PRO MATHEMATICA

VOLUMEN XXXI / No. 61 / 2020 ISSN 2305-2430

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PRO MATHEMATICA

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CONTENIDO

Estabilidad estructural de campos suaves por partes en superficies

Rode Checya[1]

Abril, 2019

Resumen

En este trabajo, consideramos campos de vectores suaves por partes definidos en una superficie compacta. El problema que estudiamos es la caracterización de la estabilidad estructural de campos de vectores suaves por partes. Después de M. Peixoto, J. Palis y A. F. Filippov, vemos que las condiciones necesarias y suficientes son: hiperbolicidad de puntos singulares, genericidad de tangencias, no conexión de sillas singulares y sólo órbitas recurrentes triviales. Estas condiciones fueron adaptadas por Brouke, Pugh y Simic para campos de vectores suaves por partes. Mostramos que para campos de vectores suaves por partes la estabilidad estructural es una propiedad genérica local desde un punto de vista diferente, y de ahí que caracterizamos al conjunto de los campos suaves por partes que son estructuralmente estables.

MSC(2010): Primary 34D30; Secondary 37C20.

Palabras clave: Estabilidad estructural, singularidades, genericidad.

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1 Introducción

Los sistemas dinámicos que describen procesos físicos donde el campo es suave por partes poseen por lo general un comportamiento discontinuo. Los dispositivos electrónicos con diodos, transistores o interruptores, los dispositivos mecánicos con engranaje, así como fenómenos físicos que involucran fricción, deslizamiento o colisión son buenos ejemplos de lo anterior. Encontramos en los sistemas —en principio evolutivos— que pueden ser interrumpidos por eventos instantáneos una fuente inagotable de tales fenómenos.

El concepto de estabilidad estructural se inició con los trabajos de Andronov y Pontrjagin [1]. Ambos pretendían caracterizar los campos de vectores en términos de órbitas periódicas, singularidades, puntos hiperbólicos u otros elementos simples frecuentes. Smale generalizó la idea al definir un comportamiento estable como aquel que no desaparece bajo pequeños cambios en el sistema. Introdujo la noción de "estabilidad estructural", entendida ahora como la característica de un flujo por la cual su topología se preserva bajo modificaciones controladas de las ecuaciones que lo describen, digamos, a pesar de perturbaciones suficientemente pequeñas en los coeficientes.

Peixoto refina el concepto y prueba la densidad de los sistemas estructuralmente estables definidos en regiones compactas del plano y en superficies orientables compactas. Para ello construye explícitamente un espacio de Banach en donde formula cuestiones de apertura y densidad asociados a sistemas dinámicos. También impone condiciones, además de las de Andronov-Pontrjagin, para garantizar estabilidad estructural. Entre ellas cabe destacar la no existencia de soluciones recurrentes no triviales; es decir, el sistema presenta apenas puntos singulares y órbitas periódicas [10].

Para campos planares suaves por partes, el problema de estabilidad fue atendido por Kozlova [6] y Filippov [4]. Para el caso de superficies compactas, la referencia básica es el trabajo de Broucke, Pugh y Simic (ver [3]). En el presente artículo analizamos, verificamos y reformulamos tales resultados. Abordamos específicamente el problema de estabilidad estructural del espacio de sistemas discontinuos definidos sobre superficies.

En la sección 2 exhibimos una colección de resultados básicos relacionados con la topología C^r, con la equivalencia topológica y con la estabilidad estructural.

En la sección 3, de acuerdo con la convención de Filippov, definimos los conceptos de órbita solución regular y de desliz. Se establece las condiciones necesarias para la estabilidad estructural local.

En la sección 4 estudiamos los equilibrios singulares hiperbólicos.

Finalmente, en la sección 5, a partir del análisis del comportamiento local trabajado en las secciones previas, se establece la caraterización de los campos suaves por partes estructuralmente estables.

2 Campos de vectores suaves por partes

Un sistema dinámico descrito por un **campo de vectores suave por partes** es un conjunto finito de ecuaciones diferenciales ordinarias, denotado por $\dot{x} = X_i(x)$, donde $x \in S_i \subset \mathbb{R}^n$, tal que los S_i son regiones abiertas no yuxtapuestas separadas entre sí por subvariedades de codimensión 1; es decir $\Sigma_{ij} = \overline{S_i} \cap \overline{S_j}$ es una hipersuperficie de dimensión $n-1$. Asumimos a $X_i : \overline{S_i} \to \mathbb{R}^n$ de clase C^1. Las discontinuidades del campo se encuentran en las fronteras Σ_{ij}. La unión de las fronteras y los S_i ocupan todo el espacio.

De aquí hasta el final M denota una superficie orientable compacta, suave y sin frontera; además, K denota un complejo simplicial finito 1-dimensional contenido en M; es decir, K es un conjunto de segmentos unidos por sus extremos e incrustado de manera lisa en M. En adelante, nos referimos a K como la **región de discontinuidad del campo**. Por motivos técnicos asumimos que cada lado pertenece a la clausura de exactamente dos componentes de $M - K$ referidas como **adyacentes**.

También asumimos que el ángulo entre dos lados incidentes en un mismo vértice es no nulo.

Un **campo de clase C^r por partes** ($1 \leq r \leq \infty$) en M es una familia $\{X_i\}$ con X_i de clase C^r y definido sobre la clausura $\overline{G_i}$ de la i-ésima componente conexa G_i de $M - K$, para $i = 1, 2, \ldots, k$. Los correspondientes X_i son las **ramas** de X (ver figura 1).

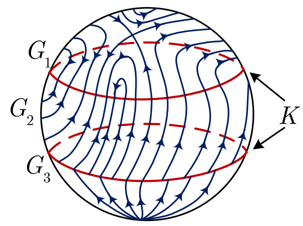
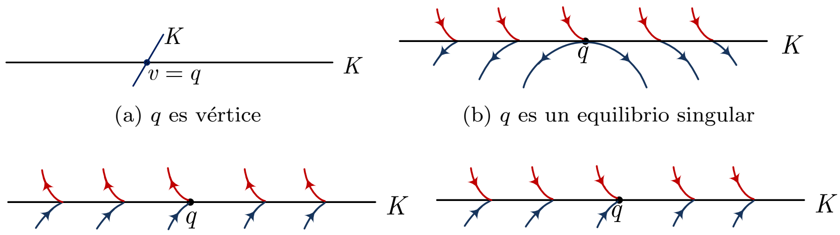


Figura 1: Un campo de vectores suave por partes en la esfera S^2.

Sea $K \subset M$ un complejo simplicial como arriba. Numeramos las componentes conexas de $M - K$ cual $G_1, \ldots, G_k$. Sea $\mathfrak{X}_K^r$ el conjunto de todos los campos suaves por partes $X = (X_1, X_2, ..., X_k)$ de clase C^r con región de discontinuidad K.

Una vez fija la inmersión de M en $\mathbb{R}^3$, los X_i pueden realizarse como campos tangentes a M. Es más, el conjunto $\mathfrak{X}_K^r$ adquiere estructura de espacio de Banach con la norma del máximo (entre las restricciones a M de las derivadas de orden hasta r inclusive).

Para analizar el comportamiento de las órbitas de estos elementos precisamos conocer la naturaleza de los puntos en K. En primer lugar están los **vértices**. Aquellos puntos en K que no son vértices, es decir los lados, pertenecen por definición a dos regiones adyacentes. Los distinguimos por el comportamiento a cada uno sus dos lados. Un punto de discontinuidad es de **tangencia** cuando al menos una de las dos ramas de X es tangente a K (esto incluye la posibilidad de que una rama se anule en este punto). Será punto de **cruce** si las ramas de X son transversales a K y ambas apuntan hacia un mismo lado. Es punto de **oposición** cuando las ramas son transversales a K pero apuntan a lados opuestos. En la figura 2 se presenta un esquema de las cuatro posibilidades.



(a) q es vértice (b) q es un equilibrio singular (c) q es punto de cruce (d) q es punto de oposición

Figura 2: Los diferentes tipos de puntos del conjunto K.

Por **doble tangencia** nos referimos a aquella en la que la tangencia aparece en ambas ramas adyacentes.

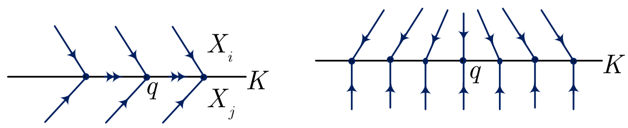
Gracias a la compacidad de K, cada rama X_i puede extenderse a una vecindad de $\overline{S_i}$. Si bien las extensiones pueden diferir, las órbitas del campo están bién definidas mientras permanezcan en la región $\overline{S_i}$.

Sea $q \in K$ un punto de oposición. Un **campo deslizante** X^* **en** q es la única combinación estrictamente convexa

$$X^*(q) = \lambda X_i(q) + (1 - \lambda) X_j(q), \qquad 0 < \lambda < 1$$

que resulta tangente a K en q. Resaltamos que X^* en este caso está bien definido apenas en un entorno de q en K.

Un punto de oposición es **deslizante** si $X^*(q) \neq 0$, y es **equilibrio singular** si $X^*(q) = 0$. Ver figura 3.



(a) q es punto deslizante (b) q es un equilibrio singular

Figura 3

Una **singularidadad** de $X \in \mathfrak{X}^r_K$ es bien un equilibrio singular, bien un punto de tangencia, o bien un vértice de K. Por motivos que se harán obvios en los párrafos siguientes, excluimos a los puntos (interiores) de G_i donde el campo X_i se anula y la órbita es estacionaria.

Una **órbita regular** de $X \in \mathfrak{X}^r_K$ es una curva suave por partes γ en M de modo que la intersección $\gamma \cap G_i$ es una trayectoria de X_i, acá se sobreentiende que la intersección con la frontera $\gamma \cap K$ consiste solamente de puntos de cruce. Nos interesan sobremanera aquellas órbitas regulares maximales respecto a estas condiciones.

Una **órbita singular** de $X \in \mathfrak{X}^r_K$ es una curva suave $\gamma \subset K$ tal que γ es una órbita de X^*, o una singularidad de X.

Observación 2.1. Es posible concatenar órbitas regulares con órbitas singulares; mecanismo conocido por **convención de Filippov**. Esto ocurre, por ejemplo, si γ llega a un punto singular q en tiempo finito y β se aleja de q en tiempo finito. Como consecuencia, pueden coexistir órbitas distintas a través de una singularidad $q \in K$. Con el fin de asegurar la unicidad de órbitas a través de un singularidades, deberemos restringir en lo posible el uso de concatenaciones.

Sean dos campos X, $Y \in \mathfrak{X}^r_K$. Una **equivalencia topológica** entre X e Y es un homeomorfismo $h : (M, K) \to (M, K)$ que lleva órbitas de X en órbitas de Y y lleva órbitas en K en órbitas de K, a la vez que preserva la orientación de las mismas. En este caso decimos que X e Y son **topológicamente equivalentes**.

Una órbita γ **sale de** $q \in K$ si se tiene $\lim\limits_{t \to 0^+} \gamma(t) = q$. Por otro lado, **entra a** $q \in K$ cuando $\lim\limits_{t \to 0^-} \gamma(t) = q$. (Ver figura 4).

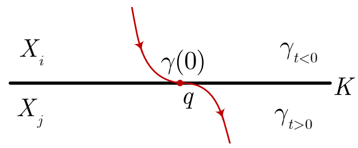


Figura 4: Una órbita γ que entra al punto q y sale del mismo.

Las separatrices se describen de manera heurística como lugares alrededor de los cuales muchas trayectorias pasan cierto tiempo para finalmente abandonarlo.

Una **separatriz inestable** es una órbita regular cuyo conjunto α-límite es un punto silla o se aparta de una singularidad del campo X (ver figura 5). Una **separatriz estable** es una órbita regular cuyo conjunto ω-límite es un punto silla o llega a una singularidad del campo X.

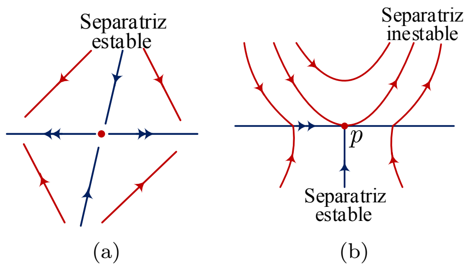


Figura 5: Se muestran separatrices inestables y estables: (a) alrededor de un punto silla singular, (b) alrededor de un punto de tangencia.

Si una separatriz es estable e inestable a la vez, se le denomina **separatriz conectora**. Dos separatrices inestables son **relacionadas** si llegan a un mismo punto (ver figura 6). Otros ejemplos se aprecian en la figura 7.

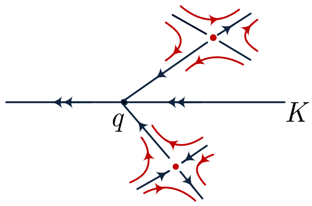
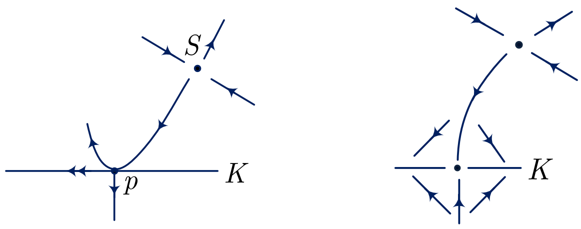


Figura 6: Separatrices relacionadas



(a) Conexión entre silla y tangencia

(b) Conexión entre silla y silla singular

Figura 7

3 Estabilidad estructural

Uno de los objetivos de la teoría de los sistemas dinámicos es clasificar la dinámica según la estructura geométrica de las órbitas; es decir, vía el estudio de su retrato de fase. Bajo un punto de vista geométrico la estabilidad estructural de un campo vectorial se entiende como la reticencia de su retrato de fase a no alterarse topológicamente ante pequeñas perturbaciones. Para campos suaves por partes, la estructura de las órbitas depende, entre otros factores, del comportamiento en la región de discontinuidad. Veremos en breve cómo la parte determinante del estudio recae sobre las propiedades del llamado campo deslizante, definido en la región de oposición de K.

Broucke, Pugh y Simic [3] plantean el caso para la esfera. En ese contexto, tomemos por ejemplo X como un campo suave por partes con dos ramas X_+ y X_-, definidas en el hemisferio boreal y austral, respectivamente: un punto en el hemisferio norte se mueve por X_+, caso contrario lo hace por X_-. Si el punto está en el ecuador (la región de discontinuidad) su futuro dependerá del contexto. Siempre es posible convenir un movimiento natural; ver por ejemplo la figura 8.

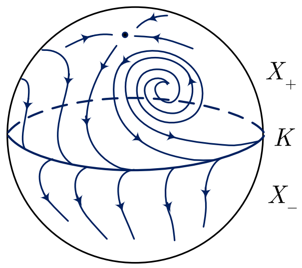


Figura 8: Estructura de las órbitas sobre S^2.

De ser el conjunto de los campos estructuralmente estables "grande" en algún sentido, entonces éste adquirirá preponderancia, pues nos acerca a lo que en términos topológicos es una propiedad genérica. Recuérdese que para campos de vectores no discontinuos, la estabilidad estructural es una propiedad genérica, es decir, se verifica en un abierto denso. Sin embargo, para campos suaves discontinuos algunas condiciones tendrán que ser reconsideradas.

Un campo $X \in \mathfrak{X}^r_K$ es **estructuralmente estable** si existe una vecindad $U \subset \mathfrak{X}^r_K$ de X tal que cada campo $Y \in U$ es topológicamente equivalente a X: en otras palabras, existe un homeomorfismo ambiental que transforma órbitas de X en órbitas de Y. Denotamos por Ω^r_K al conjunto de todos los campos $X \in \mathfrak{X}^r_K$ estructuralmente estables.

Ejemplo 3.1. Consideremos la estructura de órbitas determinada por un campo suave por partes sobre la esfera, como aquel mostrado en el lado izquierdo de la figura 9. Acá se aprecia una separatriz conectora que surca desde un punto silla ubicado en el hemisferio norte hasta un punto de tangencia. Al lado derecho observamos una ligera perturbación que carece de la conexión entre la silla y una tangencia. Como el campo y su perturbado no son equivalentes, resulta que el campo de referencia no merece ser llamado estructuralmente estable.

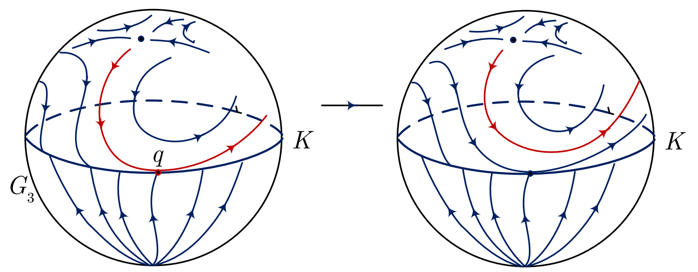


Figura 9: Estructura de órbitas inestable.

Ejemplo 3.2. Sea el cuadrado $[0,1]\times[0,1]$ dividido por el segmento L_1 con extremos $\left(\frac{1}{2},0\right)$ y $\left(\frac{1}{2},1\right)$ (ver figura 10).

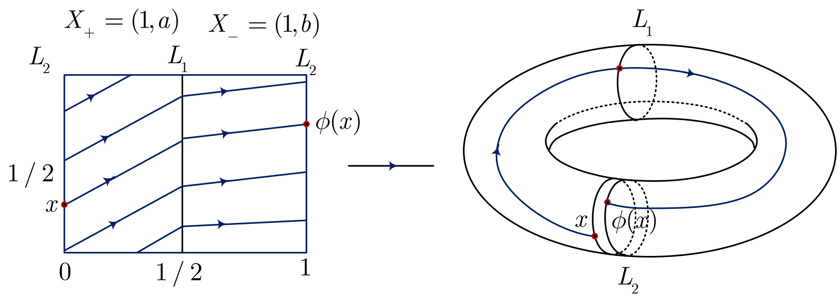


Figura 10: Estructura de órbitas sobre el toro.

El segmento L_1 divide al cuadrado en dos partes, en ellas se definen los campos constantes $X_+ = (1, a)$ y $X_- = (1, b)$. El cuadrado representa un toro, que se obtiene al identificar lados opuestos del cuadrado.

De ese modo, la región de discontinuidad consta de dos círculos $\mathcal{C}_0$, $\mathcal{C}_{1/2}$ identificados con los segmentos verticales indicados a la izquierda. Obsérvese que las órbitas de las dos regiones se concatenan. Por ejemplo, si empezamos en el segmento vertical correspondiente a $x = 0$ (es decir desde $\mathcal{C}_0$), nos habremos elevado $a/2$ unidades al llegar a $x = 1/2$ (es decir, a $\mathcal{C}_{1/2}$), de ahí tras $1/2$ unidad de tiempo adicionales acabaremos $b/2$ unidades "más arriba". En resumen, tras una unidad en el reloj, se completa una vuelta al círculo, mas al regresar a $\mathcal{C}_0$ habremos rotado $(a + b)/2$ vueltas a lo largo del meridiano; función que denotamos por $\phi : \mathcal{C}_0 \to \mathcal{C}_0$. Si se considera pequeñas perturbaciones en a o b (por ejemplo, el tránsito de rotación racional a irracional), la estructura de órbitas cesa de coincidir y el campo resulta inestable.

Por motivos técnicos, dado un segmento $E \subset K$, conviene fijar un sistema de coordenadas (U, ϕ), con $E \subset U$, de modo que $\phi(E)$ sea un intervalo sobre el eje x, digamos $\phi(E) = [-1, 1]$ (ver figura 11).

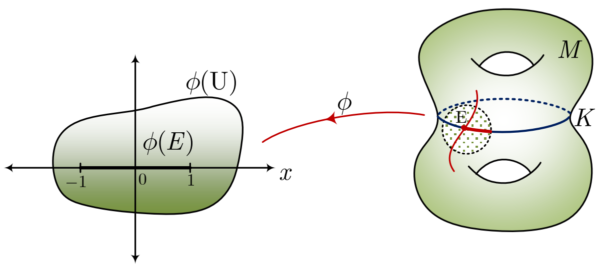


Figura 11: Una coordenada que cubre parte de una arista de K.

En este sistema coordenado las ramas se expresan "localmente" cual X_1 y X_2, definidas una sobre el semiplano cerrado superior, la otra sobre el inferior:

$$X(x,y) = \begin{cases} X_1(x,y) = (f_1(x,y), g_1(x,y)); & y \geq 0 \\ X_2(x,y) = (f_2(x,y), g_2(x,y)); & y \leq 0. \end{cases} \tag{3.1}$$

El conjunto de los puntos de oposición de K es relativamente abierto. De acuerdo con la convención anterior, para uno de ellos, digamos $q = (x,0)$, las funciones g_i y g_j no se anulan, mas poseen signos opuestos. De este modo cualquier combinación convexa toma la forma

$$\begin{aligned} \lambda X_i(x,0) + (1-\lambda) X_j(x,0) &= \lambda \left(f_i(x), g_i(x)\right) + (1-\lambda)\left(f_j(x), g_j(x)\right) \\ &= (\lambda f_i(x) + (1-\lambda) f_j(x), \lambda g_i(x) + (1-\lambda) g_j(x)). \end{aligned}$$

Al ser el campo deslizante X^* la única combinación estrictamente convexa y tangente a K, su componente vertical $\lambda g_i(x) + (1-\lambda) g_j(x)$ debe anularse. De ese modo, se tiene la igualdad

$$\lambda \quad = \quad \frac{g_j(x)}{g_j(x) - g_i(x)} \tag{3.2}$$

y el campo deslizante X^*_{ij} queda expresado cual

$$\begin{aligned} X^*_{ij}(x,0) &= \frac{g_j(x)}{g_j(x) - g_i(x)} X_i(q) + \left(1 - \frac{g_j(x)}{g_j(x) - g_i(x)}\right) X_j(q) \\ &= \frac{g_j(x)}{g_j(x) - g_i(x)} X_i(q) + \left(\frac{-g_i(x)}{g_j(x) - g_i(x)}\right) X_j(q) \\ &= \frac{g_j(x) X_i(q) - g_i(x) X_j(q)}{g_j(x) - g_i(x)}. \end{aligned}$$

Necesitamos un concepto adicional prestado del análisis real básico. Una función diferenciable $f : [a,b] \to \mathbb{R}$ tiene **ceros genéricos** si no se anula en los extremos y $f(x) = 0$ implica $f'(x) \neq 0$. El siguiente resultado justifica el nombre.

Lema 3.3. *Una función genérica de clase C^r posee ceros genéricos.*

Prueba. Obsérvese que se desprende de inmediato de la definición que todos los ceros de f son interiores y están aislados. En otras palabras, 0 es un valor regular de f. El resultado se sigue de técnicas estándar de teoría de Morse (cf. [7]). $\square$

Proposición 3.4. *El conjunto Ω de campos $X \in \mathfrak{X}_K^r$ que satisfacen las propiedades*

a) las restricciones X_i son de Morse-Smale,
b) los X_i no se anulan en K,
c) las tangencias de X_i con K aparecen en número finito y lejos de los vértices, además ninguna es doble (es decir, por ambos costados), pero todas son parabólicas (es decir, cuadráticas) y
d) los X_i son colineales a lo mucho en un número finito de puntos de K, ninguno de los cuales es un vértice

es abierto y denso en $\mathfrak{X}_K^r$.

Prueba. Fijemos un sistema de coordenadas (U, ϕ) con $\phi(E) = [-1, 1]$ (ver figura 11) y tomemos X con ramas descritas por (3.1). Definimos los conjuntos

$$\begin{aligned}
\Omega_a &= \{X \in \mathfrak{X}_K^r : \text{cumple la propiedad a}\},\\
\Omega_{ab} &= \{X \in \Omega_a : \text{cumple la propiedad b}\},\\
\Omega_{abc} &= \{X \in \Omega_{ab} : \text{cumple la propiedad c}\},\\
\Omega_{abcd} &= \{X \in \Omega_{abc} : \text{cumple la propiedad d}\}.
\end{aligned}$$

Sea $X \in \Omega_a$. El teorema de Peixoto [10] afirma que cuando las ramas X_i de X son campos Morse-Smale, el campo es genérico. Así, Ω_a es abierto y denso en $\mathfrak{X}_K^r$.

El conjunto $\Omega_{ab} = \{X \in \Omega_a : X_i(p) \neq 0\,, p \in K\}$ es abierto y denso ya que este conjunto coincide con $\{X \in \Omega_a : X_i|_K$ es transversal a $\{0\}\}$. En efecto, si $X_i|_K$ es transversal a $\{0\}$, o bien se tiene $X_i|_K(q) \neq 0$, o en su defecto se cumple $X_i|_K(q) = 0$ y $DX_i|_K(q) \cdot \mathbb{R} + T_q\{0\} = T_q\mathbb{R}^2$. Pero lo último no ocurre ya que las dimensiones no coinciden. Luego, por el teorema de transversalidad de Thom el conjunto Ω_{ab} es abierto y denso en Ω_a.

El conjunto Ω_{abc} está conformado por campos $X \in \Omega_{ab}$ con un número finito de tangencias, ninguna doble pero todas cuadráticas. Una rama $X_i(q)$ será tangente en $q \in E \subset K$ si y sólo si se tiene $g_i(q) = 0$; y será cuadrática única y exclusivamente cuando se tenga en simultáneo $f_i(q) \neq 0$ y $\dfrac{\partial g_i}{\partial x}(q) \neq 0$. En efecto, por definición se cumple

$$X_i h(x,0) = \langle X_i(q), \nabla h(q)\rangle = \langle (f_i(q), g_i(q)), (0,1)\rangle = g_i(q) = 0.$$

Y de ahí se pasa a

$$\begin{aligned} X_i^2 h(x,0) &= \langle X_i(q), \nabla X_i h(q)\rangle \\ &= \left\langle (f_i(x), g_i(x)) ; \left(\frac{\partial g_i}{\partial x}(x), \frac{\partial g_i}{\partial y}(x)\right)\right\rangle \\ &= f_i(x)\frac{\partial g_i}{\partial x}(q) + g_i(x)\frac{\partial g_i}{\partial y}(q) \\ &= f_i(x)\frac{\partial g_i}{\partial x}(x) \neq 0. \end{aligned}$$

Es decir, X_i no acepta tangencias en vértices de K si y sólo si g_i no se anula en los extremos de E, y no acoge doble tangencia en q si y sólo si $g_i(q) = 0$ implica $g_j(q) \neq 0$. Es decir, el conjunto Ω_{abc} coincide con el conjunto de los campos para los cuales h_{ij}, donde $h_{ij} : E \to \mathbb{R}^2$ está dada por $h_{ij}(x) = (g_i(x), g_j(x))$, es transversal a $\{0\}$. Por el teorema de transversalidad, el conjunto Ω_{abc} es abierto y denso en Ω_{ab}.

El conjunto Ω_{abcd} es denso y abierto en Ω_{abcd}. En efecto, sea $x_0 \in \phi(E) = [-1,1]$. Si $X \in \Omega_{abc}$, entonces por c) o $g_i(x_0) \neq 0$ o $g_j(x_0) \neq 0$. Supongamos que se teng $g_j(x_0) \neq 0$. Entonces existe un intervalo $I \subset [-1,1]$ que contiene a x_0 en donde se tiene $g_j \neq 0$ (es decir X_j no es tangente a K). En uso de la carta ϕ asociada a la caja de flujo para X_j en I, podemos asumir que se cumple $f_j(x) = 0$ y $g_j(x) > 0$ para todo $x \in I$ (en estas coordenadas de la caja del flujo el vector X_j apunta directamente hacia arriba). De este modo una colinealidad entre X_i y X_j ocurre exclusivamente cuando se tiene $f_i(x) = 0$. Por el lema (3.3), ello acontece para X solo en número finito y nunca en los extremos de I. Luego, por el teorema de transversalidad, se concluye el resultado.

Finalmente, en virtud de la transitividad, resulta que Ω es denso y abierto. □

A modo de ilustración consideramos el campo $Y = (Y_1, Y_2)$ de la figura 12. Las ramas de Y no satisfacen la propiedad 1, es decir, no son Morse-Smale. Tampoco se respeta la propiedad 2 pues se exhibe una silla que muere en $q \in K$.

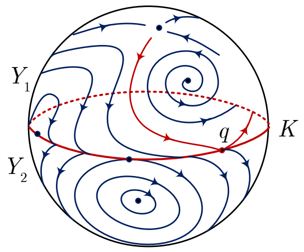


Figura 12: Un posible estructura de órbitas de $Y \notin \Omega$.

Proposición 3.5. *Dado $X \in \Omega$, existe una vecindad $\mathcal{V}$ de X en $\mathfrak{X}^r_K$ tal que cada $Y \in \mathcal{V}$ posee en común con X*

- *el mismo número de puntos críticos y del mismo tipo topológico,*
- *igual número de tangencias del mismo tipo topológico,*
- *igual cantidad de equilibrios singulares del mismo tipo e*
- *igual número de órbitas cerradas.*

Afirmar que son del mismo tipo topológico significa geométricamente que son equivalentes, es decir, que son topológicamente conjugados.

Prueba. La mayoría de propiedades son explícitas de la proposición 3.5. Las restantes se amparan en la definición de campo Morse-Smale en la parte a) del mismo resultado. □

4 Equilibrios singulares hiperbólicos

Analizamos algunas propiedades de los campos genéricos. Observamos que el campo deslizante X^* está definido en la región de oposición mas no así en vértices de K.

Proposición 4.1. *El campo deslizante X^* asociado a un campo genérico X es de clase C^r. Además, los puntos críticos de X^*, en caso existan, son atractores o repulsores, hiperbólicos en ambos casos.*

Prueba. Debido a la ecuación (3.2) el campo X^* resulta claro que es de clase C^r. Para demostrar que los ceros de X^* son hiperbólicos se troca la carta ϕ alrededor de $q \in K$ por una carta de caja de flujo ψ para hacer $f_j \equiv 0$. Luego, la colinealidad ocurre si $f_i(x) = 0$.

Al ser X genérico, en uso del lema 3.3 se tiene que $f_i'(x) \neq 0$ cuando $f_i(x) = 0$. De este modo, en el sistema de coordenadas (ψ, U'), el punto crítico $q = (x, 0)$ de X^* es hiperbólico repulsor cuando $f_i'(x) > 0$ y atractor si $f_i'(x) < 0$. $\square$

En el transcurso de la siguiente proposición definiremos y catalogaremos los diferentes tipos de singularidades en K.

Proposición 4.2. *El número de singularidades de un campo genérico es finito y solo admiten las siguientes posibilidades:*

- *silla singular (equilibrio singular);*
- *atractor singular (equilibrio singular);*
- *repulsor singular (equilibrio singular);*
- *nodo-silla singular (tangencia visible);*
- *montículo singular (tangencia invisible);*
- *vértice de K.*

Todas son estables bajo pequeñas perturbaciones.

Prueba. Sea X un campo genérico y $q \in K$ un equilibrio singular o un punto de tangencia pero no un vértice. Por la proposición (3.4) la genericidad implica que por lo menos una rama de X es transversal a

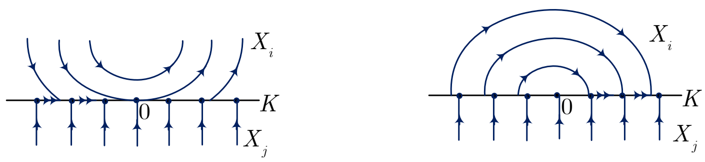
K en q, digamos X_j. Como en la prueba de la proposición (3.4), en uso de una carta de la caja flujo podemos asumir $f_j \equiv 0, g_j = 1$ y tomar $q = (0,0)$. Observemos que se tiene $g_i(0) \leq 0$, pues de no ser así X_i y X_j apuntarían hacia arriba a través del eje x en q; y resultaría que q es regular y no singular, una contradicción. También, la genericidad implica que o bien $f_i(0) = 0$ y $g_i(0) \neq 0$, o en su defecto $f_i(0) \neq 0$ y $g_i(0) = 0$, ya que la nulidad de ambos contradiría la condición b) de la proposición (3.5). De ahí que podemos destinguir dos casos bien diferenciados topológicamente. Veamos.

Caso 1. Cuando $f_i(0) = 0$, $g_i(0) < 0$, $f_i'(0) \neq 0$, aparecen como posibilidades $f_i'(0) > 0$ y $f_i'(0) < 0$. De tenerse $f_i'(x) > 0$, debido a $f_i(0) = 0$, existe una única X_i-órbita que llega al origen desde arriba: en este caso decimos que tenemos **una silla singular con separatriz regular estable**. Si $f_i'(x) < 0$ la misma X_i-órbita que llega desde arriba al origen abre paso a un **atractor singular** (ver figura 13).

(a) Silla singular (b) Atractor singular

Figura 13

Caso 2. Cuando $f_i(0) \neq 0$, $g_i(0) = 0, g_i'(0) \neq 0$, aparecen nuevamente dos posibilidades $g_i'(0) > 0$ o $g_i'(0) < 0$. Si $g_i'(0) > 0$, se obtiene una tangencia $X_i h(x,y) = \langle X_i(x,y); (0,1)\rangle = \langle (f_i, g_i); (0,1)\rangle = g_i(x,y)$ con $h(x,y) = y$ y $h^{-1} = \phi$. Este es el caso denominado **nodo-silla**. En este caso apreciamos tres separatrices: dos de ellas estables y una inestable (ver figura 14 (a)). Si $g_i'(x) < 0$, obtenemos una tangencia "invisible" comúnmente llamada **montículo singular**. Acá se vislumbra una única separatriz, la misma que es estable (ver figura 14 (b)). En ambos casos, sin embargo, el signo de $f_i(0)$ es irrelevante pues su efecto es el virar la dirección, sea hacia la izquierda o la derecha de K.



(a) Nodo-silla singular
(b) Montículo singular

Figura 14

La genericidad y finitud de las singularidades ya fue anotada en la proposición 3.5 □

5 Estabilidad estructural en $\mathfrak{X}^r_K$

Presentamos el teorema de caracterización de los campos suaves por partes estructuralmente estables definidos en una superficie compacta. Tomamos prestada de [7] la definición de órbita recurrente: una órbita γ es **recurrente** si está contenida en el conjunto ω-límite de γ o en el conjunto α-límite de γ. Las **órbitas recurrentes triviales** son los puntos críticos y las órbitas cerradas.

Escribimos Ω^r_K para denotar al conjunto de los campos estructuralmente estables. El siguiente teorema los caracteriza.

Teorema 5.1. *Un campo $X \in \mathfrak{X}^r_K$ es estructuralmente estable si y solo si*

- *satisface las condiciones de la proposición 3.5,*
- *todas sus órbitas periódicas son hiperbólicas,*
- *carece de separatrices conectadas y de separatrices relacionadas y*
- *sólo admite órbitas recurrentes triviales.*

Prueba. Si $X \in \mathfrak{X}^r_K$ es un campo que satisface las condiciones del teorema, entonces tiene un número finito de singularidades: usaremos r_i para repulsores, a_i para atractores, s_i para sillas, q_i para tangencias y

p_i para equilibrios singulares, según convenga. Cada silla, tangencia o equilibrio singular tiene asociado separatrices; las separatrices estables nacen de un repulsor y las inestables mueren en un atractor. Por la proposición 3.5 existe una vecindad $\mathcal{V}$ de X en la cual todo $\tilde{X} \in \mathcal{V}$ posee el mismo número y tipo de singularidades y separatrices que X.

Sean V_l vecindades de los r_l y U_k de los a_k. Similarmente tomemos $\tilde{V}_l$ asociados con $\tilde{r}_l$ y $\tilde{U}_k$ con $\tilde{a_k}$; las fronteras de estas vecindades han de ser transversales al campo, ver figura 15. Las separatrices cortan las fronteras ∂V_l, $\partial \tilde{V}_l$ en arcos. El homeomorfismo h es construido mediante fiel seguimiento de lo pasos que se enumeran a continuación.

Primero, como es lógico, realizamos la asignación puntual. $h(p_i) = \tilde{p_i}$, $h(r_i) = \tilde{r_i}$, $h(s_i) = \tilde{s_i}$, $h(q_i) = \tilde{q_i}$ y $h(a_i) = \tilde{a_i}$.

En segundo lugar nos aseguramos de asignar $h(\alpha) = \tilde{\alpha}$, donde α es un arco en ∂V_i y $\tilde{\alpha}$ es el correspondiente arco en $\partial \tilde{V}_i$.

En seguida, en utilización las órbitas de X, α es dejado fluir hasta $\beta \in \partial U_k$, lo mismo con$\tilde{X}$: ahora $\tilde{\alpha}$ es llevado hacia $\tilde{\beta} \in \partial \tilde{U}_k$. Hemos podido así materializar la extensión $h(\beta) = \tilde{\beta}$.

Luego extendemos h al interior de las regiones V_l y U_k.

Por último, se extiende h a las regiones limitadas por los arcos α, β y las separatrices correspondientes. $\square$

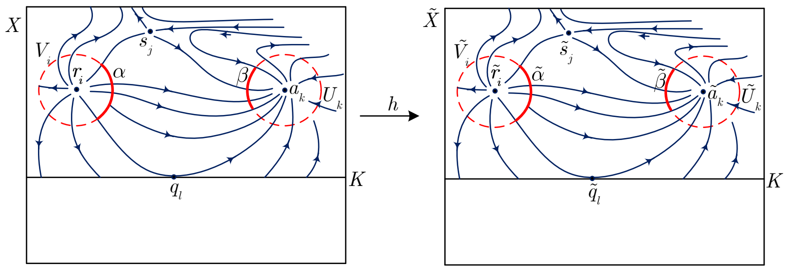


Figura 15: El homeomorfismo h lleva órbitas del campo X en órbitas de $\tilde{X}$ (la perturbación)

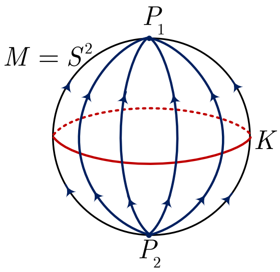


Figura 16: P_1 es un atractor hiperbólico, P_2 es un repulsor hiperbólico y K es una región de discontinuidad

Como ejemplo sean X_1 y X_2 las ramas del campo X sobre la esfera $M = S^2$ y suponemos que son transversales a $K = S^1$ como se muestra en la figura 16. Observamos que los campos son del tipo *Morse-Smale*, luego condiciones 1 y 4 del teorema se satisfacen. El campo X es genérico si y solo si los equilibrios singulares son hiperbólicos y no hay separatrices conectadas ni separatrices relacionadas entre los puntos de equilibrios singulares y sillas.

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Abstract

In this paper we present the theorem of characterization of the structural stability of piecewise smooth vector fields defined on a compact surface. It is essentially a result of the local theory, and is based on work of Brouke, Pugh & Simic [3]. They established and showed that the necessary and sufficient conditions are given by hyperbolicity of singular points, genericity of tangencies, no connection of singular saddles, and the apperance of only trivially recurrent orbits. We show the same but from a different perspective.

Keywords: Structural stability, singularities, genericity.

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About the stability between a foliation of degree two and the pencil of conics that defines it

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March, 2020

Abstract

In this paper, we study foliations on the projective plane of degree two which have a first integral with degree two. Such first integrals define a pencil of conics.

The Hilbert-Mumford criterion is a powerful tool of the Geometric Invariant Theory. An application of this theory is the characterizarion of the instability of the space of foliations of degree two, with respect to the action by a change of coordinates, and the characterization of the stability of pencils of conics, given by Alcántara.

The aim of the paper is to give another proof of the fact that a foliation of degree two defined by a pencil of conics is unstable if, and only if, the pencil is unstable.

MSC(2010): Primary 37F75; Secondary 14L24.

Keywords: Foliations, pencil of conics, unstable foliations.

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1 Introduction

The problem of classification of holomorphic foliations in complex manifolds has been the object of intense study in the last decades, see [5, 6, 8] and the references therein.

The set of holomorphic foliations on the complex projective plane of degree d, denoted by $\mathbb{F}ol(d)$, is a projective space and accepts a linear action of $\mathrm{Aut}(\mathbb{P}^2)$ under a change of coordinates. In this paper, we study foliations on $\mathbb{P}^2$ of degree two which have a first integral of degree two. Our main tool would be Geometric Invariant Theory (GIT), as developed mainly by Hilbert and Mumford [11]. We also deal with the pencil of conics defined by said first integral. In [10], Miranda found a characterization of the stability of pencils of cubic curves. Based on this result, Alcántara and Sánchez-Argáez [13] (see also [4]) proved the following characterization on the stability of a pencil of conics.

Theorem 1.1 ([13, Teorema 8]). *Let $A(x, y, z)$ and $B(x, y, z)$ be degree two homogeneous polynomials in $\mathbb{P}^2$ defining conics without common components. Let $\mathcal{L}_{A,B}$ be the pencil generated by such conics, and let $B(\mathcal{L}_{A,B})$ the set of common zeros of A and B. Then $P(\mathcal{L})$ is unstable if and only if $B(\mathcal{L}_{A,B})$ contains at most three different points.*

In [2], Alcántara obtained maximal sets of generators for unstable foliations, namely

$$CN_1 = \mathbb{P}\left\langle xy\frac{\partial}{\partial x}, xz\frac{\partial}{\partial x}, y^2\frac{\partial}{\partial x}, yz\frac{\partial}{\partial x}, z^2\frac{\partial}{\partial x}, y^2\frac{\partial}{\partial y}, yz\frac{\partial}{\partial y}, z^2\frac{\partial}{\partial y}, y^2\frac{\partial}{\partial z}\right\rangle$$
$$CN_2 = \mathbb{P}\left\langle xz\frac{\partial}{\partial x}, y^2\frac{\partial}{\partial x}, yz\frac{\partial}{\partial x}, z^2\frac{\partial}{\partial x}, xz\frac{\partial}{\partial y}, yz\frac{\partial}{\partial y}, z^2\frac{\partial}{\partial y}\right\rangle.$$

Moreover, Alcántara also achieved a characterization of the instability of a degree two foliation.

Theorem 1.2 ([2]). *Let $X \in \mathbb{F}ol(2)$ be a foliation with isolated singularities. Then X is unstable if and only if it has one of the following properties:*

1. *there exists a singular point p of multiplicity 2, or*

2. *it has an invariant line with a unique singular point with multiplicity 1 and Milnor number 5.*

Moreover, a foliation X satisfies (1) if and only there exists $g \in SL(3, \mathbb{C})$ such that $gX \in CN_1$, and satisfies (2) if and only if there exist $g \in SL(3, \mathbb{C})$ such that $gX \in CN_2 \setminus CN_1$.

In [7], Cerveau *et al.* proved that there exist, up to the action, three foliations of degree two on $\mathbb{P}^2$ defined by a pencil of conics. On the other hand, in [3], Alcántara constructs a stratification (based on GIT, see [11]) of the space of foliations with respect to the action by a change of coordinates and presents the following corollary: *a foliation of degree two defined by a pencil of conics is unstable if and only if the pencil is unstable*. In this work, we present a new proof of this corollary, where, instead of using the aforementioned stratification, we rely on Proposition 3.2, due to Darboux, by studying the singular fibers of the first integral defined by the pencil. We also use the characterization given in Theorem 1.2 for the instability of a foliation and the characterization given in Theorem 1.1 for the instability of a pencil of conics. In other words, we will establish the folllowing result.

Theorem 1.3. *Let $\mathcal{F}$ be a foliation of degree two in $\mathbb{P}^2$ with fist integral $H = \dfrac{F}{G}$, here F, G are polynomials of degree two. Then $\mathcal{F}$ is unstable if and only if $\mathcal{L}_{F,G}$ is unstable.*

2 Preliminaries

Let k be an algebraically closed field. An **algebraic group** G over k is a group that by own right is a variety over k and is such that the multiplication and inversion operations are morphisms of the variety.

Classical examples of algebraic groups include

$$GL(n,k) = \{A \in M_n(k) \, : \, \det(A) \neq 0\},$$
$$SL(n,k) = \{A \in M_n(k) \, : \, \det(A) = 1\},$$

respectively called the **general** and **special linear** groups. Given G, H algebraic groups, a **morphism** $\varphi : G \to H$ should be both a group homomorphism and a morphism of varieties.

A **linear algebraic group** is an algebraic group that is isomorphic to an algebraic subgroup of $GL(n,k)$. Note that both $GL(n,k)$ and $SL(n,k)$ are linear algebraic groups.

From now on, π will denote the projection from $\mathbb{A}_k^{n+1} \setminus \{0\}$ to $\mathbb{P}^n$. Let G be an algebraic group acting on an affine variety $X \subset \mathbb{A}_k^{n+1}$ (respectively, a projective variety $X \subset \mathbb{P}^n$). We say that the action is **linear** if there exists a group homomorphism

$$\rho : G \to GL(n+1,k)$$

with $g \cdot x = \rho(g)(x)$ (respectively, $g \cdot x = \pi(\rho(g)(\bar{x}))$, where $\bar{x} \in \pi^{-1}(x)$).

Remark 2.1. Note that if an action over a projective variety X is linear, then ρ induces an action in the affine cone

$$\check{X} = \{0\} \cup \bigcup_{x \in X} \pi^{-1}(x).$$

of X.

Denote by $\mathbb{C}[x,y,z]_d$ the space of homogeneous polynomials of degree d. Given $F \in \mathbb{C}[x,y,z]_d$, the set

$$V(F) = \{[x:y:z] \in \mathbb{P}^2 \, : \, F(x,y,z) = 0\}$$

is called the **locus** of F. Recall that $F, G \in \mathbb{C}[x,y,z]_d$ define the same curve (*i.e.,* locus) if and only if $F = \lambda G$, for some $\lambda \in \mathbb{C}^*$. Then

$$X = \{V(F) : F(x,y,z) \in \mathbb{C}[x,y,z]_d\}$$

is a proyective variety, because of the identification

$$X \simeq \frac{\mathbb{C}[x,y,z]_d \setminus \{0\}}{\mathbb{C}^*} \simeq \mathbb{P}^N,$$

where $N = \begin{pmatrix} d+2 \\ 2 \end{pmatrix} - 1.$

Example 2.2. Let X be the set of cubic planar curves, that is

$$X = \{V(F) : F(x,y,z) \in \mathbb{C}[x,y,z]_3\} \simeq \mathbb{P}^9.$$

The action in X given by

$$\begin{aligned} \rho : SL(3,\mathbb{C}) \times X &\longrightarrow X \\ (g, F(x,y,x)) &\longmapsto F(g^{-1}(x,y,z)). \end{aligned}$$

is linear.

Let G be an algebraic group acting on X. In general the quotient space X/G is not a variety.

Example 2.3. Let $G = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $X = \mathbb{C}^2$. Consider the action $\rho : G \times \mathbb{C}^2 \to \mathbb{C}^2$ given by

$$\rho(\lambda, (z_1, z_2)) = \lambda \cdot (z_1, z_2) = (\lambda z_1, \lambda^{-1} z_2).$$

The orbits of X are given by

$$\begin{aligned} O(0,0) &= \{(0,0)\}, \\ O(z_1,0) &= \{(\lambda z_1, 0) : \lambda \in G\}, \\ O(0,z_2) &= \{(0, \lambda^{-1} z_2) : \lambda \in G\}, \\ O(z_1,z_2) &= \{(x,y) : xy = z_1 z_2\}, \end{aligned}$$

where $z_1, z_2 \neq 0$. We conclude that X/G is not an algebraic variety.

Let G be an algebraic group acting on a variety X. Denote by $A(X)$ the algebra of morphisms $\varphi : X \to k$. The action of G on X induces an action of G on $A(X)$. We define

$$A(X)^G = \{f \in A(X) : f(g \cdot x) = f(x), \text{ for all } g \in G\},$$

called the **invariant ring** of X.

Example 2.4. For the action ρ of Example 2.3 we have

$$A(X) = \{f : \mathbb{C}^2 \to \mathbb{C} : f \text{ is a polynomial}\}.$$

Here we obtain

$$A(X)^G = k[xy] = A(\mathbb{C})$$

and, so, the ring is finitely generated.

The set $V(A(X)^G) = V(A(\mathbb{C})) = \mathbb{C}$ is called the **GIT-quotient** and is denoted by $X//G$. The question of whether or not $A(X)^G$ is finitely generated is a variation on Hilbert's 14th problem. The most general answer was given by Nagata [12] in 1963. See also [1].

Theorem 2.5 (Nagata)**.** *If G is a reductive group then $A(X)^G$ is finitely generated.* □

We say that a linear algebraic group G is **reductive** if for every linear action of G in k^n, and every invariant point $v \in k^n \setminus \{0\}$ there exists a G-invariant homogeneous polynomial f of degree ≥ 1 such that $f(v) \neq 0$.

Example 2.6 ([1, Ejemplo 10])**.** The groups $G = GL(3, \mathbb{C})$ and $G = SL(3, \mathbb{C})$ are known to be reductive.

One of the main goals of Geometric Invariant Theory (GIT) is to classify objects in Algebraic Geometry. The case of interest here is when X is the space of foliations of degree d, see Section 3.

Definition 2.7 ([11, Proposition 2.2], see also [1, Proposición 4]). Let X be a projective variety in $\mathbb{P}^n$ and G a reductive group acting linearly on X. Let $x \in X$ and $\overline{x} \in \pi^{-1}(x) \subset \check{X} \subset \mathbb{C}^{n+1}$. Then x is **semistable** when $0 \notin \overline{O(\overline{x})}$. We write X^{ss} for the set of semistable points of X.

Also, we say that x is **stable** whenever $0 \notin \overline{O(\overline{x})}$, the orbit of x is closed in X^{ss}, and $\dim O(x) = \dim G$.

In general, it is difficult to determine when a point in a projective variety is stable. We now describe a usable criterion, originally given by Hilbert for $G = SL(n)$, and later extended by Mumford for arbitrary G.

A **1-parameter subgroup of** G is a non-trivial homomorphism of algebraic groups $\lambda : \mathbb{C}^* \to G$.

Given a 1-parameter subgroup λ and a linear action G on $X \subset \mathbb{P}^n$, we introduce the representation

$$\begin{aligned} \mathbb{C}^* &\to GL_{n+1}(\mathbb{C}) \\ t &\mapsto \lambda_t : \lambda_t(v) = \lambda(t) \cdot v. \end{aligned}$$

This representation is diagonalizable.

Proposition 2.8 ([1, Proposición 5]). *There is a basis* $\{e_0, e_1, \ldots, e_n\}$ *of* $\mathbb{C}^{n+1}$ *such that* $\lambda(t)e_i = t^{r_i} e_i$*, with* $r_i \in \mathbb{Z}$.

By Proposition 2.8, given a point $\overline{x} \in \check{X}$, with $\overline{x} = \sum_{i=0}^{n} \overline{x}_i e_i$, we have

$$\lambda(t)\overline{x} = \sum t^{r_i} \overline{x}_i e_i.$$

Mumford used this 1-parameter subgroup to calculate the stability of elements of X by the action of G. In that way he introduced the now called **Mumford's function**: for $x \in X$ and λ in a given 1-parameter subgroup of G, we set

$$\mu(x, \lambda) = \min\{r_i : \overline{x}_i \neq 0\}.$$

Theorem 2.9 ([11, Theorem 2.1], see also [1, Teorema 12])**.** *Let X be a projective variety in $\mathbb{P}^n$ and G a reductive group acting linearly on X. Then, for any $x \in X$ we have that*

- *x is semistable if and only if $\mu(x, \lambda) \leq 0$ for all 1-parameter subgroups λ of G.*
- *x is stable if and only if $\mu(x, \lambda) < 0$ for all 1-parameter subgroups λ of G.*

Remark 2.10. Almost directly from the definition we get

$$\mu(g \cdot x, \lambda) = \mu(x, g^{-1}\lambda g),$$

for any $g \in G$.

Proposition 2.11 ([1, Proposición 6])**.** *Every 1-parameter subgroup $\lambda : \mathbb{C}^* \to SL(3, \mathbb{C})$ is conjugated to a diagonal one. In other words, we have*

$$\lambda(t) = g \begin{pmatrix} t^{n_0} & 0 & 0 \\ 0 & t^{n_1} & 0 \\ 0 & 0 & t^{n_2} \end{pmatrix} g^{-1},$$

for some $g \in SL(3, \mathbb{C})$, where $n_0 \geq n_1 \geq n_2$, $n_0 + n_1 + n_2 = 0$.

Example 2.12. Let $X = \{V(F) : F \in \mathbb{C}[x, y, z]_3\}$ and consider the linear action

$$\begin{aligned} \rho : SL(3, \mathbb{C}) \times X &\longrightarrow X \\ (g, F(x, y, x)) &\longmapsto F(g^{-1}(x, y, z)). \end{aligned}$$

We now calculate the Mumford's function associated to X and ρ. For that purpose let

$$B = \{x^3, y^3, z^3, x^2y, x^2z, xy^2, xz^2, yz^2, y^2z, xyz\}$$

be a base of X. By Remark 2.10, in order to analyze the stability of $F \in X$, it is enough to consider the 1-parameter subgroups given by

$$\begin{aligned}\lambda : \mathbb{C}^* &\to SL(3,\mathbb{C})\\ t &\mapsto \operatorname{diag}(t^{n_0}, t^{n_1}, t^{n_2}),\end{aligned}$$

where $n_0 \geq n_1 \geq n_2$ and $n_0 + n_1 + n_2 = 0$. A few calculations yield $\mu(x^{3-i-j}y^i z^j, \lambda) = (i + j - 3)n_0 - in_1 - jn_2$ and

$$\mu(F, \lambda) = \min\{(i + j - 3)n_0 - in_1 - jn_2 : a_{ij} \neq 0\}.$$

where

$$F(x, y, x) = \sum_{i,j=0}^{3} a_{ij} x^{3-i-j} y^i z^j.$$

3 Foliations in the complex projective plane

An holomorphic foliation $\mathcal{F}$ on a complex compact surface X is a family of holomorphic 1-forms $\{\omega_i\}_{i\in I}$ defined on an open covering $\{U_i\}_{i\in I}$ of X, such that $\omega_i = g_{ij}\omega_j$, for some holomorphic functión g_{ij} without zeroes on $U_i \cap U_j$. A **foliation** of degree d in $\mathbb{P}^2$ is determined by either

- a projective 1-form $\Omega = Pdx+Qdy+Rdz$, with P, Q, R homogeneous polynomials of degree $d + 1$ subject to $xP + yQ + zR = 0$ (called **Euler's condition**), that satisfies $\Omega \wedge d\Omega = 0$, the **integrabitility condition** or

- a vector field $X = A\dfrac{\partial}{\partial x} + B\dfrac{\partial}{\partial y} + C\dfrac{\partial}{\partial z}$, with A, B, C homogeneous polynomials of degree d, modulo a product GX_R of the **radial vector field** $X_R = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y} + z\dfrac{\partial}{\partial z}$ and a homogeneous polynomial G.

The **singular set** of a foliation $\mathcal{F}$, denoted by $\mathrm{Sing}(\mathcal{F})$, is given by the set

$$\mathrm{Sing}(\mathcal{F}) = \{p \in \mathbb{P}^2 \,:\, P(p) = Q(p) = R(p) = 0\}$$

when $\mathcal{F}$ is defined via a 1-form, and by

$$\mathrm{Sing}(\mathcal{F}) = \{p = [x : y : z] \in \mathbb{P}^2 \,:\, X(p) = \lambda X_R(p), \text{ for some } \lambda \in \mathbb{C}\}$$

when $\mathcal{F}$ is defined via a vector field.

Example 3.1. Let P, Q be homogeneous polynomials of degree k in $\mathbb{P}^2$ without common factors. Then the 1-form $\Omega = PdQ - QdP$ satisfies $i_{X_R}(PdQ - QdP) = 0$, where i_{X_R} is the contraction of the 1-form, and defines a foliation $\mathcal{F}_\Omega$ of degree $2k - 2$.

Let P, Q be homogeneous polynomials of degree k in $\mathbb{P}^2$. For $\alpha = [a : b] \in \mathbb{P}^1$, let $L_\alpha = aP + bQ$ be a fiber of P/Q, whose decomposition in irreducible factors is $L_\alpha = f_{\alpha,1}^{n_1} \cdots f_{\alpha,j}^{n_j}$, with $n_1, \ldots, n_j \in \mathbb{N}$ and $j \in \mathbb{N}$, all depending on α. In this setting, we write $G_\alpha = f_{\alpha,1}^{n_1-1} \cdots f_{\alpha,j}^{n_j-1}$. We have the following proposition due to Darboux [9].

Proposition 3.2 (Darboux)**.** *Let $\Omega = PdQ - QdP$. Then there exist $\Delta = G_{\beta_1} \cdots G_{\beta_n}$, with $\beta_1, \ldots, \beta_n \in \mathbb{P}^1$, and a 1-form ω such that*

$$\Omega = \Delta \cdot \omega.$$

Moreover, ω defines a foliation $\mathcal{F}(P, Q)$ of degree $2k - 2 - \deg(\Delta)$ with isolated singularities, where $k = \deg(P) = \deg(Q)$.

Remark 3.3. In Proposition 3.2, the quotient $H = \dfrac{P}{Q}$ is called a **first integral** of the foliation $\mathcal{F} = \mathcal{F}(P, Q)$. When this is the case, the quotient $\dfrac{P + \lambda Q}{P + \mu Q}$, whenever $\lambda \neq \mu$, is also a first integral of $\mathcal{F}$.

Remark 3.4. By Proposition 3.2, if P and Q are polynomials of degree two, we have $PdQ - QdP = \Delta\omega$, $\Delta = G_{\beta_1} \cdots G_{\beta_n}$, with $\beta_1, \ldots, \beta_n \in \mathbb{P}^1$, ω defines $\mathcal{F}$ and

$$\deg(\mathcal{F}) = 2 = 2 \cdot 2 - 2 - \deg(\Delta),$$

which implies $\deg(\Delta) = 0$. Then, if $G_\beta = f_{\beta,1}^{n_1-1} \ldots f_{\beta,k}^{n_k-1}$ is a factor of Δ then $n_j = 1$, for $j = 1 \ldots n$, and $L_\alpha = f_{\alpha,1}^{n_1} \cdots f_{\alpha,k}^{n_k}$ is a singular fiber of H. In particular, every singular fiber of H is a product of two lines.

Lemma 3.5. *Let $\mathcal{F}$ be a foliation of degree two in $\mathbb{P}^2$ with isolated singularities and with $H = \dfrac{F}{G}$ as first integral, where F and G are polynomials of degree 2. Then, under an automorphism of $\mathbb{P}^2$ we can reach one of the following three forms*

- $H = \dfrac{z(ax+by+cz)}{xy}$, *with at most one of a, b, c equal to zero,*
- $H = \dfrac{(x-y)(ax+cz)}{xy}$, *where $a \cdot c \neq 0$,*
- $H = \dfrac{xy}{Q(x,y,z)}$, *with $Q = ax^2 + cy^2 + mxz + nyz + pz^2$ irreducible.*

Proof. For $\alpha = [a : b] \in \mathbb{P}^1$, the quadratic form $H_\alpha = aP + bQ$ is a fiber of H. Let A_α be the associated matrix of H_α. Given $\beta, \gamma \in \mathbb{P}^1$ we have the identity

$$\det(A_\beta + tA_\gamma) = \det(A_\gamma)\det(A_\gamma^{-1}A_\beta - (-t)Id) = \det(A_\gamma)q(-t),$$

where $q(x)$ is the characteristic polynomial of $A_\gamma^{-1}A_\beta$. It follows that there is at least one singular fiber of H, a product of two lines by Remark 3.4. Let us call L_α this singular fiber.

By Remark 3.3, any two fibers of H determine a first integral of $\mathcal{F}$. Therefore, we have two possibilities:

- if H has two singular fibers, then $H = \dfrac{\ell_1\ell_2}{\ell_3\ell_4}$, where $\ell_1\ell_2$ and $\ell_3\ell_4$ are the singular fibers of H;

- if H has one singular fiber, then $H = \dfrac{\ell_1 \ell_2}{Q}$, with Q irreducible.

In the first case, after a change of coordinates H assumes one of the following shapes

- $H = \dfrac{z\ell}{xy} = \dfrac{z(ax+by+cz)}{xy}$, where at most one of a, b, c is zero, since $\mathcal{F}$ has isolated singularities.
- $H = \dfrac{(x-y)\ell}{xy} = \dfrac{(x-y)(ax+by+cz)}{xy}$. By Remark 3.3, we can further assume $b = 0$. Note that $a \neq 0$ returns us to the first case, so we can asume $a \neq 0$. Finally, we should have $c \neq 0$ if we wish $\mathcal{F}$ to have only isolated singularities.

In the second case, it is clear that under a linear change of coordinates H takes the form

$$H = \frac{xy}{Q} = \frac{xy}{ax^2 + cy^2 + mxz + nyz + qxy + pz^2},$$

and, by Remark 3.3, we can assume that $q = 0$. □

4 Pencils of curves of degree d

To $\mathbb{C}[x, y, z]_d$, the space of homogeneous polynomials of degree d, we associate the corresponding projective space $\mathbb{P}(\mathbb{C}[x, y, z]_d) \simeq \mathbb{P}^{n-1}$, here $n = \dbinom{d+2}{2} = \dfrac{(d+1)(d+2)}{2}$.

Given $F, G \in \mathbb{C}[x, y, z]_d$, with $F \neq G$, the **pencil of plane curves of degree d generated by F and G** is defined by

$$\mathcal{L} = \mathcal{L}_{F,G} = \{aF + bG \ : \ [a : b] \in \mathbb{P}^1\},$$

to which we associate the **base locus**

$$B(\mathcal{L}_{F,G}) = V(F) \cap V(G).$$

The set of pencils of plane curves of degree d is denoted by $\mathbb{G}_d$.

On the other hand, writing

$$F = \sum_{i=0}^{d}\sum_{j=0}^{d-i} a_{(i,j)}x^i y^j z^{d-i-j}, \qquad G = \sum_{i=0}^{d}\sum_{j=0}^{d-i} b_{(i,j)}x^i y^j z^{d-i-j},$$

we consider the $2 \times n$-matrix

$$\begin{bmatrix} a_{(0,0)} & a_{(0,1)} & \cdots & a_{(0,d)} & \cdots & a_{(d,0)} \\ b_{(0,0)} & b_{(0,1)} & \cdots & b_{(0,d)} & \cdots & b_{(d,0)} \end{bmatrix}.$$

Note that the indices of the columns of the previous matrix are lexicographically ordered. Define the determinants

$$\mathcal{M}_{i,j,k,l} = \det \begin{bmatrix} a_{ij} & a_{kl} \\ b_{ij} & b_{kl} \end{bmatrix}$$

and the set of N-tuples

$$(\mathcal{M}_{i,j,k,l})_{(i,j)<_{\text{lex}}(k,l)},$$

where $N = \binom{n}{2} = \frac{n(n-1)}{2}$. The function

$$\begin{aligned} P : \mathbb{G}_d &\longrightarrow \mathbb{P}^{N-1} \\ \mathcal{L} &\longmapsto (\mathcal{M}_{i,j,k,l}) \end{aligned}$$

determines what are called the **Plucker coordinates of $\mathcal{L}$**.

Theorem 4.1. *With the notations above, P is an embedding and $P(G_d)$ is a projective variety.*

Proof. See [13, Teorema 7]. □

In Theorem 1.1, the action of the group $SL(3,\mathbb{C})$ on $\mathbb{G}_d$ is

$$\begin{aligned} \sigma : SL(3,\mathbb{C}) \times \mathbb{G}_d &\longrightarrow \mathbb{G}_d \\ \sigma(g, \{k_1A + k_2B\}_{(k_1:k_2)\in\mathbb{P}(\mathbb{C})^1}) &\longmapsto \{k_1Ag + k_2Bg\}_{(k_1:k_2)\in\mathbb{P}(\mathbb{C})^1}. \end{aligned}$$

5 Relating the pencil and the foliation

The group $\mathbb{P}GL(3,\mathbb{C})$ of automorphisms of $\mathbb{CP}^2$ is a reductive group that acts linearly on $\mathbb{F}ol(d)$ by the change of coordinates

$$\begin{aligned} \mathbb{P}GL(3,\mathbb{C}) \times \mathbb{F}ol(d) &\longrightarrow \mathbb{F}ol(d) \\ (g, X) &\longmapsto g \cdot X = DgX \circ (g^{-1}), \end{aligned}$$

or more specifically, on $\mathcal{F}_2$, as

$$\left(g, \begin{pmatrix} P(x,y,z) \\ Q(x,y,z \\ R(x,y,z) \end{pmatrix} \right) \longmapsto g \begin{pmatrix} P(g^{-1}(x,y,z)) \\ Q(g^{-1}(x,y,z)) \\ R(g^{-1}(x,y,z)) \end{pmatrix}.$$

Lemma 5.1. *Let $\mathcal{L}_{F,G}$ be the pencil of conics for $F = z(ax+by+cz)$ and $G = xy$. Then $\mathcal{L}_{F,G}$ is unstable if and only if we have $abc = 0$.*

Proof. The base locus $B(\mathcal{L})$ of F and G is obtained by solving

$$\begin{aligned} xy &= 0 \\ z(ax+by+cz) &= 0. \end{aligned}$$

For $c = 0$ we obtain

$$B(\mathcal{L}) = \{[1:0:0], [0:1:0], [0:0:1]\}$$

while for $c \neq 0$ we get

$$B(\mathcal{L}) = \{[1:0:0], [0:1:0], [0:c:-b], [c:0:-a]\}.$$

Thus, $B(\mathcal{L})$ has at most three points provided $a = 0$ or $b = 0$ or $c = 0$ (or $abc = 0$ in short). Therefore, by Theorem 1.1, the pencil $\mathcal{L}$ is unstable if and only if $abc = 0$. $\square$

Lemma 5.2. *Let $\mathcal{L}_{F,G}$ be the pencil of conics for $F = (x-y)(ax+cz)$ and $G = xy$. Then $\mathcal{L}_{F,G}$ is unstable if and only if $c \neq 0$.*

Proof. When $c = 0$ (hence $a \neq 0$), the base locus $B(\mathcal{L})$ of F and G is

$$B(\mathcal{L}) = \{[0 : y : z]\} \simeq \mathbb{P}^1.$$

This in turn implies that the pencil $\mathcal{L}$ is stable. On the other hand, when $c \neq 0$ we obtain

$$B(\mathcal{L}) = \{[0 : 0 : 1], [0 : 1 : 0], [c : 0 : -a]\}.$$

Since $B(\mathcal{L})$ has at most three points, the pencil $\mathcal{L}$ is unstable. Therefore, by Theorem 1.1 the pencil $\mathcal{L}$ is unstable if and only if we have $c \neq 0$. □

Lemma 5.3. *Let $\mathcal{L}_{F,G}$ be the pencil of conics for $F = xy$ and $G = ax^2 + cy^2 + mxz + nyz + pz^2$, irredutible. Then $\mathcal{L}_{F,G}$ is unstable if and only if $p = 0$, or $p \neq 0$ and $\Delta_1\Delta_2 = 0$, here $\Delta_1 = n^2 - 4pc$ and $\Delta_2 = m^2 - 4pa$.*

Proof. A point $[x : y : z]$ of the base locus of the pencil must be either $[0 : y : z]$, where $cy^2+nyz+pz^2 = 0$, or $[x : 0 : z]$, where $ax^2+mxz+pz^2 = 0$. If $p = 0$, these points reduce to $[0 : 0 : 1]$, $[0 : m : -a]$ and $[m : 0 : -a]$, so the base locus has at most three points. When $p \neq 0$, if $\Delta_1 = n^2-4pc = 0$ or $\Delta_2 = m^2 - 4pa = 0$, then the base locus also has at most three points. On the other hand, if $\Delta_1 \neq 0$ and $\Delta_2 \neq 0$, the base locus has four points. The lemma now follows by Theorem 1.1. □

Theorem 5.4. *Let $\mathcal{F}$ be a foliation of degree two in $\mathbb{P}^2$ with fist integral $H = F/G$, here F, G are polynomials of degree 2. Then $\mathcal{F}$ es unstable if and only if $\mathcal{L}_{F,G}$ is unstable.*

Proof. By Lemma 3.5, the first integral $H = \dfrac{F}{G}$ must be of one of the following three types:

- $H = \dfrac{z(ax + by + cz)}{xy}$, where at most one among a, b, c is zero;
- $H = \dfrac{(x - y)(ax + cz)}{xy}$, with $a \cdot c \neq 0$;

- $H = \dfrac{xy}{Q} = \dfrac{xy}{ax^2 + cy^2 + mxz + nyz + pz^2}$, with Q irreducible.

For the first case set $L = ax+by+cz$. Then the foliation $\mathcal{F}$ induced by

$$\begin{aligned}\Omega_1 &= zLd(xy) - xyd(zL)\\ &= yz(by+cz)dx + xz(ax+cz)dy - xy(ax+by+2cz)dz\end{aligned}$$

has singularities in the solution set of the system

$$\begin{aligned}xy(ax+by+2cz) &= 0,\\ xz(ax+cz) &= 0,\\ xyz(by+cz) &= 0.\end{aligned}$$

Thus, the singular set of $\mathcal{F}$ is

$$\mathrm{Sing}(\mathcal{F}) = \left\{\begin{matrix}[1:0:0],[0:1:0],[0:0:1],[c:0:-a],\\ [-b:a:0],[0:c:-b],[bc:ac:-ab]\end{matrix}\right\}.$$

The foliation $\mathcal{F}$ is induced around the singularities $[1:0:0]$, $[0:1:0]$, $[0:0:1]$ by

$$\Omega_1|_{x=1} = z(a+cz)dy - y(a+by+2cz)dz, \tag{5.1}$$
$$\Omega_1|_{y=1} = z(b+cz)dx - x(ax+b+2cz)dz, \tag{5.2}$$
$$\Omega_1|_{z=1} = y(by+c)dx + x(ax+c)dy, \tag{5.3}$$

respectively. Then, by Equations 5.1, 5.2, and 5.3, if $abc = 0$ the foliation has a singularity with multiplicity 2 and $\mathcal{F}$ is unstable. On the other hand, if $abc \neq 0$, the foliation $\mathcal{F}$ has 7 singularities of multiplicity one, so $\mathcal{F}$ is semistable. In conclusion, in the first case $\mathcal{F}$ is unstable if and only if $abc = 0$. Hence, by Lemma 5.1, the foliation $\mathcal{F} = \mathcal{F}(F,G)$ turns out to be unstable if and only if the pencil $\mathcal{L}_{F,G}$ is unstable.

In the second case the foliation $\mathcal{F}$ is induced by

$$\Omega_2 = -y(cyz + ax^2)dx + x^2(ax+cz)dy - cxy(x-y)dz$$

with singular set

$$\mathrm{Sing}(\mathcal{F}) = \{[0:1:0], [0:0:1], [c:0:-a], [c:c:-a]\}.$$

The foliation $\mathcal{F}$ around $p_1 = [0:0:1]$ is induced by

$$\Omega_1|_{z=1} = -y(cy + ax^2)dx + x^2(ax + c)dy,$$

so it is unstable because the multiplicity of p_1 is 2. Therefore again by Lemma 5.2 we conclude that $\mathcal{F} = \mathcal{F}(F, G)$ is unstable and the pencil $\mathcal{L}_{F,G}$ is unstable.

In the third case the foliation $\mathcal{F} = \mathcal{F}(F, G)$ (here $F = xy$ and $G = Q$) is induced by

$$\begin{aligned}\Omega_3 = (xy)dQ - Qd(xy) &= y(xQ_x - Q)dx + x(yQ_y - Q)dy + xyQ_z dz \\ &= y(ax^2 - cy^2 - nyz - pz^2)dx + x(cy^2 - ax^2 - mxz - pz^2)dy + \\ &\quad + xy(mx + ny + 2pz)dz,\end{aligned}$$

and so now we get $\mathrm{Sing}(\mathcal{F}) = \{[0:0:1]\} \cup \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$, where

$$\begin{aligned}\mathcal{S}_1 &= \{[0:y:z] \,:\, cy^2 + nyz + pz^2 = 0\}, \\ \mathcal{S}_2 &= \{[x:0:z] \,:\, ax^2 + mxz + pz^2 = 0\}, \\ \mathcal{S}_3 &= \{[x:y:z] \,:\, Q = xQ_x = yQ_y,\, Q_z = 0,\, xy \neq 0\}.\end{aligned}$$

We now prove that $\mathcal{F}$ is unstable if and only if $p = 0$ or $p \neq 0$ and $\Delta_1\Delta_2 = 0$, where $\Delta_1 = n^2 - 4pc$ and $\Delta_2 = m^2 - 4pa$.

First assume $p = 0$. In this case, we get $m \neq 0$ or $n \neq 0$, otherwise Q becomes $Q = ax^2 + cy^2$, which is not irreducible. The foliation near $[0:0:1]$ is induced by

$$\Omega_3|_{z=1} = y(ax^2 - cy^2 - ny)dx + x(cy^2 - ax^2 - mx)dy,$$

so $[0:0:1]$ has multiplicity two. This implies that $\mathcal{F}$ is unstable.

Now assume $p \neq 0$ and $\Delta_1 = 0$. In this case we have the singular point $[0:1:-\frac{n}{2p}] \in \mathcal{S}_1$ and the foliation is given by

$$\Omega|_{y=1} = (ax^2 - c - nyz - pz^2)dx + x(mx + n + 2pz)dz.$$

After a translation to the origin, the foliation can be rewritten as

$$\omega = (ax^2 - pz^2)dx + x(mx + 2pz)dz.$$

It is clear now that $\mathcal{F}$ is unstable, because the multiplicity at $[0 : 1 : -\frac{n}{2p}]$ is two. The case $p \neq 0$ and $\Delta_2 = 0$ is handled analogously.

Finally, assume $p \neq 0$ and $\Delta_1 \cdot \Delta_2 \neq 0$. In this case $\mathcal{S}_1$ and $\mathcal{S}_2$ have two singularities each. The singularities in $\mathcal{S}_3$ are obtained from the equations

$$Q = xQ_x, \quad Q = yQ_y, \quad Q_z = 0.$$

Note that $Q_z = mx + ny + 2pz = 0$ implies $z = -\dfrac{mx + ny}{2p}$. Replacing this in either $Q = xQ_x$ or $Q = yQ_y$, we obtain

$$\Delta_1 y^2 + \Delta_2 x^2 = 0,$$

so $\mathcal{S}_3$ has two singularities, totalling seven of them. This implies that $\mathcal{F}$ is semistable.

Therefore, if $p \neq 0$, then $\mathcal{F}$ is unstable if and only if $\Delta_1 = 0$ or $\Delta_2 = 0$. By Lemma 5.3 the foliation $\mathcal{F} = \mathcal{F}(F, G)$ is unstable if and only if the pencil $\mathcal{L}_{F,G}$ is unstable. $\square$

Remark 5.5. It follows from the proof of the previous theorem, when $H = \dfrac{(x - y)(ax + cz)}{xy}$, with $a \cdot c \neq 0$, that both $\mathcal{F}$ and $\mathcal{L}_{F,G}$ are unstable (see Lemma 5.2).

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Resumen
En este artículo estudiamos foliaciones de grado dos en el plano proyectivo que acepten integral primera, también, de grado dos. Tales integrales primera definen una familia lineal de cónicas. El criterio de Hilbert-Munford es una poderosa herramienta de la teoría de invariantes geométricos. Una aplicación de esta teoría es la caracterización de la inestabilidad en el espacio de foliaciones de grado dos respecto a la acción por un cambio de coordenadas, y asimismo la caracterización de la estabilidad de las familias lineales de cónicas, ambas dadas por Alcántara. El objeto de este artículo es presentar una prueba alternativa del hecho de que una foliación de grado dos definida por una familia lineal de cónicas es inestable si y solo si la correspondiente familia lineal es inestable.

Palabras clave: Foliaciones, pincel de cónicas, inestabilidad.

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Local dynamics of parabolic skew-products

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August, 2019

Abstract

The local dynamics around a fixed point has been extensively studied for germs of one and several complex variables. In dimension one, there exist a complete picture of the trajectory of the orbits on a full neighbourhood of the fixed point. In greater dimensions some partial results are known. In this paper we analyze a case that lies between one and several variables. We consider skew product maps of the form $F(z, w) = (\lambda(z), f(z, w))$ and deal with the parabolic case, that is, when $DF(0, 0) = Id$. We describe the behaviour of orbits around a neighbourhood of the origin. We establish formulas for conjugacy maps in different regions of these neighbourhoods.

MSC(2010): Primary 37F10; Secondary 33C45.

Keywords: Skew-products, Fatou coordinates.

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1 Introduction

The dynamics of skew-product maps $F(z,w) = (\lambda(z), f(z,w))$ has been studied by several authors [11, 13, 16, 17, 18]. In this work we focus on local aspects of the theory, namely, we look at the dynamics of F close to a fixed point. For the sake of simplicity, we take said fixed point at the origin $(0,0)$. We turn our attention to a class of skew-product maps that we call **parabolic**, defined as those subject to $\lambda(z) = z + O(|z|^2)$ and $f(z,w) = w + O(|(z,w)|^2)$.

Skew-product maps are suitable to test general aspects of the dynamics of self-maps on several dimensions. Since the first coordinate depends only on one variable, we can borrow results from one dimensional complex dynamics to gain information. Nonetheless, they provide a richer theory than in dimension one. An instance of this fact can be seen in a recent article by Astorg *et al.* [3] where they describe a polynomial skew-product map in two dimensions that has a wandering Fatou component.

We center our study on maps given by

$$\begin{aligned} F :&(\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0) \\ F(z,w) &= (\lambda(z), f(z,w)), \end{aligned} \tag{1.1}$$

where $\lambda(z) = z + a_2 z^2 + O(z^3)$, with $a_2 \neq 0$, and $f(z,w) = w + b_2 w^2 + O((z,w)^3)$, with $b_2 \neq 0$.

Our goal is to describe the dynamics of such maps in a neighbourhood of the origin. We divide our program into two categories.

A. Describe regions in which F is conjugated to a simpler map.

B. Find formulas for the conjugation map in each region, as in the one dimensional case.

Finding a conjugacy map to a simpler map depends strongly on the type of map we are studying and the dimension of the space.

Consider a holomorphic germ $F : (\mathbb{C}^n, p) \to (\mathbb{C}^n, p)$ with a fixed point p. A local conjugacy of F to G is a one-to-one map $\phi : U_p \to \mathbb{C}^n$, from an open neighbourhood U_p around p, in such way that the conjugation $G = \phi^{-1} \circ F \circ \phi$ holds. In general, the main goal is to obtain a conjugacy to an easier map to study than F. This is a rich history that goes back to Schroeder. We redirect the reader to [1] or [15] for a list of helpful results.

In the case of a parabolic map F, that is, when $DF(p) = \mathrm{Id}$, only partial results are available when $n \geq 2$. The dynamics of parabolic maps in several dimensions is in general unpredictable [2, 8], and although some results have been proven for generic maps, much less is known in comparison with the one dimensional case.

One common feature of the study of parabolic maps is partial conjugacy to a translation. While usually this conjugacy cannot be realized on a whole neighbourhood around p, it is well defined on certain open sets with p at its boundary. The conjugacy is commonly referred to as a **Fatou coordinate**.

Fatou coordinates are useful for the study of parabolic maps. In dimension one for instance it is a crucial tool in the understanding of parabolic bifurcations.

Let us recall standard facts in dimension one. Consider the map $f(z) = z + a_2 z^2 + O(z^3)$, with $a_2 \neq 0$, where the origin is a parabolic fixed point. The Leau Fatou flower theorem states that there exists a parabolic basin B for the origin, that is, an open set with the origin at its boundary, where every point converges to the origin after iteration by f. There exists in fact a conjugacy of f to the translation map $g(w) = w+1$ in the set B. Similarly, there exists a repelling basin R converging to 0 under backward iteration. Likewise, we can construct a conjugacy to the translation. In this particular case, the union of B and R contains a full pinched neighbourhood of the origin [15].

Our goal is to describe the dynamics of parabolic maps in two dimensions in a similar fashion. That is, we would like to divide an entire neighbourhood of the origin into several open sets, in such a way that we can conjugate our parabolic map to a simpler map.

Our main results are stated as Theorem 5.1 and Theorem 5.2 which can be sumarized as follows.

Theorem. *Let F be as in* (1.1)*. Then, after a change of coordinates, the set*

$$U = \{(z, w) \in \mathbb{C}^2, |z| < \epsilon, |w| < \epsilon, |w| < |z|^M\},$$

(where M can be chosen as large as desired) can be divided into regions where F is conjugated to a translation map.

One immediate consequence of the theorem above is that the set U is foliated by invariant curves.

While most of the conjugacy maps for the hyperbolic case can be obtained as a limit of iterates, Fatou coordinates are in general not so easily computed. In this article we provide formulas for Fatou coordinates for the class of skew-product parabolic maps as in (1.1).

This paper is organized as follows. In Section 2 we write down properties of Fatou coordinates, namely the way they are modified after we perform change of coordinates. In Section 3 we recall results from dimension one. Section 4 gathers results from [26], where we find a complete description of the dynamics of a more particular class of parabolic maps on a whole neighbourhood of the origin. In Section 5 we work the main theorem.

2 Fatou coordinates

Since we use Fatou coordinates of different maps throughout our work, we write down here the main definitions and properties. Set $F^1 = F$ and $F^k = F \circ F^{k-1}$ for all $k \geq 2$.

Let $F : (\mathbb{C}^n, p) \to (\mathbb{C}^n, p)$ be a holomorphic germ with a fixed point p subject to $DF(p) = \mathrm{Id}$ and $F \neq \mathrm{Id}$. We will alternate between our fixed point p being the origin and the point at infinity. The hypotheses on the derivative of F guarantees that there is a local well defined inverse holomorphic germ F^{-1} on a neighbourhood of p.

Given $\zeta \in \mathbb{C}^k$ we write $T_\zeta : \mathbb{C}^k \to \mathbb{C}^k$ for the translation map $T_\zeta(z) = z + \zeta$. Unless otherwise stated, we take $\zeta \neq 0$.

Let $U^{\mathrm{i},F} \subset \mathbb{C}^n$ be an open connected set such that $p \in \partial U^{\mathrm{i},F}$ and such that for any $z \in U^{\mathrm{i},F}$, we have *(i)* $F(z) \in U^{\mathrm{i},F}$ and *(ii)* $\lim_{k\to\infty} F^k(z) = p$. When this is possible, we say $U^{\mathrm{i},F}$ is a **parabolic attracting basin** of F.

Let $U^{\mathrm{i},F} \subset \mathbb{C}^n$ be an attracting basin of F. Assume there is a holomorphic map $\phi^{\mathrm{i},F} : U^{\mathrm{i},F} \to \mathbb{C}^k$ such that the diagram

$$\begin{tikzcd} U^{\mathrm{i},F} \arrow[r, "F"] \arrow[d, "\phi^{\mathrm{i},F}"'] & U^{\mathrm{i},F} \arrow[d, "\phi^{\mathrm{i},F}"'] \\ \mathbb{C}^k \arrow[r, "T_\zeta"] & \mathbb{C}^k \end{tikzcd} \tag{2.1}$$

commutes. Then we say $\phi^{\mathrm{i},F}$ is an **incoming Fatou map** for F and $U^{\mathrm{i},F}$ with translation T_ζ.

Remark 2.1. If $\phi^{\mathrm{i},F}$ is an incoming Fatou map for F and $U^{\mathrm{i},F}$ with translation T_ζ, then $\lambda\phi^{\mathrm{i},F}$ is an incoming Fatou map for F and $U^{\mathrm{i},F}$ with translation $T_{\lambda\zeta}$ once we choose $\lambda \in \mathbb{C}^*$.

Remark 2.2. Note that we do not require $\phi^{\mathrm{i},F}$ to have an inverse map. In fact, in some cases, k, the target dimension of $\phi^{\mathrm{i},F}$, can be strictly smaller than n; in such case $\phi^{\mathrm{i},F}$ cannot be injective. In the literature this is sometimes referred as a **semi-conjugacy**.

Repelling basins as well as repelling Fatou maps are defined by considering the local inverse map F^{-1}. We do this next.

Whenever $U^{\mathrm{o},F} \subset \mathbb{C}^n$ is an open set such that $p \in \partial U^{\mathrm{o},F}$ and $U^{\mathrm{o},F}$ is an open attracting basin of F^{-1}, we will call $U^{\mathrm{o},F}$ a **repelling basin** of F.

Assume there exists an incoming Fatou map ψ for F^{-1} and $U^{\mathrm{o},F}$ with translation $T_{-\zeta}$ so that the diagram

$$\begin{tikzcd} U^{\mathrm{o},F} \arrow[r, "F^{-1}"] \arrow[d, "\psi"'] & U^{\mathrm{o},F} \arrow[d, "\psi"'] \\ \mathbb{C}^k \arrow[r, "T_{-\zeta}"] & \mathbb{C}^k \end{tikzcd}$$

commutes. Assume in addition that ψ has a holomorphic inverse map (this will imply $n = k$). Then we call the inverse $\phi^{\mathrm{o},F} = \psi^{-1}$ defined on $\psi(U^{\mathrm{o},F})$ the **outgoing Fatou map** for F and U^{o} with respect to T_ζ. A closer look at the functional equation satisfied by ψ and F^{-1} yields

$$F(\phi^{\mathrm{o},F}(z)) = \phi^{\mathrm{o},F}(z+\zeta) \text{ or } F \circ \phi^{\mathrm{o},F} = \phi^{\mathrm{o},F} \circ T_\zeta. \tag{2.2}$$

We point out some basic facts related to the concepts above.

Remark 2.3. When $n = 1$, we can assume without loss of generality $\zeta = 1$. The incoming or outgoing change of coordinates are usually referred to as, respectively, incoming or outgoing Fatou coordinates.

Remark 2.4. When $n \geq 2$ and $k = 1$, the incoming change of coordinate has been used in the past to prove the existence of Fatou-Bieberbach maps for automorphisms of $\mathbb{C}^n$ (compare [8], [25]).

Remark 2.5. It is easy to see that Fatou coordinates are not unique. From the functional relations we see that compositions (respectively pre-compositions) of translations with incoming (respectively outgoing) Fatou coordinates are also incoming (respectively outgoing) Fatou coordinates.

When there is no risk of confusion, we simply write ϕ^{i} and ϕ^{o}. For now, though, we stick to the superscript for referring to the maps in question since we want to establish how they behave when changing coordinates.

Proposition 2.6. *Let F be a parabollic germ as above. Assume F and F^{-1} have attracting basins $U^{i,F}$ and $U^{o,F} = U^{i,F^{-1}}$. Then F^{-1} has also a repelling basin, namely we can take $U^{o,F^{-1}} = U^{i,F}$. Let also $\phi^{i,F}$ (respectively $\phi^{o,F}$) be an incoming (respectively outgoing) Fatou coordinate for F and $U^{i,F}$ with respect to T_ζ. Assume further that $\phi^{i,F}$ has a well defined inverse map. Then the following formulas*

$$\phi^{o,F^{-1}}(z) = (\phi^{i,F})^{-1}(-z), \qquad \phi^{i,F^{-1}}(z) = -(\phi^{o,F})^{-1}(z) \tag{2.3}$$

yield outgoing (respectively incoming) Fatou maps for F^{-1} and $U^{o,F^{-1}}$ (respectively $U^{i,F^{-1}}$) with respect to T_ζ.

Proof. The proof follows easily by verifying directly the respective equations and using Remark 2.1. □

Although the following proposition is trivial, we will use the transformation between Fatou coordinates for different maps repeatedly on the following sections.

Proposition 2.7. *Let η be a (local) change of coordinates between F and G as in*

$$\begin{array}{ccc} (\mathbb{C}^n, q) & \xrightarrow{G} & (\mathbb{C}^n, q) \\ \eta \big\downarrow & & \eta \big\downarrow \\ (\mathbb{C}^n, p) & \xrightarrow{F} & (\mathbb{C}^n, p). \end{array} \tag{2.4}$$

Assume $U^{i,F}$ and $U^{o,F}$ are attracting and repelling basins for F along with $\phi^{i,F}$ and $\phi^{o,F}$ injective Fatou coordinates for F. Then the following provides attracting and repelling basins for G as well:

$$\begin{aligned} U^{i,G} &= \eta^{-1}(U^{i,F}), & U^{o,G} &= \eta^{-1}(U^{o,F}), \\ \phi^{i,G} &= \phi^{i,F} \circ \eta, & \phi^{o,G} &= \eta^{-1} \circ \phi^{o,F}. \end{aligned} \tag{2.5}$$

here $\phi^{i,G}$ (respectively $\phi^{o,G}$) is defined in $U^{i,G}$ (respectively $U^{o,G}$). □

Remark 2.8. One observation that we will use repeatedly on the next sections is that we do not need η to be defined on a whole neighbourhood of the origin. In fact, it is enough for η to be defined only on $U^{\mathrm{i},G}$.

3 Fatou coordinates in one dimension

Consider a parabolic germ at the origin of the form

$$f(z) = z + az^2 + O(|z|^3),$$

with $a \neq 0$. By the standard change of coordinates $Z = -az$ we reduce to the case $a = -1$. The following is the classic theorem of Leau and Fatou. See [15] for details.

Theorem 3.1. *(Leau-Fatou theorem) Take f as above. Then there exist $U^{i,f}$ and $U^{o,f}$ such that $U^{i,f} \cup U^{o,f}$ forms a punctured neighbourhood of the origin. In each of these open sets we can define incoming and outgoing Fatou coordinates $\phi^{i,f} : U^{i,f} \to \mathbb{C}$, $\phi^{o,f} : \psi(U^{o,f}) \to U^{o,f}$.* □

We can write down an explicit choice for the sets $U^{\mathrm{i},f}$ and $U^{\mathrm{o},f}$. Indeed, for any f there exist $\epsilon > 0$ so that $V_\epsilon = \{\zeta \in \mathbb{C}, |\zeta| < \epsilon, |\mathrm{Arg}(\zeta)| < 3\pi/4\}$ is an attracting basin and $-V_\epsilon$ is a repelling basin.

Next we translate all the action to a neighbourhood of ∞ using the reciprocal involution $S(z) = 1/z$. We then obtain

$$g(w) = w + 1 + \frac{\alpha}{w} + O(1/w^2),$$

where $g = S \circ f \circ S$. Our fixed point is relocated now at infinity. Let $S_R = S(V_\epsilon) = \{|w| > R, |\mathrm{Arg}(w)| < 3\pi/4\}$, where $R = 1/\epsilon$. Then we see that $U^{\mathrm{i},g}$ and $U^{\mathrm{o},g}$ can be chosen as S_R and $-S_R$, respectively.

We first start with a lemma as preparation.

Lemma 3.2. *Let $g(w) = w + 1 + \frac{\alpha}{w} + O(1/w^{1+\delta})$, where $\alpha \in \mathbb{C}$ and $\delta > 0$, be a holomorphic map defined on a neighbourhood of infinity. Take $U^{i,g}$ and $U^{o,g}$ as, respectively, attracting and repelling basins for g. Define $L_\alpha(w) = w + \alpha\log(w)$. Then $\rho = L_{-\alpha} \circ g \circ (L_{-\alpha})^{-1}$ is defined on $W = L_{-\alpha}(U^{i,g})$ and sastifies $\rho(W) \subset W$ and $\rho(w) = w+1+O(1/w^{1+\delta})$. Similarly, for $\tau = (L_\alpha)^{-1} \circ g \circ L_\alpha$ defined on $V = (L_\alpha)^{-1}(U^{o,g})$ we have $\tau(V) \supset V$ and $\tau(w) = w + 1 + O(1/w^{1+\delta})$.*

Proof. Note that L_α and $L_{-\alpha}$ are one-to-one maps on $U^{\mathrm{i},g}$ and $U^{\mathrm{o},g}$. The rest of the assertions are immediate. $\square$

Now consider the maps $\phi^{\mathrm{i},\rho}(w) = \lim_{n\to\infty} \rho^n(w) - n$. We see that this map is well defined on all of W (since $\rho(W) \subset W$) and from the estimate

$$|\rho^{n+1}(w) - \rho^n(w) - 1| = |O(1/(\rho^n(w))^{1+\delta}| = O(1/n^{1+\delta})$$

we deduce that $\{\rho^n(w) - n\}$ forms a Cauchy sequence. Hence the convergence is uniform on compact subsets of $U^{\mathrm{i},g}$.

Similarly $\phi^{o,\tau}(w) = \lim_{n\to\infty} \tau^n(w - n)$ is well defined on the open subset of $\mathbb{C}$ where it converges. Using the relations $\phi^{\mathrm{i},g} = \phi^{\mathrm{i},\rho} \circ L_{-\alpha}$ and $\phi^{\mathrm{o},g} = L_\alpha \circ \phi^{o,\tau}$ we can establish the following.

Proposition 3.3. *Let $g(w) = w+1+\frac{\alpha}{w}+O\left(\frac{1}{w^2}\right)$ be a germ at infinity. Then we take the incoming Fatou coordinate $\phi^{i,g} : U^{i,g} \to \mathbb{C}$ of g as the limit*

$$\phi^{i,g}(w) = \lim_{n\to\infty} L_{-\alpha}(g^n(w)) - n. \tag{3.1}$$

Similarly, we can find an outgoing Fatou coordinate $\phi^{i,g} : U^{o,g} \to \mathbb{C}$ by means of the formula

$$\phi^{o,g}(w) = \lim_{n\to\infty} g^n(L_\alpha(w-n)) \tag{3.2}$$

(recall the definition $L_\alpha(w) = w + \alpha \log(w)$).

4 Fatou coordinates in two dimensions

Let us recall our results from [26] for skew parabolic maps of the particular form

$$F(z,w) = \left(\frac{z}{1+z}, f_z(w)\right) = \left(\frac{z}{1+z}, w - w^2 + w^3 + O(w^4, zw^4)\right). \tag{4.1}$$

As usual, set $V_\epsilon = \{\zeta \in \mathbb{C}, |\zeta| < \epsilon, |\mathrm{Arg}(\zeta)| < 3\pi/4\}$. In dimension one, the union $V_\epsilon \cup (-V_\epsilon)$ forms a punctured neighbourhood of the origin. In dimension two, we use the four subsets

$$\begin{aligned} U^{\mathrm{i}} &= V_\epsilon \times V_\epsilon, \\ U^{\mathrm{o}} &= (-V_\epsilon) \times (-V_\epsilon), \\ U^{\mathrm{a}} &= (-V_\epsilon) \times V_\epsilon, \\ U^{\mathrm{b}} &= V_\epsilon \times (-V_\epsilon). \end{aligned} \tag{4.2}$$

Note that their union covers a full neighbourhood of the origin with the exception of the two axis $\{zw = 0\}$. (Anyway, since we have $F(0,w) = (0, f_0(w)))$ and $F(z,0) = (\frac{z}{1+z}, 0)$, the orbits of F on both axis are fully understood.)

As in the one dimensional case, we change variables so that the fixed point is at infinity by using the conjugation map $S(z,w) = (1/z, 1/w)$. In this way $G = S \circ F \circ S$ can be explicitly written as

$$G(u,v) = (u+1, g_u(v)) = \left(u+1, v+1+O\left(\frac{1}{v^2}, \frac{1}{uv^2}\right)\right). \tag{4.3}$$

Let $S_R = \{|\zeta| > R, |\mathrm{Arg}(\zeta)| < 3\pi/4\}$ with $R = 1/\epsilon$ (so that we have $I(\pm V_\epsilon) = \pm S_R$). We focus our attention on the sets

$$\begin{aligned} W^{\mathrm{i}} &= \quad S_R \times S_R, \\ W^{\mathrm{o}} &= -S_R \times -S_R, \\ W^{\mathrm{a}} &= -S_R \times S_R, \\ W^{\mathrm{b}} &= \quad S_R \times -S_R. \end{aligned} \tag{4.4}$$

Let $T_{(a,b)} : \mathbb{C}^2 \to \mathbb{C}^2$ be defined as $T_{(a,b)}(z, w) = (z + a, w + b)$.

Theorem 4.1. *Let G be as in* (4.3).
(a) For any $p \in W^i$, the iterates $G^n(p)$ converge to infinity. We have a Fatou coordinate given by $\Phi^{i,G} = \lim_{n\to\infty} T_{(-n,-n)} \circ G^n$ so that the diagram

$$\begin{array}{ccc} W^i & \xrightarrow{G} & W^i \\ {\scriptstyle \Phi^{i,G}}\big\downarrow & & {\scriptstyle \Phi^{i,G}}\big\downarrow \\ \mathbb{C}^2 & \xrightarrow{T_{(1,1)}} & \mathbb{C}^2 \end{array}$$

commutes.
(b) For any $p \in W^o$, the backward iterates $G^{-n}(p)$ converges to infinity. We have a Fatou coordinate given by $\Phi^{o,G} = \lim_{n\to\infty} G^n \circ T_{(-n,-n)}$ so that the diagram

$$\begin{array}{ccc} G^{-1}(W^o) & \xrightarrow{G} & W^o \\ {\scriptstyle \Phi^{o,G}}\big\uparrow & & {\scriptstyle \Phi^{o,G}}\big\uparrow \\ N \subset \mathbb{C}^2 & \xrightarrow{T_{(1,1)}} & T_{(1,1)}(N) \subset \mathbb{C}^2 \end{array}$$

commutes. Here we have $N = (\Phi^{o,G})^{-1}(G^{-1}(W^o))$ and $T_{(1,1)}(N) = (\Phi^{o,G})^{-1}(W^o)$.

Proof. We prove first the existence of $\Phi^{\mathrm{i},G}$. Then we will apply this same result to prove the analogue for $\Phi^{\mathrm{o},G}$.

Let $(u, v) \in W^{\mathrm{i}}$, that is $u \in S_R$ and $v \in S_R$, and write $(u_n, v_n) = G^n(u, v)$. We have $u + 1 \in S_R$ and $|v_1 - v - 1| < 1/10$ for R large enough. In this way we have $(u_1, v_1) \in W^{\mathrm{i}}$ and therefore, by induction, also $(u_n, v_n) \in W^{\mathrm{i}}$ for positive n. In fact we have $u_n = u_0 + n$ and $|v_n - v_0| = O(n)$.

For $\Phi_n^{\mathrm{i},G} = T_{(-n,-n)} \circ G^n$, a simple computation yields

$$\begin{aligned} |\Phi_{n+1}^{\mathrm{i},G} - \Phi_n^{\mathrm{i},G}| &= |G^{n+1}(u,v) - G^n(u,v) - (1,1)| \\ &= |G(u_n, v_n) - (u_n, v_n) - (1,1)| \\ &= |(0, O(1/v_n^2, 1/(u_n v_n^2)))|. \end{aligned}$$

As u_n and v_n are of growth $O(n)$ when $(u,v) \in W^{\mathrm{i}}$, we conclude that $\Phi_n^{\mathrm{i},G}$ converges uniformly in compact sets of W^{i}.

For the outgoing coordinate we write $\Phi_n^{\mathrm{o},G} = G^n \circ T_{(-n,-n)}$. Then we have the relation

$$\Phi_n^{\mathrm{o},G} \circ \eta \circ \Phi_n^{\mathrm{i},H} \circ \eta = \mathrm{Id}, \tag{4.5}$$

here $H = \eta \circ G^{-1} \circ \eta$ and $\eta(u,v) = (-u,-v)$. Since we have $H(u,v) = (u+1, v+1+O(1/v^2))$, we note that $\Phi_n^{\mathrm{i},H}$ converges, and therefore $\Phi_n^{\mathrm{o},G}$ does also. Finally, as $\eta(W^{\mathrm{i}}) = W^{\mathrm{o}}$ and $H(W^{\mathrm{i}}) \subset W^{\mathrm{i}}$ hold, we conclude $W^{\mathrm{o}} \subset \Phi^{\mathrm{o},G}(W^{\mathrm{o}})$. $\square$

We also have the following.

Theorem 4.2. *Let G be as in* (4.3).
(a) The map $\Psi^{a,G} = \lim_{n\to\infty} T_{(n,-n)} \circ G^n \circ T_{(-2n,0)}$ converges uniformly on compact subsets of W^a and fits into the commutative diagram

$$\begin{array}{ccc} W^a & \xrightarrow{(-1,g_\infty)} & W^a \\ {\scriptstyle \Psi^{a,G}}\big\downarrow & & {\scriptstyle \Psi^{a,G}}\big\downarrow \\ \mathbb{C}^2 & \xrightarrow{T_{(-1,1)}} & \mathbb{C}^2. \end{array}$$

(b) The map $\Psi^{b,G} = \lim_{n\to\infty} T_{(-2n,0)} \circ G^n \circ T_{(n,-n)}$ converges uniformly on compact sets of W^b and we have the diagram

$$\begin{array}{ccc} L^{-1}(W^b) & \xrightarrow{L=(-1,g_\infty)} & W^b \\ {\scriptstyle \Psi^{b,G}}\big\uparrow & & {\scriptstyle \Psi^{b,G}}\big\uparrow \\ E \subset \mathbb{C}^2 & \xrightarrow{T_{(-1,1)}} & T_{(-1,1)}(E) \subset \mathbb{C}^2. \end{array}$$

Proof. Define $\Psi_n^{\mathrm{a},G} = T_{(n,-n)} \circ G^n \circ T_{(-2n,0)}$ and $\Psi_n^{\mathrm{b},G} = T_{(-2n,0)} \circ G^n \circ T_{(n,-n)}$. Unraveling the definition we arrive to

$$\Psi_n^{\mathrm{b},G}(u,v) = (u, g_{u+2n-1} \circ \ldots \circ g_{u+n+1} \circ g_{u+n}(v-n)).$$

In [26] it is proven that the sequence of functions

$$\psi_n^o(v) = g_{-v+\alpha+2n-1} \circ \ldots \circ g_{-v+\alpha+n+1} \circ g_{-v+\alpha+n}(v-n)$$

converges for any $\alpha \in \mathbb{C}$ and $v \in -S_R$ with limit $\psi^o(v+1) = g_\infty(\psi^o(v))$. Applying this result for $\alpha = u - v$, we obtain the convergence of the sequence $\Psi_n^{\mathrm{b},G}(u,v) \to (u, \psi^o(v))$, where $\psi^o(v+1) = g_\infty(\psi^o(v))$.

For the other coordinate we use the identity

$$\Psi_n^{\mathrm{a},G} \circ \eta \circ \Psi_n^{\mathrm{b},H} \circ \eta = \mathrm{Id}, \tag{4.6}$$

where $H = \eta \circ G^{-1} \circ \eta$ and $\eta(u,v) = (-u,-v)$. Since we have $H(u,v) = (u+1, v+1+O_u(1/v^2))$, we conclude that $\Psi_n^{\mathrm{b},H}$ converges, and therefore $\Psi_n^{\mathrm{a},G}$ also does. □

We sumarize the results thus obtained for our map F. We name $(0,w)$ the **invariant fiber** of F.

Theorem 4.3. *Let F be as in* (4.1) *and U^i, U^o, U^a and U^b be defined as in* (4.2).
(a) The union of U^i, U^o, U^a and U^b together with the axes form a neighbourhood of the origin in $\mathbb{C}^2$.
(b) For any $q \in U^i$ we have $F^n(q) \in U^i$. Furthermore F^n converges to the origin uniformly in compact sets of U^i.
(c) For any $q \in U^o$ we have $F^{-n}(q) \in U^o$. Furthermore F^{-n} converges to the origin uniformly in compact sets of U^o.
(d) For any $q \in U^a$ we have that $F^{-n}(q)$ converges to the w-axis, the invariant fiber of the map F.
(e) For any $p \in U^b$ we have that $F^n(p)$ converges to the w-axis, the invariant fiber of the map F.

5 The general case

We are ready to tackle the general case. Consider now the map

$$F(z,w) = (\lambda(z), f_z(w)), \tag{5.1}$$

where $\lambda(z) = z + O(z^2)$ and $f_z(w) = w + O(|(z,w)|^2)$. We focus on the particular case

$$F(z,w) = (z + a_2 z^2 + O(z^3), w + b_2 w^2 + O(|(z,w)|^3)),$$

with $a_2 \neq 0$ and $b_2 \neq 0$.

By a change of coordinates we can even assume $a_2 = -1$ and $b_2 = -1$. Using a shear polynomial as a further change of coordinates, we can increase the power of z on the second term. Similarly by using another polynomial change of variables we can increase the degree of the z term that is multiplied by w. Therefore we can assume F takes the form

$$F(z,w) = (z - z^2 + O(z^3), w - w^2 + O(w^3, zw^2, z^{M+1}w, z^{M+1})), \tag{5.2}$$

with M as large as we wish.

As we still have $F(0,w) = (0, w - w^2 + O(w^3))$, we can again refer to the w-axis as an invariant fiber of F. However, this time we have $F(z,0) = (z - z^2 + O(z^3), O(z^{M+1}))$, so the z-axis is no longer invariant. To work around this issue we use the main result of [8] which states that on V_ϵ there exists an analytic function $\phi_1(z)$ subject to $\phi_1(\lambda(z)) = f_z(\phi_1(z))$. Similarly on $-V_\epsilon$ there exists an analytic function $\phi_2(z)$ such that $\phi_2(\lambda(z)) = f_z(\phi_2(z))$.

We can therefore change coordinates on $V_\epsilon \times \{|w| < \epsilon\}$ by means of $(z,w) \mapsto (z, w - \phi_1(z))$, and on the set $(-V_\epsilon) \times \{|w| < \epsilon\}$ by $(z,w) \mapsto (z, w - \phi_2(z))$. On these new coordinates we read

$$F(z,w) = (z - z^2 + O(z^3), w - w^2 + O(w^3, zw^2, z^{M+1}w)). \tag{5.3}$$

The next step is to linearize the first coordinate. As we know from the one dimensional theory, there exist two maps, ρ_1 on V_ϵ and ρ_2 on $-V_\epsilon$, that conjugate $\lambda(z)$ to the translation $u \to u+1$ on V_ϵ and $-V_\epsilon$. We have for both the estimate $\rho(z) = z + O(z^2 \log(z))$ (see [15] for details).

One more time we use the change of coordinate $(u, w) \mapsto (u, v) = (u, 1/w)$. On the new system we have

$$G(u, v) = \left(u + 1, v + 1 + O\left(\frac{1}{u}, \frac{1}{v}, \frac{\log(u)}{u^2}, \frac{v}{u^{M+1}}, \frac{v\log(u)}{u^{M+2}}\right)\right). \quad (5.4)$$

Now, as before, we divide a punctured neighbourhood of infinity in several sets:

$$\begin{aligned} W^{\mathrm{i}} &= S_R \times S_R, & W^{\mathrm{b}} &= S_R \times (-S_R) \\ W^{\mathrm{a}} &= (-S_R) \times S_R, & W^{\mathrm{o}} &= (-S_R) \times (-S_R). \end{aligned}$$

From now on, when we refer to a region W, we mean one of the four possible $W^{\mathrm{i}}, W^{\mathrm{b}}, W^{\mathrm{a}}$ or W^{o}.

Consider the class of maps $\Theta(u, v) = (u, v + \alpha\log(u) + \beta\log(v))$. It is immediate that after choosing R large enough Θ is injective in each region W. So, by conjugating G_j by Θ and choosing α and β appropriately, we can get rid of the linear terms $O(1/u, 1/v)$.

To emphasize that each of these maps is a different conjugation of G on each set W: we use θ_{i} for the change of coordinates on W^{i}, θ_{o} on W^{o}, θ_{a} on W^{a} and θ_{b} on W^{b}. We write $G_{\mathrm{i}} = (\theta_{\mathrm{i}})^{-1} \circ G_1 \circ \theta_{\mathrm{i}}$, the corresponding map defined on W^{i}, and $G_{\mathrm{o}} = (\theta_{\mathrm{o}})^{-1} \circ G_2 \circ \theta_{\mathrm{o}}$ on W^{o}, and $G_{\mathrm{a}} = (\theta_{\mathrm{a}})^{-1} \circ G_2 \circ \theta_{\mathrm{a}}$ on W^{a}, and $G_{\mathrm{b}} = (\theta_{\mathrm{b}})^{-1} \circ G_1 \circ \theta_{\mathrm{b}}$ on W^{b}. Now, the composition $\Theta^{-1} \circ G_j \circ \Theta(u, v)$ equals

$$\left(u + 1, v + 1 + O\left(\frac{1}{u^2}, \frac{1}{v^2}, \frac{\log(v)}{v^2}, \frac{\log(u)}{v^2}, \frac{v}{u^{M+1}}, \frac{v\log(u)}{u^{M+2}}\right)\right).$$

Note the similarity with the special map on last section where we have

$$G(u, v) = (u + 1, g_u(v)) = \left(u + 1, v + 1 + O\left(\frac{1}{v^{1+\delta}}, \frac{1}{uv^{1+\delta}}\right)\right).$$

In order to control the mixed terms in u and v on our maps G we define

$$\begin{aligned} \widetilde{W^{\mathrm{i}}} &= \{(u, v) \in S_R \times S_R, |u|^{M-1} > |v|\}, \quad (5.5) \\ \widetilde{W^{\mathrm{o}}} &= \{(u, v) \in (-S_R) \times (-S_R), |u|^{M-1} > |v|\}. \end{aligned}$$

For $(u,v) \in \widetilde{W^{\mathrm{i}}}$ we have $G_{\mathrm{i}}^n(u,v) = (u+n, v+O(n))$. Therefore, eventually we reach $G_{\mathrm{i}}^n(u,v) \in \widetilde{W^{\mathrm{i}}}$. Using results from the last section, it is possible to conjugate G_{i} to a translation on $\widetilde{W^{\mathrm{i}}}$ by means of the limit $\Phi^{\mathrm{i},G_{\mathrm{i}}} = \lim_{n\to\infty} T_{(-n,-n)} \circ G_{\mathrm{i}}^n$, with $\Phi^{\mathrm{i},G_{\mathrm{i}}} : \widetilde{W^{\mathrm{i}}} \to \mathbb{C}^2$. If we unravel for G, we obtain a formula for the Fatou coordinate on the incoming basin for G as

$$\Phi^{\mathrm{i},G}(u,w) = \lim_{n\to\infty} T_{(-n,-n)} \circ \theta_{\mathrm{i}}^{-1} \circ \Psi_1^{-1} \circ G^n, \tag{5.6}$$

where Ψ_1 is the composition of the change of coordinates from above and $W^{\mathrm{i},G} = \Psi_1(\widetilde{W^{\mathrm{i}}})$.

Similarly, we obtain $G_{\mathrm{o}}(\widetilde{W^{\mathrm{o}}}) \supset G_{\mathrm{o}}$. By our work on the last section we achieve a conjugation $\Phi^{\mathrm{o},G_{\mathrm{o}}} : \widetilde{W^{\mathrm{o}}} \to \mathbb{C}^2$ of G on $W^{\mathrm{o},G}$ to the translation

$$\Phi^{\mathrm{o},G_{\mathrm{o}}} = \lim_{n\to\infty} G_{\mathrm{o}}^n \circ T_{(-n,-n)}.$$

Rewritting for G we obtain the Fatou coordinate $\Phi^{\mathrm{o},G} : W^{\mathrm{o},G} \to \mathbb{C}^2$ on the outgoing basin for G as

$$\Phi^{\mathrm{o},G}(u,w) = \lim_{n\to\infty} G^n \circ \Psi_2 \circ \theta_{\mathrm{o}} \circ T_{(-n,-n)}. \tag{5.7}$$

We have therefore established the following result.

Theorem 5.1. *Given G as in (5.4), we can find incoming and outgoing Fatou coordinates for the respective incoming and outgoing basins at infinity.* □

We also obtain information on the behavior of G on the regions W^{a} and W^{b}, since we can apply Theorem 4.2 to the maps G_{a} and G_{b} respectively.

In Theorem 4.2 we proved that on the region W^{a} the map $\Psi^{\mathrm{a}} = \lim_{n\to\infty} T_{(n,-n)} \circ G_{\mathrm{a}}^n \circ T_{(-2n,0)}$ satisfies $\Psi^{\mathrm{a}} \circ (-1, g_\infty) = T_{(-1,1)} \circ \Psi^{\mathrm{a}}$. Since $G_{\mathrm{a}} = (\theta_{\mathrm{a}})^{-1} \circ G_2 \circ \theta_{\mathrm{a}}$ holds, we get $G_{\mathrm{a}}^n = (\theta_{\mathrm{a}})^{-1} \circ (\Psi_2)^{-1} \circ G^n \circ \Psi_2 \circ \theta_{\mathrm{a}}$, and so $\Psi^{\mathrm{a}} = \lim_{n\to\infty} T_{(n,-n)} \circ (\theta_{\mathrm{a}})^{-1} \circ (\Psi_2)^{-1} \circ G^n \circ \Psi_2 \circ \theta_{\mathrm{a}} \circ T_{(-2n,0)}$ converges and fits into a corresponding commutative diagram.

Similarly, on the region W^{b}, the map $\Psi^{\mathrm{b}} = \lim_{n\to\infty} T_{(-2n,0)} \circ G_{\mathrm{b}}^n \circ T_{(n,-n)}$ satisfies $\Psi^{\mathrm{b}} \circ (-1, g_\infty) = T_{(-1,1)} \circ \Psi^{\mathrm{b}}$. From $G_{\mathrm{b}} = (\theta_{\mathrm{b}})^{-1} \circ G_1 \circ \theta_{\mathrm{b}}$, we obtain $G_{\mathrm{b}}^n = (\theta_{\mathrm{b}})^{-1} \circ (\Psi_1)^{-1} \circ G^n \circ \Psi_1 \circ \theta_{\mathrm{b}}$, and thus $\Psi^{\mathrm{b}} = \lim_{n\to\infty} T_{(-2n,0)} \circ (\theta_{\mathrm{b}})^{-1} \circ (\Psi_1)^{-1} \circ G^n \circ \Psi_1 \circ \theta_{\mathrm{b}} \circ T_{(n,-n)}$ converges and fits again into the corresponding commutative diagram.

We have thus settle the following.

Theorem 5.2. *For G as in* (5.4)*, on the regions*

$$\begin{aligned} \widetilde{W^a} &= \{(u,v) \in -S_R \times S_R, |u|^{M-1} > |v|\}, \\ \widetilde{W^b} &= \{(u,v) \in S_R \times (-S_R), |u|^{M-1} > |v|\}, \end{aligned} \tag{5.8}$$

the limits: $\Psi^a = \lim_{n\to\infty} T_{(n,-n)} \circ (\theta_a)^{-1} \circ (\Psi_2)^{-1} \circ G^n \circ \Psi_2 \circ \theta_a \circ T_{(-2n,0)}$ *and* $\Psi^b = \lim_{n\to\infty} T_{(-2n,0)} \circ (\theta_b)^{-1} \circ (\Psi_1)^{-1} \circ G^n \circ \Psi_1 \circ \theta_b \circ T_{(n,-n)}$ *exist. Furthermore, the second coordinate conjugates g_∞ to the translation T_1, that is, the diagram*

$$\begin{array}{ccc} \widetilde{W^{a,b}} & \xrightarrow{(-1,g_\infty)} & \widetilde{W^{a,b}} \\ {\scriptstyle \Psi^{a,b}}\big\downarrow & & {\scriptstyle \Psi^{a,b}}\big\downarrow \\ \mathbb{C}^2 & \xrightarrow{T_{(-1,1)}} & \mathbb{C}^2 \end{array}$$

commutes for each case $\widetilde{W^a}$ and $\widetilde{W^b}$. □

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Resumen

La dinámica local en torno a vecindades de un punto fijo ha sido ampliamente estudiada tanto para gérmenes de una como de varias variables complejas. En dimensión uno disponemos de un cuadro casi completo de la trayectoria de las órbitas en torno a una vecindad del punto fijo. No obstante, en dimensiones más altas, apenas se cuenta con resultados parciales. En este trabajo analizamos un caso intermedio entre las dinámicas de una y varias variables. Consideramos aplicaciones de productos trenzados de la forma $F(z, w) = (\lambda(z), f(z, w))$ y tratamos el caso parabólico, es decir, cuando $DF(0, 0) = \mathrm{Id}$. Describimos el comportamiento de órbitas en torno a vecindades del origen. Además, establecemos fórmulas para las aplicaciones de conjugación en diferentes regiones.

Palabras clave: Aplicaciones de productos trenzados, coordenadas de Fatou.

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Adjoint orbits of $\mathfrak{sl}(2, \mathbb{R})$ and their geometry

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September, 2018

Abstract

Let $SL(2, R)$ be the special linear group and $\mathfrak{sl}(2, \mathbb{R})$ its Lie algebra. We study geometric properties associated to the adjoint orbits. In particular, we show that just three possibilities arise: either the adjoint orbit is a one-sheeted hyperboloid, or a two-sheeted hyperboloid, or else a cone. In addition, we introduce a specific potential and study the corresponding gradient vector field and its dynamics when we restrict to the adjoint orbit. We also describe the symplectic structure on these adjoint orbits coming from the well known Kirillov-Kostant-Souriau symplectic form on coadjoint orbits.

MSC(2010): Primary 22E46; Secondary 53D05.

Keywords: Adjoint orbits, symplectic structure.

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1 Introduction

This text is of expository nature. We carry out the exercise of explicitly describing adjoint orbits of $\mathfrak{sl}(2,\mathbb{R})$ together with the equations defining them as real affine algebraic varieties, over which we also describe symplectic structures.

We then focus on a single orbit that has the shape of a one-sheeted hyperboloid, presenting it as a doubly ruled surface whose tangent bundle we describe explicitly. We add a potential carefully chosen to be a Morse function, and study the orbits of the corresponding gradient flow. For the case of compact manifolds the classical Morse–Smale theorem states that the trajectories of the gradient flow converge to critical points of the potential. Here, in contrast, we show that some trajectories are not complete, thus highlighting the importance of the hypothesis of compactness in the Morse–Smale theory. For applications to mathematical physics it is essential to consider examples where some trajectories are not complete in time.

Even though our calculations are straightforward, we believe it is useful to have the results readily available in the literature. The study of the geometry of adjoint orbits is a classical topic in Geometry and Lie theory. However, the literature is mainly presented following an abstract approach, so, in this paper, we exhibit most of the details. Some references that focus in specific cases of adjoint and coadjoint orbits are [3], for classical compact Lie groups, and [1], where there is an excellent explanation of the geometry of flag manifolds arising from the adjoint representation of compact semisimple Lie groups.

We study the geometry of those adjoint orbits which arise from the adjoint representation $\mathrm{Ad}\colon \mathrm{SL}(2,\mathbb{R}) \to \mathfrak{gl}(\mathfrak{sl}(2,\mathbb{R}))$, where for each $g \in \mathrm{SL}(2,\mathbb{R})$ and $H \in \mathfrak{sl}(2,\mathbb{R})$ the adjoint action is $\mathrm{Ad}_g(H) = gHg^{-1}$. Let A, B, C be the basis of $\mathfrak{sl}(2,\mathbb{R})$ given by

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{1.1}$$

We decompose $H = xA + yB + zC$ and then find out that the adjoint orbits are of one of the following three types:

- a **one–sheeted hyperboloid**, given by the equation

$$\mathcal{O}\colon x^2+y^2-z^2=\lambda^2, \qquad \lambda\neq 0;$$

- a **two–sheeted hyperboloid**, given by

$$\mathcal{O}_1^+ : x^2+y^2-z^2=-\lambda^2, \qquad z>0, \lambda\neq 0,$$
$$\mathcal{O}_1^- : x^2+y^2-z^2=-\lambda^2, \qquad z<0, \lambda\neq 0;$$

- a **cone**, given by

$$\mathcal{O}_2^+ : x^2+y^2-z^2=0, \qquad z>0,$$
$$\mathcal{O}_2^- : x^2+y^2-z^2=0, \qquad z<0,$$
$$\mathcal{O}_2^0=\{0\}.$$

We endow the adjoint orbit $\mathcal{O}$ with the symplectic structure arising from a coadjoint orbit, thus realizing it as a symplectic manifold. Namely, we use the isomorphism between adjoint and coadjoint orbits provided by the Killing form to give this adjoint orbit the symplectic structure pulled-back from the well known Kirillov–Kostant–Souriau form on the corresponding coadjoint orbit.

We then consider the function $f(x,y,z)=yz$ over $\mathfrak{sl}(2,\mathbb{R})$ and regard its restriction to the orbit $\mathcal{O}$ as a Morse function, calculating the trajectories of its gradient vector field. We analyse the limit points of the gradient flow, and compare the results obtained here to well known results about Morse flows for the compact case.

We observe that every orbit of the adjoint action on $\mathfrak{sl}(2,\mathbb{R})$ is of one of the three types presented here, hence we have a complete description.

In general, understanding details of the family of all adjoint orbits for a given Lie algebra is a deep question with applications to non trivial aspects of the theory. Some such research areas, among many, are: the theory of Slodowy slices, the Springer theory, and the Fukaya categories in homological mirror aymmetry. Therefore, the calculations we present here may be regarded as a warm up exercise in preparation to the study of more advanced topics.

2 Preliminaries

We start by recalling some basic definitions of Lie theory. For further details, we suggest [6, 11].

A **Lie group** is a smooth manifold G with a smooth map from $G \times G \to G$ that makes G into a group and such that the inverse map $g \mapsto g^{-1}$ is also smooth.

Let $\mathrm{M}(n, \mathbb{R})$ be the set of $n \times n$ matrices with entries in the real numbers.

The **general linear group** $\mathrm{GL}(n, \mathbb{R})$ is the subset of $\mathrm{M}(n, \mathbb{R})$ of non-singular matrices with matrix multiplication as group operation.

By definition a **matrix Lie group** is a closed subgroup of $\mathrm{GL}(n, \mathbb{R})$.

For example, the **special linear group** $\mathrm{SL}(n, \mathbb{R})$ is the subgroup of $\mathrm{GL}(n, \mathbb{R})$ of non-singular matrices of determinant 1.

A **Lie algebra** is a vector space $\mathfrak{g}$ over a field $\mathbb{F}$ together with a **Lie bracket**, that is, a bilinear map

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \qquad (x, y) \mapsto [x, y]\,,$$

satisfying

- $[x, x] = 0$ for each $x \in \mathfrak{g}$,
- Jacobi identity: $[x, [y, z]]+[y, [z, x]]+[z, [x, y]] = 0$ for every $x, y, z \in \mathfrak{g}$.

Remark 2.1. If the characteristic of the $\mathbb{F}$ is not 2, then the first condition is equivalent to anticommutativity

$$[x, y] = -\,[y, x] \text{ for each } x, y \in \mathfrak{g}.$$

The **centre** of a Lie algebra consists of all those elements x in $\mathfrak{g}$, subject to $[x, y] = 0$ for all y in $\mathfrak{g}$.

Let $\mathfrak{g}_1, \mathfrak{g}_2$ be two Lie algebras over a field $\mathbb{F}$. A map $\varphi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is a **Lie algebra homomorphism** if φ is linear and satisfies

$$\varphi([x, y]) = [\varphi(x), \varphi(y)]\,,$$

for each $x, y \in \mathfrak{g}_1$. If φ is bijective, we call it an **isomorphism**.

There are several ways to understand the Lie algebra of a Lie group. Here we consider it as the tangent space at the identity element of the group, that is, if G is a Lie group, then its Lie algebra $\mathfrak{g}$ corresponds to T_eG.

For instance, $\mathrm{SL}(n,\mathbb{R})$ is a matrix Lie group with Lie algebra $\mathfrak{sl}(n,\mathbb{R})$. In terms of matrices, $\mathfrak{sl}(n,\mathbb{R})$ is the Lie algebra of $n\times n$ matrices with trace 0 and coefficients in $\mathbb{R}$, where the Lie bracket is the usual commutator $[X,Y]=XY-YX$.

Let A be a $n\times n$ matrix over $\mathbb{R}$ or $\mathbb{C}$. The **exponential** of A is the $n\times n$ matrix

$$\exp(A)=\sum_{k=0}^{\infty}\frac{A^k}{k!}.$$

An important result in Lie theory is that if G is a matrix Lie group with algebra $\mathfrak{g}$, then $\exp(A)\in G$ holds for each $A\in\mathfrak{g}$. Below we provide a direct proof when $G=\mathrm{SL}(n,\mathbb{R})$.

Proposition 2.2. *For any $A\in\mathfrak{sl}(n,\mathbb{R})$, we have $\exp(A)\in\mathrm{SL}(n,\mathbb{R})$.*

Proof. Consider the Jordan form of A. If $\{\lambda_i\}_{i=1}^{l}$ are the eigenvalues of A, then we have

$$\begin{aligned}\exp(A)=\sum_{k=0}^{\infty}\frac{A^k}{k!}&=\begin{bmatrix}\sum_{k=0}^{\infty}\frac{\lambda_1^k}{k!} & * & * & * \\ 0 & \sum_{k=0}^{\infty}\frac{\lambda_2^k}{k!} & * & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \sum_{k=0}^{\infty}\frac{\lambda_l^k}{k!}\end{bmatrix}\\ &=\begin{bmatrix}e^{\lambda_1} & * & * & * \\ 0 & e^{\lambda_2} & * & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & e^{\lambda_l}\end{bmatrix}.\end{aligned}$$

We get immediately the equality

$$\det(\exp(A))=\prod_{i=1}^{l}e^{\lambda_i}=e^{\sum_{i=1}^{l}\lambda_i}=e^{\mathrm{tr}(A)}=e^0=1,$$

and so $\exp(A)\in\mathrm{SL}(n,\mathbb{R})$. □

Let G be a Lie group with Lie algebra $\mathfrak{g}$. The **adjoint representation of G** on $\mathfrak{g}$ is the group homomorphism

$$\begin{array}{rcl} \mathrm{Ad}\colon G & \to & \mathrm{Aut}(\mathfrak{g}) \\ g & \mapsto & \mathrm{Ad}_g. \end{array}$$

For example, for $G = \mathrm{SL}(n, \mathbb{R})$ and $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$, the group homomorphism is given by

$$\begin{array}{rcl} \mathrm{Ad}\colon \mathrm{SL}(n, R) & \to & \mathrm{Aut}(\mathfrak{sl}(n, \mathbb{R})) \\ g & \mapsto & \mathrm{Ad}_g, \end{array}$$

where $\mathrm{Ad}_g(X) = gXg^{-1}$ for every $X \in \mathfrak{sl}(n, \mathbb{R})$.

Given $H \in \mathfrak{sl}(n, \mathbb{R})$, its **adjoint orbit** is

$$\mathcal{O}(H) = \{gHg^{-1} : g \in \mathrm{SL}(n, \mathbb{R})\}.$$

We will see that the geometric structure on adjoint orbits depends strongly on the element $H \in \mathfrak{sl}(2, \mathbb{R})$. We will give a complete characterization of those orbits.

The **adjoint representation of the Lie algebra** $\mathfrak{g}$ in $\mathfrak{gl}(\mathfrak{g})$ is the homomorphism

$$\begin{array}{rcl} \mathrm{ad}\colon \mathfrak{g} & \to & \mathfrak{gl}(\mathfrak{g}) \\ x & \mapsto & \mathrm{ad}_x, \end{array}$$

here $\mathrm{ad}_x(y) = [x, y]$ for each $x, y \in \mathfrak{g}$.

It follows by bilinearity of the Lie bracket that ad_x is linear for each $x \in \mathfrak{g}$; the same is true for the correspondence $x \mapsto \mathrm{ad}_x$. In order to prove that ad_x is a homomorphism we just have to check that ad_x satisfies the identity

$$\mathrm{ad}([x, y]) = \mathrm{ad}_x \circ \mathrm{ad}_y - \mathrm{ad}_y \circ \mathrm{ad}_x, \qquad \text{for every } x, y \in \mathfrak{g}.$$

The above equality holds precisely because of the Jacobi identity. The kernel of ad is the **centre** of $\mathfrak{g}$.

3 Adjoint orbits of $\mathfrak{sl}(2, \mathbb{R})$

Here we study the geometry of orbits of $\mathfrak{sl}(2, \mathbb{R})$ given by the adjoint action, namely, the action induced by the adjoint representation of $\mathrm{SL}(2, \mathbb{R})$ in its associated Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. We will classify them into three classes: either the adjoint orbit is a one–sheeted hyperboloid, or a two–sheeted hyperboloid, or else a cone, depending on the choice of the element that we take in the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. Recall the basis of $\mathfrak{sl}(2,\mathbb{R})$ introduced in (1.1), namely

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

3.1 The one–sheeted hyperboloid

Here we study the orbit of λA in $\mathfrak{sl}(2,\mathbb{R})$ for $\lambda \in \mathbb{R}\backslash\{0\}$.

Let be H in $\mathfrak{sl}(2,\mathbb{R})$ and consider the decomposition

$$H = xA + yB + zC; \quad x, y, z \in \mathbb{R}.$$

Proposition 3.1. *For fixed $\lambda \neq 0$, the adjoint orbit $\mathcal{O}(\lambda A)$ is the set of matrices $H = xA + yB + zC$ in $\mathfrak{sl}(2,\mathbb{R})$ that satisfy*

$$x^2 + y^2 - z^2 = \lambda^2.$$

Proof. First we prove that if H belongs to such orbit, then $x^2 + y^2 - z^2 = \lambda^2$. The adjoint orbit of λA is by definition

$$\mathcal{O}(\lambda A) = \{g\lambda A g^{-1} : g \in \mathrm{SL}(2,\mathbb{R})\}.$$

Hence, if $H \in \mathcal{O}(\lambda A)$, there exists $M \in \mathrm{SL}(2,\mathbb{R})$ such that $H = M\lambda A M^{-1}$. Since the determinant of a matrix is invariant under conjugation, we have

$$\det(H) = \det(\lambda A).$$

Thus, we obtain

$$\det(H) = \det\left(\begin{bmatrix} y & x+z \\ x-z & -y \end{bmatrix}\right) = z^2 - x^2 - y^2,$$
$$\det(\lambda A) = \det\left(\begin{bmatrix} 0 & \lambda \\ \lambda & 0 \end{bmatrix}\right) = -\lambda^2,$$

which implies

$$x^2 + y^2 - z^2 = \lambda^2. \tag{3.1}$$

Thus we conclude the first part of the proof. It is a well known fact that Equation (3.1) defines a surface in $\mathbb{R}^3$ called a **one–sheeted hyperboloid**.

Now we show that, reciprocally, if $H = xA + yB + zC$ satisfies Equation (3.1), then H belongs to $\mathcal{O}(\lambda A)$. Given any matrix $N \in \mathfrak{sl}(2, \mathbb{R})$, its characteristic polynomial is completely determined by its determinant. Indeed, if ρ_N denotes the characteristic polynomial of N, we have

$$\rho_N(t) = t^2 + \det(N).$$

Thus, once H satisfies Equation (3.1), we get $\det(H) = -\lambda^2$ and therefore

$$\rho_H(t) = t^2 - \lambda^2 = (t - \lambda)(t + \lambda).$$

As soon as λ is assumed to be different than zero, we know that H has two distinct eigenvalues, and so H is diagonalizable. Let

$$D = \begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix}$$

and $P \in \mathrm{GL}(2, \mathbb{R})$ be such that $PHP^{-1} = D$. Note that we can assume $P \in \mathrm{SL}(2, \mathbb{R})$ by multiplying its first column by $\frac{1}{\det(P)}$ if necessary. By the same argument, we find $P_0 \in \mathrm{SL}(2, \mathbb{R})$ such that $P_0 \lambda A P_0^{-1} = D$. Thus, we get

$$(P_0^{-1}P)H(P_0^{-1}P)^{-1} = \lambda A$$

with $P_0^{-1}P \in \mathrm{SL}(2, \mathbb{R})$. We conclude that if $H = xA + yB + zC$ satisfies Equation (3.1), then H belongs to the orbit $\mathcal{O}(\lambda A)$ and we are done. $\square$

Remark 3.2. Since $\det(\lambda B)$ satisfies Equation (3.1), the above argument implies $\mathcal{O}(\lambda A) = \mathcal{O}(\lambda B)$.

Remark 3.3. In the complex case, i.e., for $\mathfrak{sl}(2, \mathbb{C})$, if we consider

$$H_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

then we get that its adjoint orbit $\mathcal{O}(H_0)$ is diffeomorphic to $T^*\mathbb{P}^1$, specifically, the cotangent bundle of the complex projective line. So, the geometric structure of the adjoint orbit is quite different. Moreover, in [7], Gasparim, Grama, and San Martin gave a complete description of the diffeomorphism type of adjoint orbits for diagonal matrices in $\mathfrak{sl}(n,\mathbb{C})$.

3.2 The two–sheeted hyperboloid

Now we turn to the geometric structure of the adjoint orbit of λC.

Proposition 3.4. *Fix $\lambda \in \mathbb{R}\backslash\{0\}$. The adjoint orbit $\mathcal{O}(\lambda C)$ is the set of matrices $H = xA + yB + zC$ in $\mathfrak{sl}(2,\mathbb{R})$ subject to*

$$x^2 + y^2 - z^2 = -\lambda^2.$$

Proof. For $H \in \mathcal{O}(\lambda C)$ there exists $N \in \mathrm{SL}(2,\mathbb{R})$ such that $N\lambda CN^{-1} = H$. Therefore we have

$$\det(\lambda C) = \det(NHN^{-1}) = \det(H),$$

and so we get

$$\det(H) = \det\left(\begin{bmatrix} y & x+z \\ x-z & -y \end{bmatrix}\right) = z^2 - x^2 - y^2,$$
$$\det(\lambda C) = \det\left(\begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix}\right) = \lambda^2,$$

which implies

$$x^2 + y^2 - z^2 = -\lambda^2. \tag{3.2}$$

For the reciprocal, we start by showing that there is no $M \in \mathrm{SL}(n,\mathbb{R})$ such that $M(\lambda C)M^{-1} = -\lambda C$. Without loss of generality take $\lambda > 0$. Then, for

$$M = \begin{bmatrix} u & v \\ s & t \end{bmatrix}$$

we reach

$$M(\lambda C)M^{-1} = \lambda \begin{bmatrix} -us - tv & u^2 + v^2 \\ -s^2 - t^2 & us + tv \end{bmatrix}.$$

As $u^2 + v^2 \geq 0$, we easily conclude that there is no $M \in \mathrm{SL}(n, \mathbb{R})$ such that $M(\lambda C)M^{-1} = -\lambda C$. The bottom line is that we have $\lambda C \in \mathcal{O}_1^+$ if $\lambda > 0$ and $\lambda C \in \mathcal{O}_1^-$ if $\lambda < 0$.

Next we show that if $H = xA + yB + zC$ is such that x, y, z satisfy (3.2), then H belongs to $\mathcal{O}_1^+$ or $\mathcal{O}_1^-$. To verify this, we use an argument similar to the one we used in the previous subsection. Once H is such that Equation (3.2) holds, its characteristic polynomial is given by

$$\rho_H(t) = t^2 + \lambda^2 = (t + i\lambda)(t - i\lambda).$$

So, we can write H in its real Jordan form in either of two different ways

$$PHP^{-1} = \begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix}$$

or

$$RHR^{-1} = \begin{bmatrix} 0 & -\lambda \\ \lambda & 0 \end{bmatrix},$$

always with $R, P \in \mathrm{GL}(2, \mathbb{R})$. The structural difference between these two cases is that if $\det(R) > 0$ then $\det(P) < 0$, and vice versa. Assume $\det(P) > 0$. Then we can define $\tilde{P} = \frac{1}{\sqrt{\det(P)}} P$ in order to get

$$\tilde{P} H \tilde{P}^{-1} = \lambda C,$$

where $\tilde{P} \in \mathrm{SL}(2, \mathbb{R})$ and we conclude that H belongs to the orbit $\mathcal{O}(\lambda C)$. In the case when $\det(P) < 0$, we repeat the same construction for R in order to get

$$\tilde{R} H \tilde{R}^{-1} = \lambda C.$$

□

Remark 3.5. Equation (3.2) defines a **two–sheeted hyperboloid**. In this case, we have two situations. Either $z > 0$ (the upper half part of the hyperboloid) or $z < 0$ (the lower half), which respectivelly correspond to

$$\mathcal{O}_1^+ : x^2 + y^2 - z^2 = -\lambda^2, \qquad z > 0, \lambda \neq 0,$$

$$\mathcal{O}_1^- : x^2 + y^2 - z^2 = -\lambda^2, \qquad z < 0, \lambda \neq 0.$$

3.3 The cone

Note that we have analysed adjoint orbits of matrices with determinant either positive or negative. In this section we study the remaining situation, namely, adjoint orbits of matrices with zero determinant. In order to do this, define $D = A + C$ and consider its adjoint orbit $\mathcal{O}(\lambda D)$. The main result reads as follows.

Proposition 3.6. *The adjoint orbit $\mathcal{O}(\lambda D)$ corresponds to matrices $H = xA + yB + zC \in \mathfrak{sl}(2,\mathbb{R})$ subject to $x^2 + y^2 - z^2 = 0$.*

Proof. If $H = xA + yB + zC$ belongs to $\mathcal{O}(\lambda D)$, we can write down $H = L\lambda D L^{-1}$ where $L \in \mathrm{SL}(2,\mathbb{R})$. As before we get

$$\begin{aligned} \det(H) &= \det\left(\begin{bmatrix} y & x+z \\ x-z & -y \end{bmatrix}\right) = z^2 - x^2 - y^2 \text{ and} \\ \det(\lambda D) &= \det\left(\begin{bmatrix} 0 & 2\lambda \\ 0 & 0 \end{bmatrix}\right) = 0. \end{aligned}$$

We conclude that if $H = xA + yB + zC$ belongs to $\mathcal{O}(\lambda D)$, then x, y, z satisfy the relation

$$x^2 + y^2 - z^2 = 0. \tag{3.3}$$

Next we show that if $H = xA + yB + zC$ is such that x, y, z satisfy (3.3), then H belongs either to $\mathcal{O}_2^+$, $\mathcal{O}_2^-$ or $\mathcal{O}_2^0$. To prove this, we look again to the Jordan form of H. Now, once we know $\rho_H(t) = t^2$, we have two cases: whether $H = 0$ or H has as Jordan form

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

If $H = 0$, we have $H \in \mathcal{O}_2^0$. So assume $H \neq 0$ and let $P \in \mathrm{GL}(2,\mathbb{R})$ be such that

$$PHP^{-1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

If $\det(P) > 0$, define $\tilde{P} = \frac{1}{\sqrt{\det(P)}} P \in \mathrm{SL}(2,\mathbb{R})$ in order to obtain $H \in \mathcal{O}_2^+$. On the other hand, if $\det(P) < 0$, we can define $\tilde{P} \in \mathrm{SL}(2,\mathbb{R})$

as the matrix that we obtain from P by multiplying its first column by $\dfrac{1}{\det(P)}$. And so, we reach

$$\tilde{P}H\tilde{P}^{-1} = \begin{bmatrix} 0 & \det(P) \\ 0 & 0 \end{bmatrix}.$$

Therefore, in this case we obtain $H \in \mathcal{O}_2^-$. Note that if $\lambda > 0$, then we get always $x + z > 0$ and $x - z < 0$, hence we have $z > 0$ and therefore

$$\mathcal{O}_2^+ = \{H = xA + yB + zC \in \mathfrak{sl}(2, \mathbb{R}) \colon x^2 + y^2 - z^2 = 0, z > 0\}.$$

When $\lambda < 0$, we have $x + z < 0$ and $x - z > 0$ so we get $z < 0$ and thus

$$\mathcal{O}_2^- = \{H = xA + yB + zC \in \mathfrak{sl}(2, \mathbb{R}) \colon x^2 + y^2 - z^2 = 0, z < 0\}.$$

Finally, for $\lambda = 0$ we have

$$\mathcal{O}_2^0 = \{0\}.$$

Equation (3.3) defines a **cone**. We distinguish three situations; either $z > 0$ (upper half of the cone), or $z < 0$ (lower half), or else $z = 0$ (origin); which are determined by three different orbits denoted by $\mathcal{O}_2^+$, $\mathcal{O}_2^-$ and $\mathcal{O}_2^0$, respectively. We claim that we have

- $\lambda D \in \mathcal{O}_2^+$ if $\lambda > 0$,
- $\lambda D \in \mathcal{O}_2^-$ if $\lambda < 0$, and
- $\lambda D \in \mathcal{O}_2^0$ if $\lambda = 0$.

To see this, take $M \in \mathrm{SL}(n, \mathbb{R})$ and write

$$M = \begin{bmatrix} u & v \\ s & t \end{bmatrix}.$$

We have then

$$M(\lambda D)M^{-1} = \lambda \begin{bmatrix} -2su & 2u^2 \\ -2s^2 & 2us \end{bmatrix}.$$

So, if $\lambda > 0$, we have $2\lambda u^2 \geq 0$; while if $u = 0$, necessarily $s \neq 0$, otherwise must have $\det(M) = 0$. So, we conclude that there are three exclusive orbits associated with Equation (3.3), depending on whether λ is positive, negative or zero. $\square$

Remark 3.7. Now it is trivial to see that these objects comprise the adjoint orbits of $\mathfrak{sl}(2,\mathbb{R})$. In fact, given a non-zero matrix $H \in \mathfrak{sl}(2,\mathbb{R})$ subject to $H = xA + yB + zC$, we have $x^2 + y^2 - z^2 = \alpha \in \mathbb{R}$. In this way, if $\alpha \in \mathbb{R}^+$ then $H \in \mathcal{O}(\lambda A)$. If $\alpha \in \mathbb{R}^-$ then $H \in \mathcal{O}_1^+$ or $H \in \mathcal{O}_1^-$, while for $\alpha = 0$ we have $H \in \mathcal{O}_2^+$ or $H \in \mathcal{O}_2^-$. If H is the zero matrix, of course we get $H \in \mathcal{O}_2^0$. Thus, every element in $\mathfrak{sl}(2,\mathbb{R})$ is contained in one and only one of these orbits.

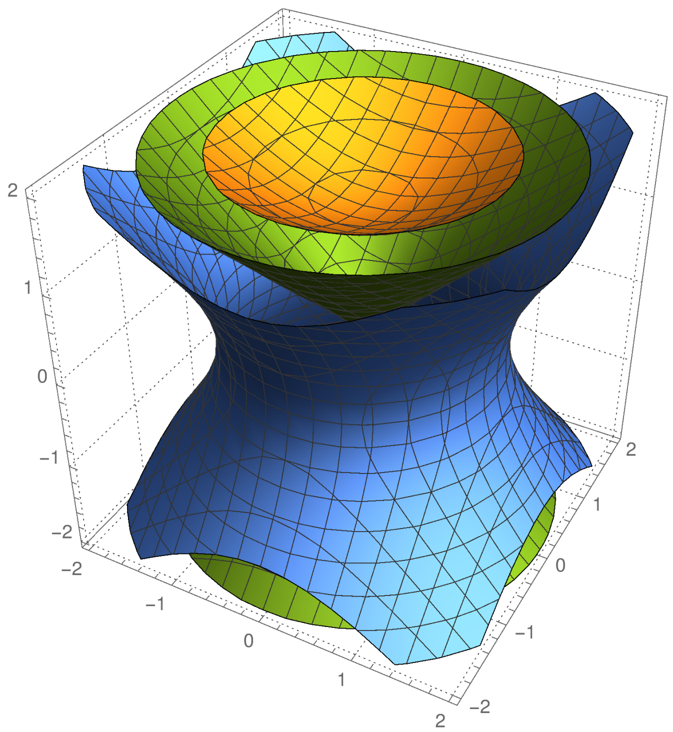

Figure 1: Orbits of $\mathfrak{sl}(2,\mathbb{R})$.

4 The geometry of the one–sheeted hyperboloid

Here we show that the one–sheeted hyperboloid is a ruled surface. Next we use this result to study the dynamics of a gradient field restricted to this surface.

Recall that a surface S is called **ruled** if it is the union of a one parameter family of lines $\{r_\alpha\}_{\alpha\in\mathcal{A}}$. More precisely, there is a family of lines $\{r_\alpha\}_{\alpha\in\mathcal{A}}$ and a parametrization r of S satisfying the following properties.

- The parametrization r is of the form $r(u,v) = c(u) + vb(u)$, for a given $v \in \mathbb{R}$, where c and b are smooth functions.

- For each u, there is $\alpha_u \in \mathcal{A}$, such that $r_u(v) = r(u,v)$ is the parametric equation of the line $r_{\alpha_u} \in \{r_\alpha\}_{\alpha\in\mathcal{A}}$.

- The association $u \mapsto \alpha_u$ is a one to one correspondence.

In this case, we say that S is ruled by the family $\{r_\alpha\}_{\alpha\in\mathcal{A}}$.

Similarly, S is called **doubly ruled** if it can be ruled in different ways by two disjoint families of lines.

For now on, denote by S the one–sheeted hyperboloid given by the equation $x^2 + y^2 - z^2 = \lambda^2$, with fixed $\lambda \neq 0$. In order to show that S is doubly ruled, we will construct explicitly such families.

Lemma 4.1. *Let S be the one–sheeted hyperboloid given by equation $x^2 + y^2 - z^2 = \lambda^2$, with $\lambda \neq 0$. There exist two disjoint families of lines F_1, F_2 contained in S.*

Proof. Consider the cylinder $C\colon x^2 + y^2 = \lambda^2$ which intersect S in the plane $z = 0$. Let (x_0, y_0) be a point in the circle $C \cap S$. Think of $y = f(x) = \pm\sqrt{\lambda^2 - x^2}$ as describing the cylinder. Notice that the tangent space to C at (x_0, y_0) is given by

$$y - y_0 = \frac{\partial f}{\partial x}(x_0)(x - x_0) + \frac{\partial f}{\partial z}(x_0)(z - 0).$$

In particular for $y_0 > 0$, we get

$$y + \left(\frac{x_0}{\sqrt{\lambda^2 - x_0^2}}\right) x = \frac{\lambda^2}{\sqrt{\lambda^2 - x_0^2}}, \tag{4.1}$$

while, for $y_0 < 0$ the equation is

$$y - \left(\frac{x_0}{\sqrt{\lambda^2 - x_0^2}}\right) x = \frac{-\lambda^2}{\sqrt{\lambda^2 - x_0^2}}. \tag{4.2}$$

Let us analyze each case separately.

Case $y_0 > 0$**.** We describe the intersection of the tangent space with S. Rewriting Equation (4.1) as

$$y = \frac{\lambda^2}{\sqrt{\lambda^2 - x_0^2}} - \left(\frac{x_0}{\sqrt{\lambda^2 - x_0^2}}\right) x,$$

squaring both sides, and substituting y^2 by $\lambda^2 + z^2 - x^2$, we find

$$(x - x_0)^2 = \left(\frac{\lambda^2 - x_0^2}{\lambda^2}\right) z^2.$$

The above equation gives two planes containing $(x_0, y_0, 0)$, namely

$$\begin{cases} x - \left(\frac{\sqrt{\lambda^2 - x_0^2}}{\lambda}\right) z = x_0 \\ \\ x + \left(\frac{\sqrt{\lambda^2 - x_0^2}}{\lambda}\right) z = x_0. \end{cases}$$

Once again, considering the intersection with the plane (4.1), we get two sets of systems of equations

$$\begin{cases} y - \left(\frac{x_0}{\sqrt{\lambda^2 - x_0^2}}\right) x = \frac{-\lambda^2}{\sqrt{\lambda^2 - x_0^2}} \\ \\ x - \left(\frac{\sqrt{\lambda^2 - x_0^2}}{\lambda}\right) z = x_0, \end{cases} \tag{4.3}$$

$$\begin{cases} y - \left(\frac{x_0}{\sqrt{\lambda^2 - x_0^2}}\right) x = \frac{-\lambda^2}{\sqrt{\lambda^2 - x_0^2}} \\ \\ x + \left(\frac{\sqrt{\lambda^2 - x_0^2}}{\lambda}\right) z = x_0. \end{cases} \tag{4.4}$$

Let us find the line determined by the planes in Equation (4.3). Note that $v_1 = \left(\frac{x_0}{\sqrt{\lambda^2 - x_0^2}}, 1, 0\right)$ and $v_2 = \left(1, 0, \frac{-\sqrt{\lambda^2 - x_0^2}}{\lambda}\right)$ are normal vectors to the planes. We need the explicit value

$$v_1 \times v_2 = \left(\frac{-\sqrt{\lambda^2 - x_0^2}}{\lambda}, \frac{x_0}{\lambda}, -1\right).$$

In this way, the parametric equation of the intersection line determined by (4.3) is

$$r_1(t) = (x_0, y_0, 0) + t\left(\frac{-\sqrt{\lambda^2 - x_0^2}}{\lambda}, \frac{x_0}{\lambda}, -1\right).$$

We check now why r_1 is contained in S. Using $y_0 = \sqrt{\lambda^2 - x_0^2}$, we get

$$\left(x_0 - \frac{t\sqrt{\lambda^2 - x_0^2}}{\lambda}\right)^2 + \left(y_0 + \frac{tx_0}{\lambda}\right)^2 - t^2 = \lambda^2.$$

Similarly, fashion the line determined by the planes in Equation (4.4) is

$$r_2(t) = (x_0, y_0, 0) + t\left(\frac{\sqrt{\lambda^2 - x_0^2}}{\lambda}, \frac{-x_0}{\lambda}, -1\right).$$

Since $y_0 = \sqrt{\lambda^2 - x_0^2}$, for every $t \in \mathbb{R}$, we obtain

$$\left(x_0 + \frac{t\sqrt{\lambda^2 - x_0^2}}{\lambda}\right)^2 + \left(y_0 + \frac{tx_0}{\lambda}\right)^2 - t^2 = \lambda^2,$$

and hence r_2 is contained in S.

Case $y_0 < 0$. Substituting Equation (4.2) in $x^2 + y^2 - z^2 = \lambda^2$, we reach

$$\begin{cases} y - \left(\frac{x_0}{\sqrt{\lambda^2 - x_0^2}}\right) x = \frac{-\lambda^2}{\sqrt{\lambda^2 - x_0^2}} \\ \\ x - \left(\frac{\sqrt{\lambda^2 - x_0^2}}{\lambda}\right) z = x_0, \end{cases} \tag{4.5}$$

$$\begin{cases} y - \left(\frac{x_0}{\sqrt{\lambda^2 - x_0^2}}\right) x = \frac{-\lambda^2}{\sqrt{\lambda^2 - x_0^2}} \\ \\ x + \left(\frac{\sqrt{\lambda^2 - x_0^2}}{\lambda}\right) z = x_0. \end{cases} \tag{4.6}$$

Working as in the previous case, the intersection plane is

$$s_1(t) = (x_0, y_0, 0) + t\left(\frac{-\sqrt{\lambda^2 - x_0^2}}{\lambda}, \frac{-x_0}{\lambda}, -1\right),$$

which shows that s_1 is contained in S.

Equation (4.6) yields

$$s_2(t) = (x_0, y_0, 0) + t\left(\frac{\sqrt{\lambda^2 - x_0^2}}{\lambda}, \frac{x_0}{\lambda}, -1\right),$$

also contained in S.

For $(\lambda, 0, 0)$ and $(-\lambda, 0, 0)$, namely the point when $y_0 = 0$, the tangent spaces are given by the equations $x = \lambda$ and $x = -\lambda$, respectively. When $x = \lambda$ both

$$\begin{aligned} l_1(t) =& (\lambda, 0, 0) + t(0, 1, -1), \\ l_2(t) =& (\lambda, 0, 0) + t(0, -1, -1). \end{aligned} \tag{4.7}$$

are contained in S.

For $x = -\lambda$, the same is true for

$$\begin{aligned} l_1'(t) =& (-\lambda, 0, 0) + t(0, -1, -1), \\ l_2'(t) =& (-\lambda, 0, 0) + t(0, 1, -1). \end{aligned} \tag{4.8}$$

Observe that we can equally well get the lines from Equation (4.6) by a rotation of π radians of the lines obtained in Equation (4.3) around z–axis, which is to be expected since we are looking at diametrically opposite points in the cylinder. In fact, rotating $r_1(t)$ we achieve

$$\begin{bmatrix} \cos\pi & -\sin\pi & 0 \\ \sin\pi & \cos\pi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 - \frac{t\sqrt{\lambda^2 - x_0^2}}{\lambda} \\ y_0 + \frac{tx_0}{\lambda} \\ -t \end{bmatrix} = \begin{bmatrix} -x_0 + \frac{t\sqrt{\lambda^2 - x_0^2}}{\lambda} \\ -y_0 - \frac{tx_0}{\lambda} \\ -t \end{bmatrix},$$

which is exactly $s_2(t)$. For $r_2(t)$ rotated by π radians around of z–axis we get

$$\begin{bmatrix} \cos\pi & -\sin\pi & 0 \\ \sin\pi & \cos\pi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 + \frac{t\sqrt{\lambda^2 - x_0^2}}{\lambda} \\ y_0 - \frac{tx_0}{\lambda} \\ -t \end{bmatrix} = \begin{bmatrix} -x_0 - \frac{t\sqrt{\lambda^2 - x_0^2}}{\lambda} \\ -y_0 + \frac{tx_0}{\lambda} \\ -t \end{bmatrix},$$

exactly $s_1(t)$. By the same argument, we see that $l_1'(t)$ is a rotation of $l_1(t)$ and $l_2'(t)$, a rotation of $l_2(t)$.

Let us define F_1 as the union of the lines obtained from (4.3) and (4.6) together with $l_1(t)$ and $l_1'(t)$. Similarly, let F_2 be the union of the families of the lines obtained from (4.4) together with (4.5), this time appending $l_2(t)$ and $l_2'(t)$. These families are disjoint since they come from different linearly independent systems of equations. $\square$

Proposition 4.2. *The families F_i, for $i = 1, 2$, satisfy the following properties.*

- *For any two lines $a, b \in F_i$, there exists a rotation R_θ around the z–axis such that $R_\theta a = b$.*
- *If $a \in F_i$ and b is such that there exists a rotation R_θ around the z–axis such that $R_\theta a = b$, then $b \in F_i$.*

Proof. We prove the proposition just for F_1, the case F_2 is completely analogous.

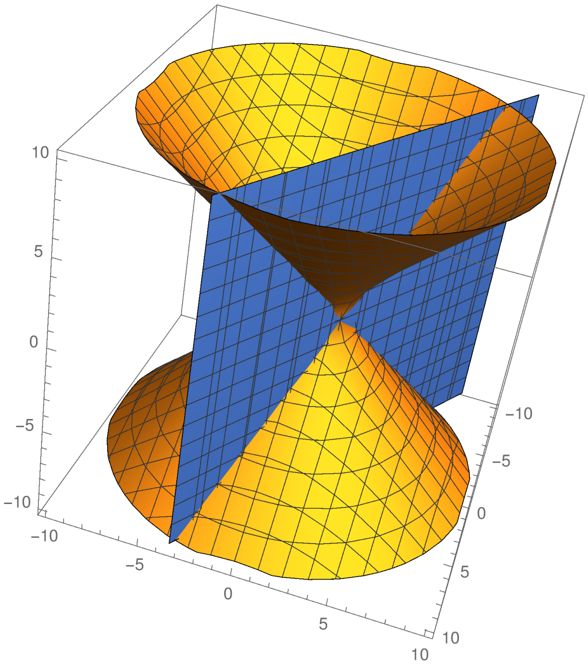

Figure 2: The intersection of the plane in (4.1) with the one–sheeted hyperboloid.

By Lemma 4.1, if $a, b \in F_1$, then the lines pass through a point $(x_0, y_0, 0) \in S$, so they have the shape

$$r_1(t) = (x_0, y_0, 0) + t\left(-\frac{\sqrt{\lambda^2 - x_0^2}}{\lambda}, \frac{x_0}{\lambda}, -1\right), \qquad y_0 > 0,$$

$$s_2(t) = (x_0, y_0, 0) + t\left(\frac{\sqrt{\lambda^2 - x_0^2}}{\lambda}, \frac{x_0}{\lambda}, -1\right), \qquad y_0 < 0.$$

Note that it is enough to show that for each $a \in F_1$ there exists a rotation R_θ such that $R_\theta l_1 = a$. This is so because $R_\phi l_1 = b$ implies $R_\phi R_\theta^{-1} a = b$. The case $a = l_2'$ was the content of Lemma 4.1, so we suppose $a \neq l_2'$.

Direct calculation yields then

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda \\ t \\ -t \end{bmatrix} = \begin{bmatrix} \lambda\cos\theta - t\sin\theta \\ \lambda\sin\theta + t\cos\theta \\ -t \end{bmatrix}. \tag{4.9}$$

If a passes through $(x, y, 0) \in S$, we choose θ so that $x = \lambda\cos\theta$ and $y = \lambda\sin\theta$ (this is possible given that for each $(x_0, y_0, 0) \in S$ we have $x_0^2 + y_0^2 = \lambda^2$). By using $y = \sqrt{\lambda^2 - x^2}$ whenever $y > 0$, we get

$$\begin{bmatrix} x + \frac{t\sqrt{\lambda^2 - x^2}}{\lambda} \\ y + \frac{tx_0}{\lambda} \\ -t \end{bmatrix};$$

while for $y < 0$, we use $y = -\sqrt{\lambda^2 - x^2}$ and obtain

$$\begin{bmatrix} x - \frac{t\sqrt{\lambda^2 - x^2}}{\lambda} \\ y + \frac{tx_0}{\lambda} \\ -t \end{bmatrix}.$$

To verify the second statement it is enough to show that if there exists a rotation R_θ such that $R_\theta l_1 = b$, then $b \in F_1$, since by Item 4.2 there exists R_ϕ such that $R_\phi l_1 = a$. If this is so, there exists α such that $R_\alpha a = b$ if and only if there exists ϕ such that $R_\phi R_\alpha l_1 = b$. By changing variables on Equation (4.9), with $x_0 = \lambda\cos\theta$ and $y_0 = \lambda\sin\theta$, we reach

$$\begin{bmatrix} x_0 \pm \frac{t\sqrt{\lambda^2 - x^2}}{\lambda} \\ y_0 + \frac{tx_0}{\lambda} \\ -t \end{bmatrix};$$

which is exactly the expression of the lines given by the planes (4.3) and (4.6). Finally, by definition of F_1, we have $R_\theta l_1 \in F_1$. □

Proposition 4.3. *The one–sheeted hyperboloid S is a doubly ruled surface.*

Proof. Fix $(x_0, y_0, z_0) \in S$. We look again at the line $l_1 \in F_1$, where $l_1(t) = (\lambda, 0, 0) + t(0, 1, -1)$. We will show that there exists a rotation R_θ around z–axis such that $R_\theta l_1(t) = (x_0, y_0, z_0)$ for some $t \in \mathbb{R}$ (observe that by the second item in the above proposition, we already have $R_\theta l_1 \in F_1$). By Equation (4.9), we get

$$R_\theta l_1(t) = (\lambda \cos\theta - t\sin\theta, \lambda\sin\theta + t\cos\theta, -t),$$

and by letting $t = -z_0$, we obtain

$$R_\theta l_1(-z_0) = (\lambda\cos\theta + z_0\sin\theta, \lambda\sin\theta - z_0\cos\theta, z_0). \tag{4.10}$$

Varying θ in Equation (4.10) ables us to trace the entire level curve S at $z = z_0$, which is a circle of radius $\lambda^2 - z_0^2$. Therefore, since $(x_0, y_0, z_0) \in S$ holds, there exist θ subject to $\lambda\cos\theta + z_0\sin\theta = x_0$ and $\lambda\sin\theta - z_0\cos\theta = y_0$. Thus $R_\theta l_1$ is a line in F_1 subject to $(x_0, y_0, z_0) \in R_\theta l_1$. In the same way, it is not hard to show that there exists a rotation R_ϕ such that $R_\phi l_2$ contains the point (x_0, y_0, z_0). We conclude that $r(\theta, t) = R_\theta l_1(t)$ and $s(\theta, t) = R_\theta l_2(t)$ are parametric equations for S. Hence, S is ruled by both F_1 and F_2. □

5 The tangent spaces to $\mathcal{O}(\lambda A)$

The adjoint orbit $\mathcal{O}(\lambda A)$ is a surface in $\mathbb{R}^3$. The goal of this section is to depict the tangent space to $\mathcal{O}(\lambda A)$ and determine its relation with the image of the adjoint representation of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. Our starting point is the following proposition, which provides an identification of said tangent space.

Proposition 5.1. *Let* ad *be the adjoint representation of* $\mathfrak{g}$. *For* $H \in \mathcal{O}(\lambda A)$ *we have*

$$\mathrm{Im}(\mathrm{ad}(H)) = T_H\mathcal{O}(\lambda A).$$

Proof. Notice that every curve passing through H in $\mathcal{O}(\lambda A)$ has the form $g_t H g_t^{-1}$, where $g\colon [-\epsilon, \epsilon] \to \mathrm{SL}(2,\mathbb{R})$ smoothly satisfies $g_0 = \mathrm{Id}$. Thus, every tangent vector $v \in T_H\mathcal{O}(\lambda A)$ can be written as

$$v = \frac{d}{dt} g_t H g_t^{-1}\bigg|_{t=0},$$

for some smooth curve g_t. However, if $\frac{d}{dt} g_t \Big|_{t=0} = X \in \mathfrak{sl}(2,\mathbb{R})$, then

$$\frac{d}{dt} g_t H g_t^{-1} \Big|_{t=0} = \frac{d}{dt} \mathrm{Ad}_{g_t} H \Big|_{t=0} = d(\mathrm{Ad}_{\mathrm{Id}})(X)H = \mathrm{ad}(X)H = [X,H]. \tag{5.1}$$

Hence we get $v \in \mathrm{Im}(\mathrm{ad}(H))$, which forces the inclusion $T_H \mathcal{O}(\lambda A) \subset \mathrm{Im}(\mathrm{ad}(H))$.

Conversely, given $X_0 = [X,H]$, just take some smooth curve g_t subject to

$$\frac{d}{dt} g_t \Big|_{t=0} = X_0 \quad \text{with,} \quad g_0 = \mathrm{Id} \in \mathrm{SL}(2,\mathbb{R}),$$

and plug it into Equation (5.1): we obtain the desired result. □

Let A, B, C be the basis of $\mathfrak{sl}(2,\mathbb{R})$ given in (1.1). Since they satisfy the relations $[B,A] = C, [C,A] = B$, and $[B,C] = A$, for any $H \in \mathfrak{g}$ written as $H = xA + yB + zC$ we get

$$\mathrm{ad}(H) = \begin{bmatrix} 0 & -z & -y \\ z & 0 & x \\ y & -x & 0 \end{bmatrix}.$$

Since $\dim \ker \mathrm{ad}(H) = 1$, by taking any two column vectors in $\mathrm{ad}(H)$ we have

$$T_H \mathcal{O}(\lambda A) = span\{zB + yC, xB - yA\},$$

whenever $H \in \mathcal{O}(\lambda A)$. As $H = xA + yB + zC$ implies that x, y, z satisfy $x^2 + y^2 - z^2 = \lambda^2$, we get

$$T_H \mathcal{O}(\lambda A) = span\{zB + yC, xB - yA \colon x^2 + y^2 - z^2 = \lambda^2\}.$$

6 Morse theory on the adjoint orbit $\mathcal{O}(A)$

We use the adjoint orbit $\mathcal{O}(\lambda A)$ to construct an example which shows how the compactness hypothesis is essential to the Morse-Smale theorem. We take $\lambda = 1$ to ease computations.

Let M be a manifold and $f\colon M \to \mathbb{R}$ a smooth function. A critical point p of f is **non-degenerate** if the Hessian matrix of f in p is non-degenerate. If all critical points of f are non-degenerate, we say f is a **Morse function**.

Theorem 6.1 (Morse–Smale)**.** *[10, Lem. 2.23]. Let M be a compact manifold without boundary and $f\colon M \to \mathbb{R}$ a Morse function. If $\phi_p(t)$ is the trajectory of the gradient vector field ∇f at p, then both limits*

$$\lim_{t\to\infty} \phi_p(t) \qquad \text{and} \qquad \lim_{t\to-\infty} \phi_p(t)$$

exist, in fact, they are critical points of f. □

Dynamics of the gradient vector field We study the behaviour of a gradient vector field restricted to $\mathcal{O}(A)$. For that, we consider the function $f(x,y,z) = yz$. The gradient of f (with respect to the canonical inner product) is

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x}(x,y,z)e_1 + \frac{\partial f}{\partial y}(x,y,z)e_2 + \frac{\partial f}{\partial z}(x,y,z)e_3 = ze_2 + ye_3.$$

Therefore, the gradient matrix of f in the canonical basis is

$$\nabla f = \begin{pmatrix} 0 \\ z \\ y \end{pmatrix}.$$

Proposition 6.2. *The gradient vector field ∇f is tangent to $\mathcal{O}(A)$.*

Proof. Let us consider the relation $g(x,y) = \pm\sqrt{x^2+y^2-1}$ which describes the one–sheeted hyperboloid. We look first at the case $z_0 \geq 0$ (hence $g(x,y) = \sqrt{x^2+y^2-1}$). The normal vector to the surface at a given point (x_0, y_0, z_0) is

$$\vec{n} = \left(\frac{x_0}{\sqrt{x_0^2+y_0^2-1}}, \frac{y_0}{\sqrt{x_0^2+y_0^2-1}}, -1 \right).$$

Thus, taking the inner product between $\vec{n}$ and $\nabla f(x_0,y_0,z_0) = (0, z_0, y_0)^T$, yields

$$\langle \vec{n}, \nabla f(x_0,y_0,z_0)\rangle = \frac{y_0 z_0}{\sqrt{x_0^2+y_0^2-1}} - y_0;$$

and after using $z_0 = \sqrt{x_0^2 + y_0^2 - 1}$ we reach

$$\langle \vec{n}, \nabla f(x_0, y_0, z_0) \rangle = 0.$$

Hence, ∇f takes the point (x_0, y_0, z_0) to a vector tangent to the surface, and so, ∇f is tangent to $\mathcal{O}(A)$ for $z_0 \geq 0$.

Similarly for $z_0 \leq 0$, we use $g(x, y) = -\sqrt{x^2 + y^2 - 1}$ instead and get

$$\vec{n} = \left(-\frac{x_0}{\sqrt{x_0^2 + y_0^2 - 1}}, -\frac{y_0}{\sqrt{x_0^2 + y_0^2 - 1}}, -1 \right).$$

With $z_0 = -\sqrt{x_0^2 + y_0^2 - 1}$, we obtain

$$\langle \vec{n}, \nabla f(x_0, y_0, z_0) \rangle = -\frac{y_0 z_0}{\sqrt{x_0^2 + y_0^2 - 1}} - y_0 = 0.$$

In either case, we conclude that the gradient vector field ∇f is tangent to $\mathcal{O}(A)$. $\square$

Proposition 6.3. *The function $f(x, y, z) = yz$ restricted to $\mathcal{O}(A)$ is a Morse function.*

Proof. Notice that $(1, 0, 0)$ and $(-1, 0, 0)$ are the singularities of the restriction gradient vector field to the orbit $\mathcal{O}(A)$. Let $\operatorname{Hess}(f)$ be the Hessian matrix of f, namely,

$$\operatorname{Hess}(f) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Note that the restriction of $\operatorname{Hess}(f)$ to each of the tangent spaces at $(1, 0, 0)$ and $(-1, 0, 0)$ is non-degenerate. In fact, using the results of Section 5 and identifying $(1, 0, 0)$ and $(-1, 0, 0)$ with the matrices A and $-A$ in $\mathcal{O}(A)$, respectively, we get $T_A\mathcal{O}(A) = span\{(0, 0, 1), (0, 1, 0)\}$ and $T_{-A}\mathcal{O}(A) = span\{(0, 0, -1), (0, -1, 0)\}$. It is not hard to conclude the equalities

$$T_A\mathcal{O}(A) \cap \ker \operatorname{Hess}(f) = 0,$$
$$T_{-A}\mathcal{O}(A) \cap \ker \operatorname{Hess}(f) = 0,$$

which implies that $f|_{\mathcal{O}(A)}$ is a Morse function. $\square$

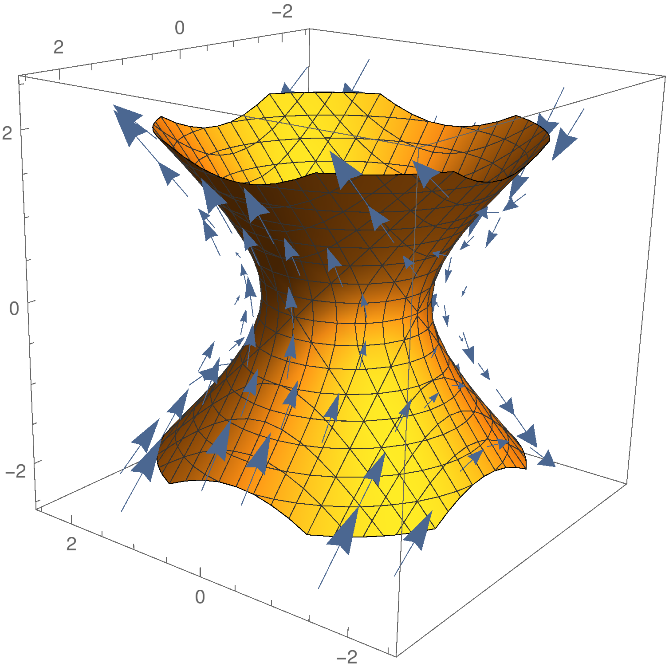

Figure 3: The vector field restricted to $\mathcal{O}(A)$.

Using the dynamics above, we obtain trajectories of ∇f which are not complete, thus showing that the hypothesis of compactness in Theorem 6.1 is fundamental.

The trajectories of the gradient ∇f restricted to $T_A\mathcal{O}(A)$ are solutions of the following linear system of differential equations

$$\begin{bmatrix} y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y(t) \\ z(t) \end{bmatrix}.$$

Since 1 and 1 are eigenvalues of the linear part, with eigenvectors $v_1 =$

$(-1, 1)$ and $v_2 = (1, 1)$, respectively, it follows that the general solution has the form

$$\begin{bmatrix} y(t) \\ z(t) \end{bmatrix} = c_1 e^{-t} v_1 + c_2 e^{t} v_2.$$

Setting $c_1 = 0$ and $c_2 = 1$, we obtain $\gamma_1(t) = e^t(1, 1)$. Note that $\lim_{t\to-\infty} \gamma_1(t) = (0, 0)$ but the limit when $t \to \infty$ does not make sense in $\mathcal{O}(A)$.

On the other hand, taking $c_2 = 0$ and $c_1 = 1$, we get $\gamma_2(t) = e^{-t}(-1, 1)$. Where $\lim_{t\to\infty} \gamma_2(t) = (0, 0)$ but the limit when $t \to -\infty$ does not exist.

Considering γ_1 and γ_2 in the tangent space $T_A\mathcal{O}(A)$, we have that these are the lines $(1, t, t)$ and $(1, -t, t)$. Moreover, they correspond to the lines l_1 and l_2 as described in Equation (4.7) in Lemma 4.1 of Section 4. So they are contained in $\mathcal{O}(A)$. Summarizing, we obtain two trajectories $\gamma_1(t)$ and $\gamma_2(t)$ of the gradient ∇f whose limit points do not belong to the orbit, namely, do not satisfy the conclusions of Theorem 6.1. This happens because the one–sheeted hyperboloid $\mathcal{O}(A)$ is a non-compact submanifold of $\mathbb{R}^3$.

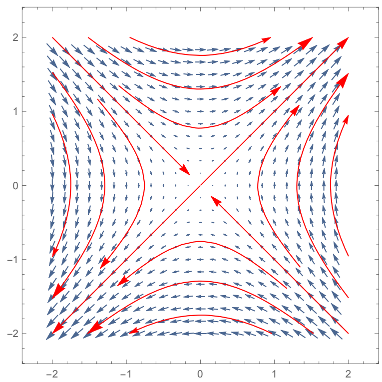

Figure 4: The gradient ∇f restricted on $T_A\mathcal{O}(A)$ at $A = (1, 0, 0)$.

7 A symplectic structure in $\mathcal{O}(A)$

Here we realize the adjoint orbit $\mathcal{O}(A)$ as a symplectic manifold. We follow the construction by Kirillov–Kostant–Souriau [8, 9]. First we construct the symplectic form on the coadjoint orbit and then, using the Killing form, we induce the symplectic structure on the adjoint orbit $\mathcal{O}(A)$. For a more general study of symplectic geometry on adjoint orbits we refer the reader to [2] and [7].

In order to perform the construction, we start by recalling some basic definitions of symplectic geometry.

Let V be a real vector space and $\omega\colon V \times V \to \mathbb{R}$ a skew–symmetric bilinear form. We say that ω is a **symplectic form** if it is non-degenerate, that is, $\omega(u, v) = 0$ for all v implies $u = 0$. In this case, we say that (V, ω) is a **symplectic vector space**.

Let M be a manifold. We say that a 2–form $\omega \in \Omega^2(M)$ is **non-degenerate** if the 1–form $\omega_x = \omega(x, \cdot)$ is non-degenerate for each $x \in M$. Thus, for every $x \in M$, the tangent space T_xM is a symplectic vector space.

A **symplectic structure** on M is a 2–form $\omega \in \Omega^2(M)$ which is non-degenerate and closed. In this case, we say that (M, ω) is a **symplectic manifold**.

Now we define the coadjoint representation, which is the dual of the adjoint representation and will allow us to define coadjoint orbits. First, let us consider the natural pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$ given by

$$\begin{aligned}\langle\ ,\ \rangle\colon \mathfrak{g}^* \times \mathfrak{g} &\to \mathbb{R}\\ (\xi, X) &\mapsto \langle \xi, X\rangle = \xi(X).\end{aligned}$$

For $\xi \in \mathfrak{g}^*$, we define $\mathrm{Ad}_g^*\xi$ by the rule

$$\langle \mathrm{Ad}_g^*\xi, X\rangle = \langle \xi, \mathrm{Ad}_{g^{-1}}X\rangle, \qquad X \in \mathfrak{g}.$$

The **coadjoint representation of G on $\mathfrak{g}^*$** is by definition

$$\begin{aligned}\mathrm{Ad}^*\colon G &\to \mathrm{Aut}(\mathfrak{g}^*)\\ g &\mapsto \mathrm{Ad}_g^*.\end{aligned}$$

Similarly, we have a coadjoint representation of $\mathfrak{g}$ on $\mathfrak{g}^*$ given by

$$\begin{aligned} \mathrm{ad}^* \colon \mathfrak{g} &\to \mathfrak{gl}(\mathfrak{g}^*) \\ u &\mapsto \mathrm{ad}^*_u. \end{aligned}$$

To be more explicit, given $u \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$, we have $\langle \mathrm{ad}^*_u(\xi), v\rangle = -\xi([u,v])$. Here $[\ ,\ ]$ is the Lie bracket on $\mathfrak{g}$.

Let us consider $\xi \in \mathfrak{g}^*$ and denote by

$$\mathcal{O}^* = \{\varphi \in \mathfrak{g}^* \colon \text{ there is } u \in G, \mathrm{Ad}^*_u(\xi) = \varphi\}$$

the **coadjoint orbit** of ξ. Since the vectors $\mathrm{Ad}^*_u(\xi)$ span the tangent space $T_\xi\mathcal{O}^*$ we have

$$T_\xi\mathcal{O}^* = \{\mathrm{ad}^*_u(\xi) \colon u \in \mathfrak{g}\}.$$

Note that for a fixed $\xi \in \mathfrak{g}^*$, the value of $\xi[u,v]$ depends just on ad^*_u and ad^*_v at the point ξ. In fact, if $\mathrm{ad}^*_u(\xi) = \mathrm{ad}^*_{u'}(\xi)$, then

$$\xi(u-u',v) = (\mathrm{ad}^*_u - \mathrm{ad}^*_{u'})(\xi)(v) = 0,$$

for each $v \in \mathfrak{g}$. Thus, the following definition of a skew-symmetric bilinear form on $T_\xi\mathcal{O}^*$ makes sense.

For $\xi \in \mathfrak{g}^*$ fixed, we define a skew-symmetric bilinear form on $T_\xi\mathcal{O}^*$ by

$$\omega_\xi(\mathrm{ad}^*_u(\xi), \mathrm{ad}^*_v(\xi)) = \xi([u,v]).$$

Lemma 7.1. *For each $\xi \in \mathfrak{g}^*$ the form ω_ξ is non-degenerate.*

Proof. Note that if

$$\omega_\xi(\mathrm{ad}^*_u(\xi), \mathrm{ad}^*_v(\xi)) = 0,$$

for all $v \in \mathfrak{g}$, then $\xi([u,v]) = 0 = -\xi([u,v])$ and therefore $\mathrm{ad}^*_u(\xi) = 0$. □

Since we have $[\mathrm{Ad}_g(u), \mathrm{Ad}_g(v)] = \mathrm{Ad}_g([u,v])$, we get the equality

$$\mathrm{Ad}^*_g\xi([\mathrm{Ad}_g(u), \mathrm{Ad}_g(v)]) = \mathrm{Ad}^*_g\xi(\mathrm{Ad}_g([u,v])) = \xi([u,v]).$$

Thus, ω_ξ defines a point-wise form ω on $\mathcal{O}$.

Lemma 7.2. *The 2–form ω is closed, that is, satisfies* $\mathrm{d}\omega = 0$.

Proof. We analyse ω point-wise. For any $\xi \in \mathfrak{g}^*$, given x, y and z in $\mathfrak{g}$ set $X = \mathrm{ad}_x^*(\xi)$, $Y = \mathrm{ad}_y^*(\xi)$ and $Z = \mathrm{ad}_z^*(\xi)$. We have then

$$\begin{aligned}
\mathrm{d}\omega_\xi(X,Y,Z) =& \frac{1}{3}\left(X\omega_\xi(Y,Z) - Y\omega_\xi(X,Z) + Z\omega_\xi(X,Y)\right) + \\
&+ \frac{1}{3}(-\omega_\xi([X,Y],Z) + \omega_\xi([X,Z],Y) - \omega_\xi([Y,Z],X)).
\end{aligned}$$

Note that all the directional derivatives vanish, since ω_ξ is constant relative to ξ. Thus using Jacobi identity, we reach

$$\begin{aligned}
\mathrm{d}\omega_\xi(X,Y,Z) &= \frac{1}{3}(-\omega_\xi([X,Y],Z) + \omega_\xi([X,Z],Y) - \omega_\xi([Y,Z],X)), \\
&= \frac{1}{3}(-\xi([[x,y],z]) + \xi([[x,z],y]) - \xi([[y,z],x])), \\
&= \frac{1}{3}\xi(-[[x,y],z]) + [[x,z],y] - [[y,z],x], \\
&= \frac{1}{3}\xi(0), \\
&= 0.
\end{aligned}$$

Therefore ω is closed. □

Theorem 7.3. *Let $\mathcal{O}^* \subset \mathfrak{g}^*$ be a coadjoint orbit. Then ω_ξ defines a symplectic structure on $\mathcal{O}^*$.*

Proof. This is a direct consequence of Lemmas 7.1 y 7.2. □

Remark 7.4. This symplectic structure on the coadjoint orbit is canonical and is called the **Kirillov–Kostant–Souriau form**.

Now we show below how to endow the adjoint orbit with a symplectic structure which comes from the symplectic form constructed above.

Proposition 7.5. *Suppose $\mathfrak{g}$ admits an* Ad*–invariant inner product, that is, one subject to*

$$\langle \mathrm{Ad}_g(u), \mathrm{Ad}_g(v) \rangle = \langle u, v \rangle,$$

for each $g \in G$. Then the identification $\mathfrak{g} \cong \mathfrak{g}^*$ induced by this inner product also provides an isomorphism between the adjoint and coadjoint representations.

Proof. The isomorphism of vector spaces $\mathfrak{g} \cong \mathfrak{g}^*$ is given by

$$\begin{array}{rcl} \varphi\colon \mathfrak{g} & \to & \mathfrak{g}^* \\ v & \mapsto & I_v, \end{array} \tag{7.1}$$

where $I_v(u) = \langle u, v\rangle$, for each $u \in \mathfrak{g}^*$. We want to show that φ is an isomorphism of representations as well, namely, an isomorphism of Lie algebras for which the following diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\varphi"] \arrow[d, "\mathrm{Ad}_g"'] & \mathfrak{g}^* \arrow[d, "\mathrm{Ad}_g^*"] \\ \mathfrak{g} \arrow[r, "\varphi"'] & \mathfrak{g}^* \end{tikzcd}$$

is commutative.

Since φ is an isomorphism of vector spaces, whenever we get $\mathfrak{g} \simeq V$ as vector spaces endowed with Lie brackets, then we can make V^* into a Lie algebra by defining a Lie bracket as

$$[a, b]_* = \varphi([\varphi^{-1}(a), \varphi^{-1}(b)]), \text{ for each } a, b \in V^*.$$

Thus, we get $\mathfrak{g}^* \cong (V^*, [\cdot,\cdot]_*)$ and φ is a Lie algebra homomorphism. In fact, it is easy to check the equality

$$\varphi([a, b]) = [\varphi(a), \varphi(b)]_*.$$

Next, let $v \in \mathfrak{g}$ be a fixed element. Since Ad_g is invertible, there exists $w \in \mathfrak{g}$ such that $\mathrm{Ad}_g(w) = v$. Using Ad–invariance for the inner product, we obtain

$$\varphi(\mathrm{Ad}_g(u))(v) = \langle \mathrm{Ad}_g(u), v\rangle = \langle \mathrm{Ad}_g(u), \mathrm{Ad}_g(w)\rangle = \langle u, w\rangle.$$

In the same way, we get

$$\mathrm{Ad}_g^*(\varphi(u))(v) = \varphi(u)(\mathrm{Ad}_{g^{-1}}(v)) = \langle u, \mathrm{Ad}_{g^{-1}}(v)\rangle = \langle u, w\rangle.$$

Therefore, the diagram is commutative and φ is a representation isomorphism. $\square$

Remark 7.6. Notice that the above result holds in a more general context where the product is only non-degenerate and not necessarily positive definite, and hence not a inner product. It follows from the fact that the linear map induced by a non-degenerate product between $\mathfrak{g}$ and $\mathfrak{g}^*$ is an isomophism. The proof of the above proposition also holds in this case, since it only requires the existence of such isomorphism.

Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. The **Killing form** on $\mathfrak{g}$ is the map

$$\begin{array}{rcl} B\colon \mathfrak{g}\times\mathfrak{g} & \to & \mathbb{F} \\ (x,y) & \mapsto & B(x,y)=\operatorname{tr}(\operatorname{ad}_x\circ\operatorname{ad}_y). \end{array}$$

Proposition 7.7. *The Killing form is* Ad*-invariant.*

Proof. In fact, we have

$$\begin{aligned} B(\operatorname{Ad}_g(x),\operatorname{Ad}_g(y)) &= \operatorname{tr}(g\ \circ\ \operatorname{ad}_x\circ\operatorname{ad}_y\circ\ g^{-1}) \\ &= \operatorname{tr}(\operatorname{ad}_x\circ\operatorname{ad}_y) \\ &= B(x,y). \end{aligned}$$

□

Proposition 7.8. *The Killing form is symmetric and bilinear.*

Proof. Symmetry follows from the property $\operatorname{tr}(MN)=\operatorname{tr}(NM)$. Thus, we have

$$B(x,y)=\operatorname{tr}(\operatorname{ad}_x\circ\operatorname{ad}_y)=\operatorname{tr}(\operatorname{ad}_y\circ\operatorname{ad}_x)=B(y,x).$$

Since ad and trace are linear, we get

$$\begin{aligned} B(\alpha x+\beta y,z) &= \operatorname{tr}(\operatorname{ad}(\alpha x+\beta y)\circ\operatorname{ad}z) \\ &= \operatorname{tr}((\alpha\operatorname{ad}x+\beta\operatorname{ad}y)\circ\operatorname{ad}z) \\ &= \alpha\operatorname{tr}(\alpha\operatorname{ad}x+\operatorname{ad}z)+\beta\operatorname{tr}(\operatorname{ad}y\circ\operatorname{ad}z) \\ &= \alpha B(x,z)+\beta B(y,z), \end{aligned}$$

for each $x,y,z\in\mathfrak{g}$. Thus, B is linear on the first entry. By the symmetry we get the same for the second entry. □

Hanceforth, for simplicity we use the notation $\langle a, b\rangle = B(a, b)$.

Proposition 7.9. *The Killing form is non-degenerate.*

Proof. Since $[B, A] = C, [C, A] = B$, and $[B, C] = A$ hold, we get

$$\mathrm{ad}(A) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \ \mathrm{ad}(B) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \ \mathrm{ad}(C) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Direct computation yields

$$\langle A, A\rangle = \mathrm{tr}(\mathrm{ad}(A)\circ \mathrm{ad}(A)) = \mathrm{tr}\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 2,$$

$$\langle B, B\rangle = \mathrm{tr}(\mathrm{ad}(B)\circ \mathrm{ad}(B)) = \mathrm{tr}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 2,$$

$$\langle C, C\rangle = \mathrm{tr}(\mathrm{ad}(C)\circ \mathrm{ad}(C)) = \mathrm{tr}\left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = -2,$$

$$\langle B, A\rangle = \mathrm{tr}(\mathrm{ad}(B)\circ \mathrm{ad}(A)) = \mathrm{tr}\left(\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = 0,$$

$$\langle A, C\rangle = \mathrm{tr}(\mathrm{ad}(A)\circ \mathrm{ad}(C)) = \mathrm{tr}\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}\right) = 0,$$

$$\langle B, C\rangle = \mathrm{tr}(\mathrm{ad}(B)\circ \mathrm{ad}(C)) = \mathrm{tr}\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}\right) = 0.$$

Hence, if $H = xA + yB + zC$, expressing the operator $\langle H, \cdot\rangle$ on its matrix representation form we obtain

$$\langle H, \cdot\rangle = \begin{bmatrix} 2x & 2y & -2z \end{bmatrix}.$$

Therefore, the map $\langle H, \cdot\rangle$ is zero if and only if H is zero. □

In this way, we identify the adjoint orbit $\mathcal{O}(A)$ with the coadjoint orbit $\mathcal{O}^*(A) = \{f \in \mathfrak{g}^* : \exists u \in G, \mathrm{Ad}_u^*(\varphi(A)) = f\}$, where φ is the map in (7.1). It follows that we can induce on $\mathcal{O}(A)$ the symplectic structure built on $\mathcal{O}^*(A)$ as

$$\omega'_p(\mathrm{ad}_p(a), \mathrm{ad}_p(b)) := \omega_{\varphi(p)}(\mathrm{ad}^*_{\varphi(p)}(a), \mathrm{ad}^*_{\varphi(p)}(b)) = \langle p, [a,b] \rangle,$$

for each $a, b, p \in \mathcal{O}(A)$.

Corollary 7.10. *The pair $(\mathcal{O}(A), \omega')$ is a symplectic manifold.* □

Remark 7.11. The Killing form is non-degenerate because we are working with a semisimple Lie algebra. This is essential to achieve the identification between adjoint and coadjoint orbits and, consequently, to perform the above construction.

In [7], the authors construct another symplectic form which does not come from the Kirillov–Kostant–Souriau symplectic form. Their method involves Lie theory and the construction is performed directly on adjoint orbits of semisimple Lie groups. It remains to carry out a complete classification of adjoint orbits in higher dimension.

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Resumen

Para el grupo especial lineal $\mathrm{SL}(2,\mathbb{R})$ y su álgebra de Lie $\mathfrak{sl}(2,\mathbb{R})$ estudiamos propiedades geométricas asociadas a sus órbitas adjuntas. En particular mostramos que se presentan apenas tres alternativas para la órbita: o bien es un hiperboloide de una hoja, o un hiperboloide de dos hojas o en su defecto un cono. Además, introducimos un potencial específico y estudiamos el correspondiente campo gradiente y su dinámica al restringirnos a la órbita adjunta. También describimos la estructura simpléctica de tales órbitas que provienen de la bien conocida forma simpléctica de Kirillov–Kostant–Souriau en órbitas coadjuntas.

Palabras clave: Órbitas adjuntas, estructura simpléctica.

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