\[
\begin{align*}
((\delta(t)v_m)', w_j) + a(t; v_m, w_j) + b(t; v_m, v_m, w_j) + c(t; v_m, w_j) \\
+d(t; v_m, w_j) = (\delta(t)g, w_j), \quad j = 1, \ldots, m, \quad v_m(t) \in \sqrt{m} = [w_1, \ldots, w_m].
\end{align*}
\]

\( (AP) \)

\[
v_m(0) = v_{0m}, v_{0m} \to v_0 \text{ in } H.
\]

**First Estimate.** Considering \( w_j = v_m(t) \) in \( (AP) \), and Lemma 3.4 (i) we obtain

\[
\frac{1}{2} \frac{d}{dt} (\delta(t)|v_m|^2) - \frac{1}{2} \delta'(t)|v_m|^2 + a(t; v_m, v_m) + c(t; v_m, v_m) + d(t; v_m, v_m) = (\delta(t)g, v_m).
\]

Applying Lemmas 3.3 and 3.4, we have

\[
\frac{1}{2} \frac{d}{dt} (\delta(t)|v_m|^2) + a_0 \delta(t) ||v_m||^2 \leq (\frac{1}{2} |\delta'(t)| + c_0 + d_0 + \frac{1}{2}) \delta(t)|v_m|^2
\]

\[+ \frac{1}{2} |\delta(t)g|^2.\]
Hence there exists a positive constant \( c \), such that

\[
\frac{d}{dt} (\delta(t) |v_m|^2) + \|v_m\|^2 \leq c \delta(t) |v_m|^2 + c|g|^2.
\]

Integrating in \([0, t]\), we obtain

\[
\delta(t) |v_m|^2 + \int_0^t \|v_m\|^2 ds \leq c \int_0^t \delta(s) |v_m|^2 ds + c \int_0^t |g(s)|^2 ds + \delta(0) |v_0|^2.
\]

Then Gronwall's inequality implies that,

\[
(\sqrt{\delta}v_m) \quad \text{is bounded in } \ L^\infty(0, T; H) \text{ and}
\]

\[
v_m \quad \text{is bounded in } \ L^2(0, T; V)
\]

(12)

Second Estimate. Let \( P_m \) the orthogonal projection \( P_m : H \to \sqrt{m} \)
given by \( P_m \varphi = \sum_{j=1}^{m} \langle \varphi, w_j \rangle w_j \). By the special choice of \((w_\mu)\), we have

\[
\|P_m\|_{L(V_s, V_s)} \leq 1 \text{ and, by transposition, } \|P_m^*\|_{L(V_s', V_s')} \leq 1.
\]

We note that \( P_m v'_m = v'_m \).

Multiplying the approximate equation \((AP)\) by \( w_j \), adding from \( j = 1 \) to \( j = m \), and using the notation from Lemmas 3.3 and 3.4, we obtain

\[
(\delta(t), v_m)' = -P_m^* A(t)v_m - P_m^* B(t)v_m - P_m^* C(t)v_m + P_m^* D(t)v_m.
\]

(13)

Using again Lemmas 3.4 and 3.5, we have that each term of the second member of (13) is bounded in \( L^2(0, T; V_s'(\Omega)) \). This implies that

\[
(\delta v_m)' \quad \text{is bounded in } \ L^2(0, T; V_s'(\Omega)).
\]

(14)

From (12) and (14) there exists a subsequence of \((v_m)\) still denoted by \((v_m)\) and a function \( v \), such that,

\[
v_m \rightharpoonup v \quad \text{in } \ L^2(0, T; V)
\]

(15)

\[
\delta v_m \rightharpoonup \delta v \quad \text{in } \ L^2(0, T; H)
\]

(16)

\[
(\delta v_m)' \rightharpoonup (\delta v)' \quad \text{in } \ L^2(0, T; V_s')
\]

(17)
\[ \delta v_m \rightarrow \delta v \quad \text{in} \quad L^2(0, T; H) \quad \text{strong and (a.e) in} \quad Q. \quad (18) \]

From Lemma 3.6 and (12), we get that
\[ (\delta v_{mi} v_{mj}) \quad \text{is bounded in} \quad L^2(0, T; L^\frac{p}{2}(\Omega)) \quad (19) \]

From (18) and (19), we obtain
\[ \delta v_{mi} v_{mj} \rightarrow \delta v_i v_j \quad \text{in} \quad L^2(0, T; L^\frac{p}{2}(\Omega)) \quad (20) \]

By (20) and Lemma 3.4 (i), we have
\[ b(t; v_m, v_m, w_j) \rightarrow b(t; v, v, w_j) \quad \text{in} \quad L^2(0, T) \quad (21) \]

Using (15) - (21), we can pass to the limit in the approximate equation \((AP)\), obtaining then a solution \(v\) of the problem (31).

**Proof of Theorem 1.** Let \(u(x, t) = K(t) v(K^{-1}(t)x, t)\), where \(v\) is a weak solution of Problem (II).

Let \(\epsilon(x, t) = \psi(K^{-1}(t)x, t)\), where \(\psi \in L^2(0, T; (L^n(\Omega))^n)\), \(\psi' \in L^2(0, T; H)\), \(\psi(0) = \psi(T) = 0\). Then
\[ \epsilon \in L^2(0, T; V(\Omega_t)) \cap (L^n(\Omega_t))^n, \quad \epsilon' \in L^2(0, T; H(\Omega_t)), \quad \epsilon(0) = \epsilon(T) = 0. \]

Since, \(x = K(t)y, \ y = K^{-1}(t)x, \ x_r = \alpha_{rj}(t)y_j, \ y_l = \beta_{lr}(t)x_r\), then,
\[ \frac{\partial \epsilon_i(x, t)}{\partial t} = \beta'_{ki}(t)\psi_k(y, t) + \beta_{ki}(t)\frac{\partial \psi_k(y, t)}{\partial t} + \beta_{ki}(t)\beta'_{is}(t)\alpha_{sj}(t)y_j \frac{\partial \psi_k(y, t)}{\partial y_l}. \]

Therefore, 30
\[- \int_{\Omega_t} u_i(x, t) \frac{\partial \varepsilon_i(x, t)}{\partial t} \, dx = - \int_{\Omega} \delta(t) v_i(y, t) \frac{\partial \Psi_i(y, t)}{\partial t} \, dy + \int_{\Omega} \delta(t) \alpha_{sj}(t) \beta_{ls}(t) y_j \frac{\partial v_i(y, t)}{\partial y_j} \Psi_i(y, t) \, dy \]
\[- \int_{\Omega} \delta(t) \alpha_{ir}(t) \beta'_{ki}(t) v_r(y, t) \Psi_k(y, t) \, dy + \int_{\Omega} \delta(t) \alpha_{sl}(t) \beta'_{ls}(t) v_k(y, t) \Psi_k(y, t) \, dy \quad (22)\]

\[\hat{a}(t; u, \varepsilon) = \int_{\Omega_t} \frac{\partial u_i(x, t)}{\partial x_j} \frac{\partial \varepsilon_i(x, t)}{\partial x_j} \, dx = \int_{\Omega} \delta(t) a_{jk}(t) \frac{\partial v_i(y, t)}{\partial y_j} \frac{\partial \psi_i(y, t)}{\partial y_k} \, dy \quad (23)\]

where \(a_{jk}(t) = \beta_ji(t)\beta_{ki}(t)\).

\[\hat{b}(t; u, u, \varepsilon) = \int_{\Omega_t} u_i(x, t) \frac{\partial u_j(x, t)}{\partial x_i} \varepsilon_j(x, t) \, dx = \int_{\Omega} \delta(t) v_i(y, t) \frac{\partial v_j(y, t)}{\partial y_i} \psi_j(y, t) \, dy \quad (24)\]

\[\int_{\Omega} f^T(x, t) \varepsilon(x, t) \, dx = \int_{\Omega} \delta(t) g(y, t)^T \psi(y, t) \, dy \quad (25)\]

where \(g(y, t) = k^{-1}(t)f(K(t), t)\).

Integrating (22) - (25) in \([0, T]\) and using the definitions (7) - (10), we conclude that \(u\) is a weak solution of the problem (I). \(\square\)

**Proof of Theorem 3.** Let \((w_j)\) be a special basis of \(V_s(\Omega)\), where we fix \(s = \frac{\alpha}{2}\). We consider the approximate problem:

31
\[
(\delta(t)v_m)', w_j) + a(t; v_m, w_j) + b(t; v_m, v_m, w_j) + c(t; v_m, w_j)
\]

\[(IV) \quad +d(t; w_m, w_j) = (\delta(t)g, w_j), j = 1, \ldots m, v_m(t) \in \sqrt{m} = [w_1, \ldots w_m].
\]

\[v_m(0) = v_0, \quad v_0 \text{ arbitrary}\]

We prove that there exists \( R > 0 \), independent of \( m \), such that

\[|v_0| \leq R \Rightarrow |v_m(T)| \leq R.\]

In fact, if we consider \( w_j = v_m \) in \((IV)\), we have

\[
\frac{1}{2} \delta(t) \frac{d}{dt} |v_m|^2 + a(t; v_m, v_m) + c(t; v_m, v_m) + d(t; v_m, v_m)
\]

\[= \delta'(t)|v_m|^2 + (\delta(t)g, v_m).
\]

From \((26)\) and Lemmas 3.3 and 3.4, we have

\[
\frac{1}{2} \delta(t) \frac{d}{dt} |v_m|^2 + a_1 \delta(t)|v_m|^2 \leq \left( \frac{\delta'(t)}{\delta(t)} + c_0 + d_0 \right) \delta(t)|v_m|^2 + (\delta(t)g, v_m).
\]

From \((H3)\) and the Schwartz inequality, we have

\[
\frac{d}{dt} |v_m|^2 + \alpha|v_m|^2 \leq \frac{1}{2} |g|^2 \quad \text{then}
\]

\[
\frac{d}{dt} (e^{\alpha t} |v_m|^2) \leq \frac{1}{\alpha} e^{\alpha t} |g|^2, \quad \text{or}
\]

\[e^{\alpha t} |v_m(T)|^2 \leq |v_0|^2 + \frac{1}{\alpha} \int_0^T e^{\alpha t} |g|^2 dt \leq |v_0|^2 + c \leq R^2 + c.
\]

Hence,

\[|v_m(T)|^2 \leq \frac{R^2 + c}{e^{\alpha T}} \leq R^2. \quad \text{Therefore} \quad R^2 \geq \frac{c}{e^{\alpha T} - 1}.
\]
Then, the mapping $v_0 \rightarrow v_m(T)$ apply $B_R$ in $B_R$, where $B_R$ is the disc of radius $R$, centered in the origin contained in the $V_m$ space with the topology $| \cdot |_{H(\Omega)}$. Therefore, there exists $u_{0m} \in B_R$, such that $v_m(t) = v_{0m}$.

Let $v_m$ be the solution of approximate problem $(IV)$ such that $v_m(0) = v_{0m}$. Since $u_{0m}$ is bounded in $H$, then we have:

$$
v_m \text{ is bounded in } L^2(0,T; V)
$$

$$
\delta v_m \text{ is bounded in } L^\infty(0,T; H)
$$

$$
(\delta v_m)' \text{ is bounded in } L^2(0,T; V'_s(\Omega)).
$$

Therefore, we can extract a subsequence $(v_m)$, such that

$$
v_m \rightharpoonup v \text{ weak in } L^2(0,T; V)
$$

$$
\delta v_m \rightharpoonup \delta v \text{ weak star in } L^2(0,T; H)
$$

$$
(\delta v_m)' \rightharpoonup (\delta v)' \text{ weak in } L^2(0,T; V'_s)
$$

$$
\delta v_m \rightarrow \delta v \text{ in } L^2(0,T; H) \text{ and (a.e) in } Q
$$

Then $v_m(0) \rightarrow v(0), \ v_m(T) \rightarrow v(T)$ in $V'_s$. Since $v_m(0) = v_m(T)$, then $v(0) = v(T)$. □

**Proof of Theorem 4.** Let $u(x, t) = k(t)u(K^{-1}(t)x, t)$, where $u$ is a solution of Problem (II), such that

$$
u(0) = v(T). \quad (27)
$$

We know that $u$ is solution of (I). Then from $(H_3)$ and (27), we have $u(0) = u(T)$. □

200 Math. Subject Classification - 35Q30, 76D05

**References**


Cruz S. Q. de Caldas, Juan Limaco e Rioco K. Barreto
Departamento de Matemática Aplicada
Universidade Federal Fluminense IMUFF RJ, Brasil.
gmacruz@vm.uff.br; rikaba@vm.uff.br

Pedro Gamboa
Instituto de Matemática
Universidade Federal do Rio de Janeiro, UFRJ, Brasil
pgamboa@dmm.im.ufrj.br