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CONTENIDO

Multivariate skew-normal/independent distributions: properties and inference

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Abstract

Liu (1996) discussed a class of robust normal/independent distributions which contains a group of thick-tailed cases. In this article, we develop a skewed version of these distributions in the multivariate setting, and we call them multivariate skew normal/independent distributions. We derive several useful properties for them. The main virtue of the members of this family is that they are easy to simulate and lend themselves to an EM-type algorithm for maximum likelihood estimation. For two multivariate models of practical interest, the EM-type algorithm has been discussed with emphasis on the skew-t, the skew-slash, and the contaminated skew-normal distributions. Results obtained from simulated and two real data sets are also reported.

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1. Introduction

A normal distribution is a routine assumption for analyzing real data, but it may be unrealistic, specially when strong skewness and heavy-tailed appear. In practice, we generate a great number of data that are skewed or heavy-tailed, for instance, information on family income, the CD4 cell count from AIDS studies, etc. Thus, one needs to develop a flexible class of models that can readily adapt to the non-normality behavior of certain phenomena. Flexible models that include several known distributions, including normal distribution, are of particular importance, since such models can adapt to distributions that are in the neighborhood of the normal model (DiCiccio and Monti (2004) [13]). Lange and Sinsheimer (1993) [22] developed a normal/independent distribution which contains a group of thick-tailed distributions that is often used for robust inference of symmetrical data (Liu (1996) [25]). In this article we further generalize the normal/independent (NI) distributions and combine skewness with heavy-tailed. These new classes of distributions are attractive not only because they model both cases, but because they have a stochastic representation for easy implementation of the EM-algorithm, and so facilitate the study of many useful properties. Our proposal extends some of the recent results found in Azzalini and Capitanio (2003) [6], Gupta (2003) [17], and Wang and Genton (2006) [31].

Azzalini (1985) [4] proposed a univariate skew-normal distribution that was generalized to the multivariate case by Azzalini and Dalla–Valle (1996) [7] and Arellano–Valle et al. (2005) [2]. The multivariate skew-normal density extends the multivariate normal model by allowing a shape parameter to account for skewness. The probability density function of the generic element of a multivariate skew-normal distribution is given explicitly by

$$f(\mathbf{y}) = 2\phi_p(\mathbf{y}|\boldsymbol{\mu}, \boldsymbol{\Sigma})\Phi_1(\boldsymbol{\lambda}^\top \boldsymbol{\Sigma}^{-1/2}\,(\mathbf{y}-\boldsymbol{\mu}))\,, \quad \mathbf{y} \in \mathbb{R}^p, \qquad (1.1)$$

where $\phi_p(.|\boldsymbol{\mu}, \boldsymbol{\Sigma})$ stands for the probability density function of the p-variate normal distribution with mean vector $\boldsymbol{\mu}$ and covariate matrix $\boldsymbol{\Sigma}$, while $\Phi_1(.)$ represents the cumulative distribution function of the standard normal distribution, here $\mathbf{\Sigma}^{-1/2}$ satisfies $\mathbf{\Sigma}^{-1/2}\mathbf{\Sigma}^{-1/2} = \mathbf{\Sigma}^{-1}$. When we have $\boldsymbol{\lambda} = \mathbf{0}$, the skew normal distribution reduces to the normal distribution ($\mathbf{Y} \sim \mathrm{N}_p(\boldsymbol{\mu}, \mathbf{\Sigma})$). A p-dimensional random vector $\mathbf{Y}$ with probability density function as in (1.1) will be denoted by $\mathrm{SN}_p(\boldsymbol{\mu}, \mathbf{\Sigma}, \boldsymbol{\lambda})$. Its marginal stochastic representation, which can be used to derive several of its properties, is given by

$$\mathbf{Y} \stackrel{d}{=} \boldsymbol{\mu} + \mathbf{\Sigma}^{1/2}(\boldsymbol{\delta}|T_0| + (\mathbf{I}_p - \boldsymbol{\delta}\boldsymbol{\delta}^\top)^{1/2}\mathbf{T}_1), \quad \text{with} \quad \boldsymbol{\delta} = \frac{\boldsymbol{\lambda}}{\sqrt{1 + \boldsymbol{\lambda}^\top \boldsymbol{\lambda}}}, \tag{1.2}$$

where $T_0 \sim \mathrm{N}_1(0, 1)$ and $\mathbf{T}_1 \sim \mathrm{N}_p(\mathbf{0}, \mathbf{I}_p)$ are independent, $|T_0|$ denotes of course the absolute value of T_0, and " $\stackrel{d}{=}$ " stands for "distributed as". From (1.2) it follows that the expectation and variance of $\mathbf{Y}$ are given, respectively, by

$$E[\mathbf{Y}] = \boldsymbol{\mu} + \sqrt{2/\pi}\mathbf{\Sigma}^{1/2}\boldsymbol{\delta} \tag{1.3}$$

and

$$Var[\mathbf{Y}] = \mathbf{\Sigma} - (2/\pi)\mathbf{\Sigma}^{1/2}\boldsymbol{\delta}\boldsymbol{\delta}^\top\mathbf{\Sigma}^{1/2}. \tag{1.4}$$

Several extensions of the above model has been proposed. For example we have the skew-t distributions (Sahu et al., (2003) [27], Gupta, (2003) [17]), skew-Cauchy distributions (Arnold and Beaver (2000) [3]), skew-slash distributions (Wang and Genton (2006) [31]), skew-slash-t distributions (Tan and Peng (2006) [29]), and skew-elliptical distributions (Azzalini and Capitanio (1999) [5], Branco and Dey (2001) [9], Sahu et al. (2003) [27], Genton and Loperfido (2005) [16]). In this paper we define a new unified family of asymmetric distributions that offers a much needed flexibility by combining both skewness with heavy-tailed. This family contains, as a special case, the multivariate skew-normal distribution defined by Arellano-Valle et al. (2005) [2], the multivariate skew-slash distribution defined by Wang and Genton (2006) [31], the multivariate skew-t distribution defined by Azzalini and Capitanio (2003) [6], and all the distributions studied by Lange and Sinsheimer (1993) [22] in the symmetric context. Thus, our proposal is a more flexible class than the existing skewed distributions, since it allows easy implementation of inferences in any type of models. We point out that the results and methods provided here are not available elsewhere in the literature.

The plan of the article is as follows. In Section 2, the normal/independent distributions (NI) are reviewed for completeness. In Section 3, the skew-normal normal/independent distributions (SNI) are described, and the main results are presented. In Section 4, we derive the maximum likelihood estimates (MLE) for two important applications of SNI distributions. Analytical expressions for the observed information matrix are worked in Section 5. An illustrative example is presented in Section 6, depicting the usefulness of the proposed methodology. Our concluding remarks are presented in Section 7. We also include an appendix as Section 8.

2. Normal/independent distributions

The symmetric family of NI distributions has attracted much attention in the last few years, mainly because it includes distributions such as the Student-t, the slash, the power exponential, and the contaminated normal distributions. All these distributions have heavier tails than the normal.

We say that a p-dimensional vector $\mathbf{Y}$ has a **NI distribution** with location parameter $\boldsymbol{\mu} \in \mathbb{R}^p$ and positive definite scale matrix $\boldsymbol{\Sigma}$ (see for instance, Lange and Sinsheimer (1993) [22]) if its density function has the form

$$f(\mathbf{y}) = \int_0^\infty \phi_p(y|\boldsymbol{\mu}, u^{-1}\boldsymbol{\Sigma})dH(u;\boldsymbol{\nu}), \tag{2.1}$$

where $H(u;\boldsymbol{\nu})$ is a cumulative distribution function of a unidimensional positive random variable U indexed by the parameter vector $\boldsymbol{\nu}$. For a random vector with a probability density function as in (2.1), we shall use the notation $\mathbf{Y} \sim \mathrm{NI}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; H)$. Now, when $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \mathbf{I}_p$, we simply use $\mathbf{Y} \sim \mathrm{NI}_p(H)$.

The stochastic representation of $\mathbf{Y}$ is given by

$$\mathbf{Y} = \boldsymbol{\mu} + U^{-1/2}\mathbf{Z}, \tag{2.2}$$

with $\mathbf{Z} \sim \mathrm{N}_p(\mathbf{0}, \boldsymbol{\Sigma})$ and U a positive random variable with cumulative distribution function H independent of $\mathbf{Z}$. Examples of NI distributions are described subsequently (see also Lange and Sinsheimer (1993) [22]). For this family, the distributional properties of the **Mahalanobis distance**

$$d = (\mathbf{y} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\mathbf{y} - \boldsymbol{\mu}),$$

are also described because they are extremely useful in testing the goodness of fit and for detecting outliers.

2.1 Examples of NI distributions

• *The Student-t distribution* $\mathbf{Y} \sim t_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu)$ *with* $\nu > 0$ *degrees of freedom.* The use of the t-distribution as an alternative to the normal distribution has frequently been suggested in the literature. For example Little (1988) [24] and Lange et al. (1989) [23] use the Student-t distribution for robust modeling. The variable $\mathbf{Y}$ has density

$$f(\mathbf{y}) = \frac{\Gamma(\frac{p+\nu}{2})}{\Gamma(\frac{\nu}{2})\pi^{p/2}} \nu^{-p/2} |\boldsymbol{\Sigma}|^{-1/2} \left(1 + \frac{d}{\nu}\right)^{-(\frac{p+\nu}{2})}. \tag{2.3}$$

In this case, we have $U \sim Gamma(\nu/2, \nu/2)$, where the cumulative distribution function $H(u;\nu)$ has density

$$h(u;\nu) = \frac{(\nu/2)^{\nu/2} u^{\nu/2-1}}{\Gamma(\nu/2)} \exp\left(-\frac{1}{2}\nu u\right), \tag{2.4}$$

and finite reciprocal moments $E[U^{-m}] = \frac{(\nu/2)^m \Gamma(\nu/2-m)}{\Gamma(\nu/2)}$, for $m < \nu/2$. From Lange and Sinsheimer (1993) [22] we also get

$$d = (\mathbf{y} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\mathbf{y} - \boldsymbol{\mu}) \sim pF(p, \nu).$$

• *The slash distribution* $\mathbf{Y} \sim SL_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu)$ *with shape parameter* $\nu > 0$. This distribution presents heavier tails than the normal distribution. It also includes the limiting normal case as $\nu \uparrow \infty$. Its probability density function is given by

$$f(\mathbf{y}) = \nu \int_0^1 u^{\nu-1} \phi_p(\mathbf{y}|\boldsymbol{\mu}, u^{-1}\boldsymbol{\Sigma})\, du. \tag{2.5}$$

Here $H(u;\nu)$ has density

$$h(u;\nu) = \nu u^{\nu-1}\mathbb{I}_{(0,1)}, \tag{2.6}$$

with reciprocal moments $E[U^{-m}] = \dfrac{\nu}{\nu - m}$, for $m < \nu$. The Mahalanobis distance has cumulative distribution function given by

$$Pr(d \leq r) = Pr(\chi^2_p \leq r) - \frac{2^\nu \Gamma(p/2+\nu)}{r^\nu \Gamma(p/2)} Pr(\chi^2_{p+2\nu} \leq r).$$

• *The contaminated normal distribution* $\mathbf{Y} \sim CN_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu, \gamma)$, *with* $0 \leq \nu \leq 1$, $0 < \gamma \leq 1$ (Little (1988) [24]). This distribution may also be applied for modeling symmetric data with outlying observations. The parameter ν represents the percentage of outliers, while γ may be interpreted as a scale factor. Its probability density function is

$$f(\mathbf{y}) = \nu \phi_p(\mathbf{y}|\boldsymbol{\mu}, \frac{\boldsymbol{\Sigma}}{\gamma}) + (1-\nu)\phi_p(\mathbf{y}|\boldsymbol{\mu}, \boldsymbol{\Sigma}). \tag{2.7}$$

In this case the cumulative distribution function $H(u;\boldsymbol{\nu})$ is given by

$$h(u;\boldsymbol{\nu}) = \nu \mathbb{I}_{(u=\gamma)} + (1-\nu)\mathbb{I}_{(u=1)}, \qquad \boldsymbol{\nu} = (\nu, \gamma)^\top, \tag{2.8}$$

where here $\mathbb{I}_{(A)}$ is the indicator function of the set A. Clearly we have $E[U^{-m}] = \nu/\gamma^m + 1 - \nu$ and

$$Pr(d \leq r) = \nu Pr(\chi^2_p \leq \gamma r) + (1-\nu) Pr(\chi^2_p \leq r).$$

The power-exponential distribution is the type NI. However, the scale distribution $H(u;\boldsymbol{\nu})$ is not computationally attractive and will not be dealt with in this work.

3. Multivariate SNI distributions and main results

In this section, we define the multivariate SNI distributions and study some of their properties (*v.g.*, moments, kurtosis, linear transformations, and marginal and conditional distributions).

Definition 3.1. A p-dimensional random vector $\mathbf{Y}$ **follows a SNI distribution** with location parameter $\boldsymbol{\mu} \in \mathbb{R}^p$, scale matrix $\boldsymbol{\Sigma}$ (a $p \times p$ positive-definite matrix) and skewness parameter $\boldsymbol{\lambda} \in \mathbb{R}^p$ if its probability density function is given by

$$f(\mathbf{y}) = 2\int_0^\infty \phi_p(\mathbf{y}|\boldsymbol{\mu}, u^{-1}\boldsymbol{\Sigma})\Phi_1(u^{1/2}\boldsymbol{\lambda}^\top\boldsymbol{\Sigma}^{-1/2}(\mathbf{y}-\boldsymbol{\mu}))\, dH(u) \tag{3.1}$$
$$= 2\int_0^\infty \frac{u^{p/2}}{(2\pi)^{p/2}}|\boldsymbol{\Sigma}|^{-1/2} e^{-\dfrac{u}{2}d_\lambda}\Phi_1(u^{1/2}\boldsymbol{\lambda}^\top\boldsymbol{\Sigma}^{-1/2}(\mathbf{y}-\boldsymbol{\mu}))\, dH(u),$$

where U is a positive random variable with cumulative distribution function $H(u;\boldsymbol{\nu})$.

For a random vector with probability density function as in (3.1), we use the notion $\mathbf{Y} \sim SNI_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}; H)$. When $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \mathbf{I}_p$, we get a standard SNI distribution and denote it by $SNI_p(\boldsymbol{\lambda}; H)$.

It is clear from (3.1) that when $\boldsymbol{\lambda} = \mathbf{0}$, we get back the NI distribution defined in (2.1). For a random vector with probability density function as in (3.1), we write the **Mahalanobis distance** as

$$d_\lambda = (\mathbf{y}-\boldsymbol{\mu})^\top\boldsymbol{\Sigma}^{-1}(\mathbf{y}-\boldsymbol{\mu}).$$

In Definition 3.1, note that the cumulative distribution function $H(u;\boldsymbol{\nu})$ is indexed by the vector $\boldsymbol{\nu}$. Thus, if we suppose that $\boldsymbol{\nu}_\infty$ is such that $\boldsymbol{\nu} \uparrow \boldsymbol{\nu}_\infty$, and $H(u;\boldsymbol{\nu})$ converges weakly to the distribution function $H_\infty(u) = H(u;\boldsymbol{\nu}_\infty)$ of the unit point mass at 1, then the density function in (3.1) converges to the density function of a random vector having a skew-normal distribution. The proof of this result is similar to the one present in Lange and Sinsheimer (1993) [22] for the NI case.

For a SNI random vector, the stochastic representation given below can be used to quickly simulate pseudo-realizations of $\mathbf{Y}$, and also to study many of their properties.

Proposition 3.2. *For* $\mathbf{Y} \sim SNI_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}; H)$ *we have*

$$\mathbf{Y} \stackrel{d}{=} \boldsymbol{\mu} + U^{-1/2}\mathbf{Z}, \tag{3.2}$$

with $\mathbf{Z} \sim SN_p(\mathbf{0}, \boldsymbol{\Sigma}, \boldsymbol{\lambda})$ *and* U *a positive random variable with cumulative distribution function* H *independent of* $\mathbf{Z}$*. (Compare Equation (1.1).)*

Proof. This follows from the hypothesis $\mathbf{Y}|U = u \sim \mathrm{SN}_p(\boldsymbol{\mu}, u^{-1}\boldsymbol{\Sigma}, \boldsymbol{\lambda})$. □

Notice that the stochastic representation given in (2.2) for the NI case is a specialization of (3.2) for $\boldsymbol{\lambda} = \mathbf{0}$. Hence, we have extended the family of NI distributions to the skewed case. Besides, from (1.2) it follows that (3.2) can be written as

$$\mathbf{Y} \stackrel{d}{=} \boldsymbol{\mu} + \frac{1}{U^{1/2}}\boldsymbol{\Sigma}^{1/2}\{\boldsymbol{\delta}|X_0| + (\mathbf{I}_n - \boldsymbol{\delta}\boldsymbol{\delta}^T)^{1/2}\mathbf{X}_1\}, \tag{3.3}$$

where $\boldsymbol{\delta} = \boldsymbol{\lambda}/\sqrt{1+\boldsymbol{\lambda}^\top\boldsymbol{\lambda}}$, and U, $X_0 \sim N_1(0,1)$ and $\mathbf{X}_1 \sim N_p(\mathbf{0}, \mathbf{I}_p)$ are independent. The marginal stochastic representation given in (3.3) is very important since it allows us to implement the EM-algorithm for a wide class of linear models similar to those of Lachos et al. (2007) [20].

In the next proposition, we derive a general expression for the moment generating function of a SNI random vector.

Proposition 3.3. *For $\mathbf{Y} \sim SNI_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}; H)$ and $\mathbf{s} \in \mathbb{R}^p$, we have*

$$M_{\mathbf{y}}(\mathbf{s}) = E[e^{\mathbf{s}^\top\mathbf{Y}}] = \int_0^\infty 2e^{\mathbf{s}^\top\boldsymbol{\mu} + \frac{1}{2}u^{-1}\mathbf{s}^\top\boldsymbol{\Sigma}\mathbf{s}}\Phi_1(u^{-1/2}\boldsymbol{\delta}^\top\boldsymbol{\Sigma}^{1/2}\mathbf{s})dH(u). \tag{3.4}$$

Proof. From Proposition 3.2, we obtain $\mathbf{Y}|U = u \sim SN_p(\boldsymbol{\mu}, u^{-1}\boldsymbol{\Sigma}, \boldsymbol{\lambda})$. Next we get $M_{\mathbf{y}}(\mathbf{s}) = E_U[E[e^{\mathbf{s}^\top\mathbf{Y}}|U]]$ from well known properties of conditional expectation. As U is a positive random variable with cumulative distribution function H, we derive the proof from the fact that $\mathbf{Z} \sim SN_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda})$ implies $M_{\mathbf{z}}(\mathbf{s}) = 2e^{\mathbf{s}^\top\boldsymbol{\mu} + \frac{1}{2}\mathbf{s}^\top\boldsymbol{\Sigma}\mathbf{s}}\Phi_1(\boldsymbol{\delta}^\top\boldsymbol{\Sigma}^{1/2})$. □

The next proposition shows that a SNI random vector is invariant under linear transformations. This, in turn, implies that the marginal distributions of $\mathbf{Y} \sim \mathrm{SNI}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}; H)$ are still SNI.

Proposition 3.4. *Let $\mathbf{Y} \sim SNI_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}; H)$. Then for any fixed vector $\mathbf{b} \in \mathbb{R}^m$ and matrix $\mathbf{A} \in \mathbb{R}^{m\times p}$ of full row rank we get*

$$\mathbf{V} = \mathbf{b} + \mathbf{A}\mathbf{Y} \sim SNI_p(\mathbf{b} + \mathbf{A}\boldsymbol{\mu}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^\top, \boldsymbol{\lambda}^*; H), \tag{3.5}$$

here we have $\boldsymbol{\lambda}^* = \boldsymbol{\delta}^*/(1-\boldsymbol{\delta}^{*\top}\boldsymbol{\delta}^*)^{1/2}$ *with* $\boldsymbol{\delta}^* = (\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^\top)^{-1/2}\mathbf{A}\boldsymbol{\Sigma}^{1/2}\boldsymbol{\delta}$. *Moreover, if* $m = p$ *and* $\mathbf{A}$ *is non-singular, then we get* $\boldsymbol{\lambda}^* = \boldsymbol{\lambda}$. *Also, for any* $\mathbf{a} \in \mathbb{R}^p$, *we obtain*

$$\mathbf{a}^\top\mathbf{Y} \sim SNI_p(\mathbf{a}^\top\boldsymbol{\mu}, \mathbf{a}^\top\boldsymbol{\Sigma}\mathbf{a}, \lambda^*; H),$$

where $\lambda^* = \alpha/(1-\alpha^2)^{1/2}$, *with* $\alpha = \{\mathbf{a}^\top\boldsymbol{\Sigma}\mathbf{a}(1+\boldsymbol{\lambda}^\top\boldsymbol{\lambda})\}^{-1/2}\mathbf{a}^\top\boldsymbol{\Sigma}^{1/2}\boldsymbol{\lambda}$.

Proof. The proof of this result is direct from Proposition 3.3 since we have $M_{\mathbf{b}+\mathbf{A}\mathbf{Y}}(\mathbf{s}) = e^{\mathbf{s}^\top\mathbf{b}}M_{\mathbf{Y}}(\mathbf{A}^\top\mathbf{s})$. When $\mathbf{A}$ is non-singular, it is easy to see that $\boldsymbol{\delta}^* = \boldsymbol{\delta}$ holds. □

Applying Proposition 3.4 to $\mathbf{A} = [\mathbf{I}_{p_1}, \mathbf{0}_{p_2}]$, with $p_1 + p_2 = p$, we obtain the following additional properties of a SNI random vector, related to the marginal distribution this time.

Corollary 3.5. *Let* $\mathbf{Y} \sim SNI_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}; H)$ *and suppose* $\mathbf{Y}$ *is partitioned as* $\mathbf{Y}^\top = (\mathbf{Y}_1^\top, \mathbf{Y}_2^\top)^\top$ *of dimensions* p_1 *and* $p_2 = p - p_1$, *respectively. Let*

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}, \quad \boldsymbol{\mu} = (\boldsymbol{\mu}_1^\top, \boldsymbol{\mu}_2^\top)^\top$$

be the corresponding partitions of $\boldsymbol{\Sigma}$ *and* $\boldsymbol{\mu}$. *Then, the marginal density of* $\mathbf{Y}_1$ *is* $SNI_{p_1}(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11}, \boldsymbol{\Sigma}_{11}^{1/2}\widetilde{\boldsymbol{\upsilon}}; H)$, *where* $\widetilde{\boldsymbol{\upsilon}} = \dfrac{\boldsymbol{\upsilon}_1 + \boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\upsilon}_2}{\sqrt{1+\boldsymbol{\upsilon}_2^\top\boldsymbol{\Sigma}_{22.1}\boldsymbol{\upsilon}_2}}$, *with* $\boldsymbol{\Sigma}_{22.1} = \boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}$ *and* $\boldsymbol{\upsilon} = \boldsymbol{\Sigma}^{-1/2}\boldsymbol{\lambda} = (\boldsymbol{\upsilon}_1^\top, \boldsymbol{\upsilon}_2^\top)^\top$. □

Proposition 3.6. *Let* $\mathbf{Y} \sim SNI_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}; H)$. *Then the distribution of* $\mathbf{Y}_2$, *conditionally on* $\mathbf{Y}_1 = \mathbf{y}_1$ *and* $U = u$, *has density*

$$f(\mathbf{y}_2|\mathbf{y}_1, u) = \phi_{p_2}(\mathbf{y}_2|\boldsymbol{\mu}_{2.1}, u^{-1}\boldsymbol{\Sigma}_{22.1})\frac{\Phi_1(u^{1/2}\boldsymbol{\upsilon}^\top(\mathbf{y}-\boldsymbol{\mu}))}{\Phi_1(u^{1/2}\widetilde{\boldsymbol{\upsilon}}^\top(\mathbf{y}_1-\boldsymbol{\mu}_1))}, \tag{3.6}$$

with $\boldsymbol{\mu}_{2.1} = \boldsymbol{\mu}_2 + \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}(\mathbf{y}_1 - \boldsymbol{\mu}_1)$. *Furthermore, we get*

$$E[\mathbf{Y}_2|\mathbf{y}_1, u] = \boldsymbol{\mu}_{2.1} + u^{-1/2}\frac{\phi_1(u^{1/2}\widetilde{\boldsymbol{\upsilon}}^\top(\mathbf{y}_1-\boldsymbol{\mu}_1))}{\Phi_1(u^{1/2}\widetilde{\boldsymbol{\upsilon}}^\top(\mathbf{y}_1-\boldsymbol{\mu}_1))}\frac{\boldsymbol{\Sigma}_{22.1}\boldsymbol{\upsilon}_2}{\sqrt{1+\boldsymbol{\upsilon}_2^\top\boldsymbol{\Sigma}_{22.1}\boldsymbol{\upsilon}_2}}. \tag{3.7}$$

Proof. As we have $f(\mathbf{y}_2|\mathbf{y}_1, u) = f(\mathbf{y}|u)/f(\mathbf{y}_1|u)$, Formula (3.6) for the density follows after noticing $\mathbf{Y}|U = u \sim SN_p(\boldsymbol{\mu}, u^{-1}\boldsymbol{\Sigma}, \boldsymbol{\lambda})$ and $\mathbf{Y}_1|U = u \sim SN(\boldsymbol{\mu}_1, u^{-1}\ \boldsymbol{\Sigma}_{11}, \boldsymbol{\Sigma}_{11}^{1/2}\widetilde{\boldsymbol{\upsilon}})$. The expectation suggested in (3.7) is confirmed by Lemma 8.2 (in the appendix) if we take $A = \boldsymbol{\upsilon}_1^\top(\mathbf{y}_1 - \boldsymbol{\mu}_1) - \boldsymbol{\upsilon}_2^\top\boldsymbol{\mu}_2$, $\mathbf{B} = \boldsymbol{\upsilon}_2$, $\boldsymbol{\mu} = \boldsymbol{\mu}_{2.1}$, and $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_{22.1}$. This concludes the proof. □

Note that given u, when we have $\boldsymbol{\Sigma}_{21} = \mathbf{0}$ and $\boldsymbol{\lambda}_2 = \mathbf{0}$, it is possible to obtain independence for the components $\mathbf{Y}_1$ and $\mathbf{Y}_2$ of a SNI random vector $\mathbf{Y}$. The following corollary is a by-product of Proposition 3.6, since we have $E[\mathbf{Y}_2|\mathbf{y}_1] = E_U[E[\mathbf{Y}_2|\mathbf{y}_1, U]|\mathbf{y}_1]$.

Proposition 3.7. *For* $\mathbf{Y} \sim SNI_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}; H)$ *the first moment of* $\mathbf{Y}_2$*, conditionally on* $\mathbf{Y}_1 = \mathbf{y}_1$*, is given by*

$$E[\mathbf{Y}_2|\mathbf{y}_1] = \boldsymbol{\mu}_{2.1} + \frac{\boldsymbol{\Sigma}_{22.1}\boldsymbol{\upsilon}_2}{\sqrt{1 + \boldsymbol{\upsilon}_2^\top\boldsymbol{\Sigma}_{22.1}\boldsymbol{\upsilon}_2}} E[U^{-1/2}\frac{\phi_1(U^{1/2}\widetilde{\boldsymbol{\upsilon}}^\top(\mathbf{y}_1 - \boldsymbol{\mu}_1))}{\Phi_1(U^{1/2}\widetilde{\boldsymbol{\upsilon}}^\top(\mathbf{y}_1 - \boldsymbol{\mu}_1))}\mathbf{y}_1],$$

with $\boldsymbol{\mu}_{2.1} = \boldsymbol{\mu}_2 + \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}(\mathbf{y}_1 - \boldsymbol{\mu}_1)$. □

The next result can be useful in applications to linear models. For instance, we can use it when the linear model depends on a vector of unobservable random effects and a vector of random errors (linear mixed model) in which the random effects are assumed to have a SNI distribution and the errors are assumed to have a NI distribution.

Proposition 3.8. *Suppose we have* $\mathbf{X} \sim SNI_m(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1, \boldsymbol{\lambda}, H)$ *and* $\mathbf{Y} \sim NI_p(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2, H)$*. If there is a positive random variable* U *with cumulative distribution function* H *so that we can write* $\mathbf{X} \stackrel{d}{=} \boldsymbol{\mu}_1 + U^{-1/2}\mathbf{Z}$ *and* $\mathbf{Y} \stackrel{d}{=} \boldsymbol{\mu}_2 + U^{-1/2}\mathbf{W}$*, with* $\mathbf{Z} \sim SN_m(\mathbf{0}, \boldsymbol{\Sigma}_1, \boldsymbol{\lambda})$ *independent of* $\mathbf{W} \sim N_p(\mathbf{0}, \boldsymbol{\Sigma}_2)$*, then for any matrix* $\mathbf{A}$ *of dimension* $p \times m$ *we have*

$$\mathbf{AX} + \mathbf{Y} \sim SNI_m(\mathbf{A}\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2, \mathbf{A}\boldsymbol{\Sigma}_1\mathbf{A}^\top + \boldsymbol{\Sigma}_2, \boldsymbol{\lambda}_*; H),$$

here $\boldsymbol{\lambda}_* = \boldsymbol{\delta}_*/\sqrt{1 - \boldsymbol{\delta}_*^\top\boldsymbol{\delta}_*}$*, with* $\boldsymbol{\delta}_* = (\mathbf{A}\boldsymbol{\Sigma}_1\mathbf{A}^\top + \boldsymbol{\Sigma}_2)^{-1/2}\mathbf{A}\boldsymbol{\Sigma}_1^{1/2}\boldsymbol{\delta}$.

Proof. The proof is based on Proposition 3.3. Note first that surely $\mathbf{X}$

and $\mathbf{Y}$ are independent. Next, letting $\mathbf{V} = \mathbf{AX} + \mathbf{Y}$ we obtain

$$\begin{aligned}
M_{\mathbf{V}}(\mathbf{s}) &= E_U(E[e^{\mathbf{s}^\top \mathbf{AX}}|U]E[e^{\mathbf{s}^\top \mathbf{Y}}|U]) \\
&= \int_0^\infty 2\mathrm{e}^{\mathbf{s}^\top \mathbf{A}\boldsymbol{\mu}_1 + \frac{1}{2u}\mathbf{s}^\top \mathbf{A}\boldsymbol{\Sigma}_1\mathbf{A}^\top \mathbf{s}}\Phi_1(\frac{\boldsymbol{\delta}^\top \boldsymbol{\Sigma}_1^{1/2}\mathbf{A}^\top \mathbf{s}}{\sqrt{u}})e^{\mathbf{s}^\top \boldsymbol{\mu}_2 + \frac{1}{2u}\mathbf{s}^\top \boldsymbol{\Sigma}_2\mathbf{s}}\, dH(u) \\
&= \int_0^\infty 2\mathrm{e}^{\mathbf{s}^\top (\mathbf{A}\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2) + \frac{1}{2u}\mathbf{s}^\top (\mathbf{A}\boldsymbol{\Sigma}_1\mathbf{A}^\top + \boldsymbol{\Sigma}_2)\mathbf{s}}\Phi_1(\frac{\boldsymbol{\delta}^\top \boldsymbol{\Sigma}_1^{1/2}\mathbf{A}^\top \mathbf{s}}{\sqrt{u}})\, dH(u) \\
&= \int_0^\infty 2\mathbf{e}^{\mathbf{t}^\top (\mathbf{A}\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2) + \frac{1}{2u}\mathbf{s}^\top (\mathbf{A}\boldsymbol{\Sigma}_1\mathbf{A}^\top + \boldsymbol{\Sigma}_2)\mathbf{s}}\Phi_1(\frac{\boldsymbol{\delta}_*^\top \boldsymbol{\Psi}^{1/2}\mathbf{s}}{\sqrt{u}})\, dH(u),
\end{aligned}$$

where $\boldsymbol{\Psi} = \mathbf{A}\boldsymbol{\Sigma}_1\mathbf{A}^\top + \boldsymbol{\Sigma}_2$, $\boldsymbol{\delta}_* = \boldsymbol{\Psi}^{-1/2}\mathbf{A}\boldsymbol{\Sigma}_1^{1/2}\boldsymbol{\delta}$, and the proof follows from Proposition 3.3. □

In the following proposition we derive the mean and the covariance matrix of a SNI random vector. Furthermore, we present the multidimensional kurtosis coefficient for a random vector SNI, which represent a extension of the kurtosis coefficient proposed by Azzalini and Capitanio (1999) [5].

Proposition 3.9. *Suppose we have $\mathbf{Y} \sim SNI_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}; H)$. Then the following conditions hold.*

a) If $E[U^{-1/2}] < \infty$, then we have

$$E[\mathbf{Y}] = \boldsymbol{\mu} + \sqrt{\frac{2}{\pi}}E[U^{-1/2}]\boldsymbol{\Sigma}^{1/2}\boldsymbol{\delta}.$$

b) If $E[U^{-1}] < \infty$, then we have

$$Var[\mathbf{Y}] = \boldsymbol{\Sigma}_y = E[U^{-1}]\boldsymbol{\Sigma} - \frac{2}{\pi}E^2[U^{-1/2}]\boldsymbol{\Sigma}^{1/2}\boldsymbol{\delta}\boldsymbol{\delta}^\top \boldsymbol{\Sigma}^{1/2}.$$

c) If $E[U^{-2}] < \infty$, then the multidimensional kurtosis coefficient is

$$\gamma_2(\mathbf{Y}) = \frac{E[U^{-2}]}{E^2[U^{-1}]}a_{1y} - 4\frac{E[U^{-3/2}]}{E^2[U^{-1}]}a_{2y} + a_{3y} - p(p+2),$$

here

$$
\begin{aligned}
a_{1y} &= p(p+2)+2(p+2)\boldsymbol{\mu}_y^\top\boldsymbol{\Sigma}_y^{-1}\boldsymbol{\mu}_y+3(\boldsymbol{\mu}_y^\top\boldsymbol{\Sigma}_y^{-1}\boldsymbol{\mu}_y)^2,\\
a_{2y} &= \left(p+\frac{2}{E[U^{-1/2}]}\right)\boldsymbol{\mu}_y^\top\boldsymbol{\Sigma}_y^{-1}\boldsymbol{\mu}_y\\
&\qquad +\left(1+\frac{2}{E[U^{-1/2}]}-\frac{\pi}{2}\frac{E[U^{-1}]}{E^2[U^{-1/2}]}\right)(\boldsymbol{\mu}_y^\top\boldsymbol{\Sigma}_y^{-1}\boldsymbol{\mu}_y)^2,\\
a_{3y} &= 2(p+2)\boldsymbol{\mu}_y^\top\boldsymbol{\Sigma}_y^{-1}\boldsymbol{\mu}_y+3(\boldsymbol{\mu}_y^\top\boldsymbol{\Sigma}_y^{-1}\boldsymbol{\mu}_y)^2,
\end{aligned}
$$

where $\boldsymbol{\mu}_y = E[\mathbf{Y}-\boldsymbol{\mu}] = \sqrt{\frac{2}{\pi}}E[U^{-1/2}]\boldsymbol{\Sigma}^{1/2}\boldsymbol{\delta}$.

Proof. The proof of a) and b) follows from Proposition 3.2. To obtain the expression in c) we use the definition of the multivariate kurtosis introduced by Mardia (1974) [26]. Without loss of generality we take $\boldsymbol{\mu}=\mathbf{0}$, so to get $\boldsymbol{\mu}_y = E[\mathbf{Y}] = \sqrt{\frac{2}{\pi}}E[U^{-1/2}]\boldsymbol{\Sigma}^{1/2}\boldsymbol{\delta}$. Note first that the kurtosis is defined by $\gamma_2(\mathbf{Y}) = E[\{(\mathbf{Y}-\boldsymbol{\mu}_y)^\top\boldsymbol{\Sigma}_y^{-1}\mathbf{Y}-\boldsymbol{\mu}_y)\}^2]$. Now, by using the stochastic representation of $\mathbf{Y}$ given in (2.2) we obtain

$$
(\mathbf{Y}-\boldsymbol{\mu}_y)^\top\boldsymbol{\Sigma}_y^{-1}\mathbf{Y}-\boldsymbol{\mu}_y)\stackrel{d}{=}U^{-1}\mathbf{Z}^\top\boldsymbol{\Sigma}_y^{-1}\mathbf{Z}-2U^{-1/2}\mathbf{Z}^\top\boldsymbol{\Sigma}_y^{-1}\boldsymbol{\mu}_y+\boldsymbol{\mu}_y^\top\boldsymbol{\Sigma}_y^{-1}\boldsymbol{\mu}_y,
$$

where $\mathbf{Z}\sim SN_p(\mathbf{0},\boldsymbol{\Sigma},\boldsymbol{\lambda})$. Due to the definition of $\gamma_2(\mathbf{Y})$, the proof follows after some algebraic manipulations involving the first two moments of a quadratic form (see Genton, He and Liu, (2001) [15]) and Lemma 8.1. □

Note that under the skew-normal distribution condition, i.e, when $U=1$, the multidimensional kurtosis coefficient reduces to $\gamma_2(\mathbf{Y}) = 2(\pi-3)(\boldsymbol{\mu}_y^\top\boldsymbol{\Sigma}_y^{-1}\boldsymbol{\mu}_y)^2$, which is the kurtosis coefficient for a skew-normal random vector (see for instance, Azzalini and Capitanio (1999) [5]).

Proposition 3.10. *If* $\mathbf{Y}\sim SNI_p(\boldsymbol{\mu},\boldsymbol{\Sigma},\boldsymbol{\lambda};H)$, *then for any even function* g *the distribution of* $g(\mathbf{Y}-\boldsymbol{\mu})$ *does not depend on* $\boldsymbol{\lambda}$ *and has the same distribution as* $g(\mathbf{X}-\boldsymbol{\mu})$, *where* $\mathbf{X}\sim NI_p(\boldsymbol{\mu},\boldsymbol{\Sigma};H)$. *In particular, if* $\mathbf{A}$ *is a* $p\times p$ *symmetric matrix, then* $(\mathbf{Y}-\boldsymbol{\mu})^\top\mathbf{A}(\mathbf{Y}-\boldsymbol{\mu})$ *and* $(\mathbf{X}-\boldsymbol{\mu})^\top\mathbf{A}(\mathbf{X}-\boldsymbol{\mu})$ *are identically distributed.*

Proof. The proof follows from Proposition 3.3; a similar procedure can be found in Wang et al. (2004) [30]. □

As a by-product of Proposition 3.10 we have the following interesting result.

Corollary 3.11. *Let* $\mathbf{Y} \sim SNI_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}; H)$*. Then the quadratic form*

$$d_\lambda = (\mathbf{Y} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{Y} - \boldsymbol{\mu})$$

has the same distribution as $d = (\mathbf{X} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{X} - \boldsymbol{\mu})$*, where* $\mathbf{X} \sim NI_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; H)$*.* □

Corollary 3.11 is interesting because it allows us in practice to check models (see Section 5). On the other hand, Corollary 3.11 together with a result from Lange and Sinsheimer (1993) [22, Section 2] allows us to obtain the m-th moment of d_λ.

Corollary 3.12. *Let* $\mathbf{Y} \sim SNI_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}; H)$*. Then for any* $m > 0$ *we have*

$$E[d_\lambda^m] = \frac{2^m \Gamma(m + p/2)}{\Gamma(p/2)} E[U^{-m}].$$

□

3.1 Examples of SNI distributions

We provide several examples of SNI distributions.

• *The skew-t distribution* $ST_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}, \nu)$ *with* ν *degrees of freedom.* Consider $U \sim Gamma(\nu/2, \nu/2)$. Similar procedures to those of Gupta (2003) [17, Section 2] lead us to the density function

$$f(\mathbf{y}) = 2t_p(\mathbf{y}|\boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu) T_1(\frac{\sqrt{v+p}\boldsymbol{\lambda}^\top \boldsymbol{\Sigma}^{-1/2}(\mathbf{y} - \boldsymbol{\mu})}{\sqrt{d+\nu}}|0, 1, \nu + p), \ \mathbf{y} \in \mathbb{R}^p, \tag{3.8}$$

where, as usual, $t_p(\cdot|\boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu)$ and $T_p(\cdot|\boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu)$ denote, respectively, the probability density function and cumulative distribution function of the

Student-t distribution $t_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu)$ as defined in (2.3). Absorbed by the skew-t distribution is the skew-Cauchy distribution when $\nu = 1$. Also, when $\nu \uparrow \infty$, we recover the skew-normal distribution as the limiting case; see Gupta (2003) [17] for further details. In this case, from Proposition 3.9, the mean and covariance matrix of $\mathbf{Y} \sim ST_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}, \nu)$ are given by

$$E[\mathbf{Y}] = \boldsymbol{\mu} + \sqrt{\nu/\pi}\,\frac{\Gamma(\frac{\nu-1}{2})}{\Gamma(\frac{\nu}{2})}\boldsymbol{\Sigma}^{1/2}\boldsymbol{\delta},\;\; \nu > 1$$

and

$$Var[\mathbf{Y}] = \frac{\nu}{\nu-2}\boldsymbol{\Sigma} - (\nu/\pi)\,(\frac{\Gamma(\frac{\nu-1}{2})}{\Gamma(\frac{\nu}{2})})^2\boldsymbol{\Sigma}^{1/2}\boldsymbol{\delta}\boldsymbol{\delta}^\top\boldsymbol{\Sigma}^{1/2},\;\; \nu > 2.$$

In what follows we give an important result which will be used in the implementation of the EM algorithm.

Proposition 3.13. *If $\mathbf{Y} \sim ST_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}, \nu)$, then we have*

$$E[U^r|\mathbf{y}] = \frac{2^{r+1}\nu^{\nu/2}\Gamma(\frac{p+\nu+2r}{2})(d+\nu)^{-\frac{p+\nu+2r}{2}}}{f(\mathbf{y})\Gamma(\nu/2)\sqrt{\pi^p}|\boldsymbol{\Sigma}|^{1/2}}T_1(\sqrt{\tfrac{p+\nu+2r}{d+\nu}}A|0,1,p+\nu+2r)$$

and

$$E[U^r W_{\Phi_1}(U^{1/2}A)] = \frac{2^{r+1/2}\nu^{\nu/2}\Gamma(\frac{p+\nu+2r}{2})(d+\nu+A^2)^{-\frac{p+\nu+2r}{2}}}{f(\mathbf{y})\Gamma(\nu/2)\sqrt{\pi}^{p+1}|\boldsymbol{\Sigma}|^{1/2}}.$$

where $A = \boldsymbol{\lambda}^\top\boldsymbol{\Sigma}^{-1/2}(\mathbf{y}-\boldsymbol{\mu})$ and $W_{\Phi_1}(x) = \phi_1(x)/\Phi_1(x)$, for $x \in \mathbb{R}$.

Proof. The proof follows from Lemma 1 in Azzalini and Capitanio (2003) [6, Lemma 1] as we have $f(u|\mathbf{y}) = f(\mathbf{y}, u)/f(\mathbf{y})$ plus

$$E[U^r|\mathbf{y}] = \frac{2}{f(\mathbf{y})}\int_0^\infty u^r\phi_p(\mathbf{y}|\boldsymbol{\mu}, u^{-1}\boldsymbol{\Sigma})\Phi_1(u^{1/2}A)G_u(\nu/2,\nu/2)du$$

and

$$E[U^r W_{\Phi_1}(U^{1/2}A)] = \frac{2}{f(\mathbf{y})}\int_0^\infty u^r\phi_p(\mathbf{y}|\boldsymbol{\mu}, u^{-1}\boldsymbol{\Sigma})\phi_1(u^{1/2}A)G_u(\nu/2,\nu/2)du,$$

here the probability density function of the $Gamma(\frac{\nu}{2},\frac{\nu}{2})$ distribution is given by $G_u(\nu/2,\nu/2)$. □

For a skew-t random vector $\mathbf{Y}$, partitioned as $\mathbf{Y}^\top = (\mathbf{Y}_1^\top, \mathbf{Y}_2^\top)^\top$, we have from Corollary 1 that $\mathbf{Y}_1 \sim ST_{p_1}(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11}, \boldsymbol{\Sigma}_{11}^{1/2}\widetilde{\boldsymbol{\upsilon}}, \nu)$ holds. Thus, from Proposition 3.7 we have the following result.

Corollary 3.14. *For* $\mathbf{Y} \sim ST_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}, \nu)$ *we have*

$$\begin{aligned} E[\mathbf{Y}_2|\mathbf{y}_1] &= \boldsymbol{\mu}_{2.1} + \frac{\boldsymbol{\Sigma}_{22.1}\boldsymbol{\upsilon}_2}{\sqrt{1+\boldsymbol{\upsilon}_2^\top \boldsymbol{\Sigma}_{22.1}\boldsymbol{\upsilon}_2}} \frac{\nu^{\nu/2}\Gamma(\frac{\nu+p_1-1}{2})}{\Gamma(\nu/2)\sqrt{\pi}^{(p_1+1)}|\boldsymbol{\Sigma}_{11}|^{1/2}} \times \\ & \quad \frac{1}{f(\mathbf{y}_1)}(\nu + d_{y_1} + (\widetilde{\boldsymbol{\upsilon}}^\top(\mathbf{y}_1 - \boldsymbol{\mu}_1))^2)^{-\frac{\nu+p_1-1}{2}}, \end{aligned}$$

where $d_{y_1} = (\mathbf{y}_1 - \boldsymbol{\mu}_1)^\top \boldsymbol{\Sigma}_{11}^{-1}(\mathbf{y}_1 - \boldsymbol{\mu}_1)$. □

• *The skew-slash distribution* $SSL_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}, \nu)$ *with the shape parameter* $\nu > 0$. With $h(u;\nu)$ as in (2.6), from Proposition 3.2 can easily be derived

$$f(\mathbf{y}) = 2\nu \int_0^1 u^{\nu-1}\phi_p(\mathbf{y}|\boldsymbol{\mu}, \frac{\boldsymbol{\Sigma}}{u})\Phi_1(u^{1/2}\boldsymbol{\lambda}^\top\boldsymbol{\Sigma}^{-1/2}(\mathbf{y}-\boldsymbol{\mu})), \quad \mathbf{y} \in \mathbb{R}^p, \tag{3.9}$$

The skew-slash distribution becomes the skew-normal distribution when $\nu \uparrow \infty$. See Wang and Genton (2006) [31] for further details. In this particular case, from Proposition 3.9 we get

$$E[\mathbf{Y}] = \boldsymbol{\mu} + \sqrt{\frac{2}{\pi}}\frac{2\nu}{2\nu-1}\boldsymbol{\Sigma}^{1/2}\boldsymbol{\delta}, \ \ \nu > 1/2$$

and

$$Var[\mathbf{Y}] = \frac{\nu}{\nu-1}\boldsymbol{\Sigma} - \frac{2}{\pi}(\frac{2\nu}{2\nu-1})^2\boldsymbol{\Sigma}^{1/2}\boldsymbol{\delta}\boldsymbol{\delta}^\top\boldsymbol{\Sigma}^{1/2}, \ \ \nu > 1.$$

As in the skew-t case we have the following results.

Proposition 3.15. *For* $\mathbf{Y} \sim SSL_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}, \nu)$ *we get*

$$E[U^r|\mathbf{y}] = \frac{2^{\nu+r+1}\nu\Gamma(\frac{p+2\nu+2r}{2})P_1(\frac{p+2\nu+2r}{2}, \frac{d}{2})d^{-\frac{p+2\nu+2r}{2}}}{f(\mathbf{y})\sqrt{\pi^p}|\boldsymbol{\Sigma}|^{1/2}}E[\Phi(S^{1/2}A)],$$

where $S_i \sim Gamma(\frac{p+2\nu+2r}{2}, \frac{d}{2})\mathbb{I}_{(0,1)}$, *and*

$$
\begin{aligned}
&E[U^r W_{\Phi_1}(U^{1/2}A)] = \\
&= \frac{2^{\nu+r+1/2}\nu\Gamma(\frac{2\nu+p+2r}{2})}{f(\mathbf{y})\sqrt{\pi}^{p+1}|\mathbf{\Sigma}|^{1/2}}(d+A^2)^{-\frac{2\nu+p+2r}{2}}P_1(\frac{2\nu+p+2r}{2}, \frac{d+A^2}{2})
\end{aligned}
$$

here $P_x(a,b)$ *is the cumulative distribution function of the* $Gamma(a,b)$ *distribution evaluated at* x. □

Corollary 3.16. *If* $\mathbf{Y} \sim SSL_p(\boldsymbol{\mu}, \mathbf{\Sigma}, \boldsymbol{\lambda}, \nu)$ *then we have*

$$
\begin{aligned}
E[\mathbf{Y}_2|\mathbf{y}_1] &= \boldsymbol{\mu}_{2.1} + \frac{\mathbf{\Sigma}_{22.1}\boldsymbol{\upsilon}_2}{\sqrt{1+\boldsymbol{\upsilon}_2^\top\mathbf{\Sigma}_{22.1}\boldsymbol{\upsilon}_2}} \times \\
&\quad \frac{2^\nu\nu}{f(\mathbf{y}_1)}\frac{\Gamma\frac{(p_1+2\nu-1)}{2}(d_{y1}+(\widetilde{\boldsymbol{\upsilon}}^\top(\mathbf{y}_1-\boldsymbol{\mu}_1))^2)^{-\frac{p_1+2\nu-1}{2}}}{\sqrt{\pi}^{(p_1+1)}|\mathbf{\Sigma}_{11}|^{1/2}} \times \\
&\quad P_1(\frac{p_1+2\nu-1}{2}, \frac{d_{y_1}+(\widetilde{\boldsymbol{\upsilon}}^\top(\mathbf{y}_1-\boldsymbol{\mu}_1))^2}{2}),
\end{aligned}
$$

where $d_{y_1} = (\mathbf{y}_1-\boldsymbol{\mu}_1)^\top\mathbf{\Sigma}_{11}^{-1}(\mathbf{y}_1-\boldsymbol{\mu}_1)$. □

• *The contaminated skew-normal distribution* $SCN_p(\boldsymbol{\mu}, \mathbf{\Sigma}, \boldsymbol{\lambda}, \nu, \gamma)$ *with* $0 \leq \nu \leq 1$, $0 < \gamma < 1$. Taking $h(u;\boldsymbol{\nu})$ as in (2.8), we get in a straightforward manner

$$
\begin{aligned}
f(\mathbf{y}) &= 2\{\nu\phi_p(\mathbf{y}|\boldsymbol{\mu}, \frac{\mathbf{\Sigma}}{\gamma})\Phi_1(\gamma^{1/2}\boldsymbol{\lambda}^\top\mathbf{\Sigma}^{-1/2}(\mathbf{y}-\boldsymbol{\mu})) \\
&\quad +(1-\nu)\phi_p(\mathbf{y}|\boldsymbol{\mu}, \mathbf{\Sigma})\Phi_1(\boldsymbol{\lambda}^\top\mathbf{\Sigma}^{-1/2}(\mathbf{y}-\boldsymbol{\mu}))\}. \qquad (3.10)
\end{aligned}
$$

In this case the contaminated skew-normal distribution reduces to the skew-normal distribution when $\gamma = 1$. Hence, the mean vector and the covariance matrix are given, respectively, by

$$
E[\mathbf{Y}] = \boldsymbol{\mu} + \sqrt{\frac{2}{\pi}}(\frac{\nu}{\gamma^{1/2}} + 1 - \nu)\mathbf{\Sigma}^{1/2}\boldsymbol{\delta}
$$

and

$$Var[\mathbf{Y}] = (\frac{\nu}{\gamma} + 1 - \nu)\mathbf{\Sigma} - \frac{2}{\pi}(\frac{\nu}{\gamma^{1/2}} + 1 - \nu)^2 \mathbf{\Sigma}^{1/2}\boldsymbol{\delta}\boldsymbol{\delta}^\top \mathbf{\Sigma}^{1/2}.$$

From (3.10) we derive the following results.

Proposition 3.17. *For* $\mathbf{Y} \sim SCN_p(\boldsymbol{\mu}, \mathbf{\Sigma}, \boldsymbol{\lambda}, \nu, \gamma)$ *we get*

$$E[U^r|\mathbf{y}] = \frac{2}{f(\mathbf{y})}[\nu\gamma^r \phi_p(\mathbf{y}|\boldsymbol{\mu}, \gamma^{-1}\mathbf{\Sigma})\Phi_1(\gamma^{1/2}A) + (1-\nu)\phi_p(\mathbf{y}|\boldsymbol{\mu}, \mathbf{\Sigma})\Phi_1(A)]$$

and

$$E[U^r W_{\Phi_1}(U^{1/2}A)] = \frac{2}{f(\mathbf{y})}[\nu\gamma^r \phi_p(\mathbf{y}|\boldsymbol{\mu}, \gamma^{-1}\mathbf{\Sigma})\phi_1(\gamma^{1/2}A) + (1-\nu)\phi_p(\mathbf{y}|\boldsymbol{\mu}, \mathbf{\Sigma})\phi_1(A)].$$

□

Corollary 3.18. *For* $\mathbf{Y} \sim SCN_p(\boldsymbol{\mu}, \mathbf{\Sigma}, \boldsymbol{\lambda}, \nu, \gamma)$ *we get*

$$\begin{aligned} E[\mathbf{Y}_2|\mathbf{y}_1] \quad = \quad & \boldsymbol{\mu}_{2.1} + \frac{2\mathbf{\Sigma}_{22.1}\boldsymbol{\upsilon}_2}{f(\mathbf{y}_1)\sqrt{1+\boldsymbol{\upsilon}_2^\top \mathbf{\Sigma}_{22.1}\boldsymbol{\upsilon}_2}} \times \\ & \Big[\nu\gamma^{-1/2}\phi_{p_1}(\mathbf{y}_1|\boldsymbol{\mu}_1, \gamma^{-1}\mathbf{\Sigma}_{11})\phi_1(\gamma^{1/2}\widetilde{\boldsymbol{\upsilon}}^\top(\mathbf{y}_1 - \boldsymbol{\mu}_1)) + \\ & (1-\nu)\phi_{p_1}(\mathbf{y}_1|\boldsymbol{\mu}_1, \mathbf{\Sigma}_{11})\phi_1(\widetilde{\boldsymbol{\upsilon}}^\top(\mathbf{y}_1 - \boldsymbol{\mu}_1))\Big], \end{aligned}$$

where $d_{y_1} = (\mathbf{y}_1 - \boldsymbol{\mu}_1)^\top \mathbf{\Sigma}_{11}^{-1}(\mathbf{y}_1 - \boldsymbol{\mu}_1)$. □

Remark 3.19. The stochastic representation given by Equation (2.2) can be used to obtain the slash Student. Let U_1 (with probability density function as in (2.6)), $U_2 \sim Gamma(\nu/2, \nu/2)$ (with $\nu > 0$), and $\mathbf{X} \sim N_p(\mathbf{0}, \mathbf{\Sigma})$ be all independently distributed. Then

$$\mathbf{Y} \stackrel{d}{=} \boldsymbol{\mu} + U_1^{-1/2} U_2^{-1/2}\mathbf{X} \tag{3.11}$$

has a slash student distribution (Tang and Peng (2006) [29]). The proof follows from the formula

$$\mathbf{T} = U_2^{-1/2}\mathbf{X} \sim t_p(\boldsymbol{\mu}, \mathbf{\Sigma}, \nu).$$

Remark 3.20. If $\mathbf{X} \sim \mathrm{SN}_p(\mathbf{0}, \boldsymbol{\Sigma}, \boldsymbol{\lambda})$, then $\mathbf{Y}$ in (3.11) has a skew-slash student distribution as shown by Tang and Peng (2006) [29]. Obviously, many other distributions can be constructed by choosing appropriate probability density functions (i.e, $h(.;\boldsymbol{\nu})$) for U_1 and U_2.

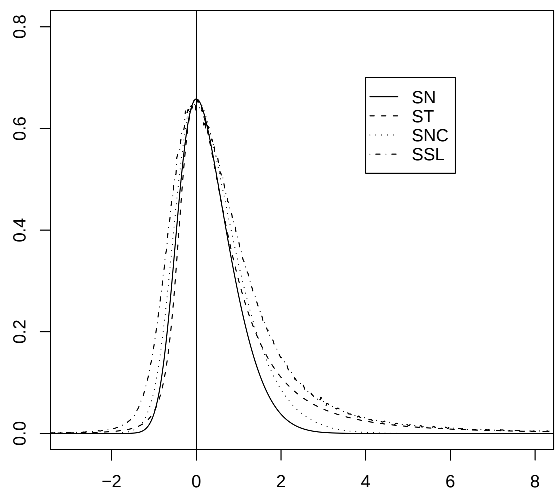

Figure 1: Density curves of the univariate skew-normal, skew-t, skew-slash and contaminated skew-normal distributions.

In Figure 1 we drew the density of the standard distribution $SN_1(3)$ together with the standard densities of the distributions $ST_1(3, 2)$, SSL_1 $(3, 1)$ and $SNC_1(3, 0.5, 0.5)$. They are rescaled to take the same value at the origin. The four densities are positively skewed. The skew-slash and skew-t distributions have much heavier tails than the skew-normal distribution. Actually, the skew-slash and the skew-t distributions do not have finite means nor variances. Figure 2 depicts some contours of the densities associated with the standard bivariate skew-normal distribution $SN_2(\boldsymbol{\lambda})$, the standard bivariate skew-t distribution $ST_2(\boldsymbol{\lambda}, 2)$, the standard bivariate skew-slash distribution $SSL_2(\boldsymbol{\lambda}, 1)$, and the standard bivariate contaminated skew-normal distribution $SCN_2(\boldsymbol{\lambda}, 0.5,\ 0.5)$, with $\boldsymbol{\lambda} = (2, 1)^\top$ for all the distributions. Note that these contours are not elliptical and they can be strongly asymmetric depending on the choice of the parameters.

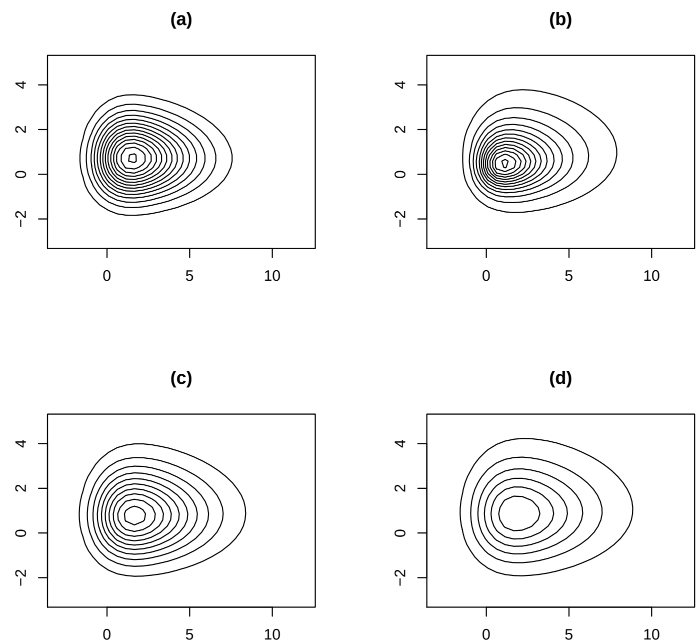

Figure 2: Contour plot of some elements of the standard bivariate SNI family. (a) $SN_2(\boldsymbol{\lambda})$, (b) $ST_2(\boldsymbol{\lambda}, 2)$, (c) $SCN_2(\boldsymbol{\lambda}, 0.5, 0.5)$, and (d) $SSL_2(\boldsymbol{\lambda}, 1)$, where $\boldsymbol{\lambda} = (2, 1)^\top$.

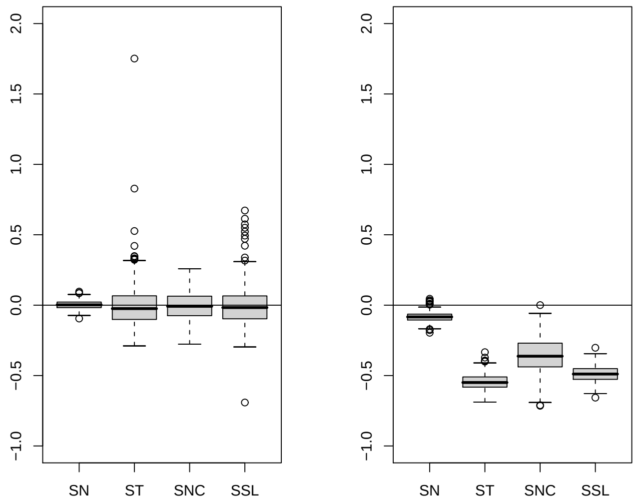

Figure 3: Box-plots of the sample mean (left panel) and sample median (right panel) on 500 samples of size n=100 from the four standardized distributions: $SN_1(3)$, $ST_1(3, 2)$, $SSL_1(3, 1)$, and $SNC_1(3, 0.9, 0.1)$. The respective means are adjusted to zero.

3.2 A Simulation study

To illustrate the usefulness of the SNI distribution, we perform a small simulation in order to study the behavior of two location estimators, the sample mean and the sample median under four different standard univariate settings. In particular, we consider a standard skew-normal $SN_1(3)$, a skew-t $ST_1(3, 2)$, a skew-slash $SSL_1(3, 1)$, and a contaminated skew-normal $SCN_1(3, 0.9, 0.1)$. The mean of all the asymmetric distributions is adjusted to zero, so that all four distributions are comparable. Thus, this setting represents four distributions with the same mean, but with different tail behaviors and skewness. Note that the skew-slash and skew-t will have infinite variance when $\nu = 1$, $\nu = 2$, respectively. We simulate 500 samples of size $n = 100$ for them. For each sample, we compute the sample mean and the sample median and report the box-plot for each distribution in Figure 3. In the left panel all box-plots of the estimated means are centered around zero but have larger variability for the heavy-tailed distributions (skew-t and skew-slash). In the right panel the box-plots of the estimated medians have a slightly larger variability for the skew-normal and skew-contaminated normal, and have a much smaller variability for skew-t and skew-slash distributions. This indicates that the median is a robust estimator of location at asymmetric light-tailed distributions. On the other hand, the median estimator becomes biased as soon as unexpected skewness and heavy-tailed arise in the underlying distribution.

4. Maximum likelihood estimation

This section presents an EM-algorithm to perform maximum likelihood estimation for two multivariate SNI models of considerable practical interest.

4.1 Multivariate SNI responses

Suppose that we have observations on n independent individuals, $\wedge_1, \ldots,$ $\wedge_n$, where $\wedge_i \sim SNI_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}; H)$, $i = 1, \ldots, n.$. The parameter vector is $\boldsymbol{\theta} = (\boldsymbol{\mu}^\top, \boldsymbol{\gamma}^\top, \boldsymbol{\lambda}^\top)^\top$, where $\boldsymbol{\gamma}$ denotes a minimal set of parameters such that $\boldsymbol{\Sigma}(\boldsymbol{\gamma})$ is well defined (e.g., the upper triangular elements of $\boldsymbol{\Sigma}$ in the unstructured case).

In what follows, we illustrate the implementation of likelihood inference for the multivariate SNI via the EM-algorithm. The EM-algorithm is a popular iterative algorithm for maximum likelihood estimation for models with incomplete data. More specifically, let $\mathbf{y}$ denote the observed data and $\mathbf{s}$ the missing one. The complete data $\mathbf{y}_c = (\mathbf{y}, \mathbf{s})$ is $\mathbf{y}$ augmented with $\mathbf{s}$. We denote by $\ell_c(\boldsymbol{\theta}|\mathbf{y}_c)$ (with $\boldsymbol{\theta} \in \boldsymbol{\Theta}$) the complete-data log-likelihood function and by $Q(\boldsymbol{\theta}|\widehat{\boldsymbol{\theta}}) = E[\ell_c(\boldsymbol{\theta}|\mathbf{y}_c)|\mathbf{y}, \widehat{\boldsymbol{\theta}}]$ the expected complete-data log-likelihood function. Each iteration of the EM-algorithm involves two steps, an E-step and a M-step, defined as follows.

- **E-step**: Compute $Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(r)})$ as a function of $\boldsymbol{\theta}$.
- **M-step**: Find $\boldsymbol{\theta}^{(r+1)}$ such that $Q(\boldsymbol{\theta}^{(r+1)}|\boldsymbol{\theta}^{(r)}) = \max_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(r)})$.

By using (3.3), the setup defined above can be written as

$$\mathbf{Y}_i|T_i = t_i, U_i = u_i, \quad \overset{\text{ind}}{\sim} \; N_p(\boldsymbol{\mu} + t_i\boldsymbol{\Sigma}^{1/2}\boldsymbol{\delta}, u_i^{-1}\boldsymbol{\Sigma}^{1/2}(\mathbf{I}_p - \boldsymbol{\delta}\boldsymbol{\delta}^\top)\boldsymbol{\Sigma}^{1/2}) \tag{4.1}$$

$$T_i|U_i = u_i \quad \overset{\text{iid}}{\sim} \; HN_1(0, \frac{1}{u_i}) \tag{4.2}$$

$$U_i \quad \overset{\text{ind}}{\sim} \; h(u_i; \boldsymbol{\nu}), \tag{4.3}$$

all independent, where $HN_1(0,1)$ denotes the univariate standard half-normal distribution (see $|X_0| = |T_0|$ in Equation (1.2) or Johnson et al. (1994) [18]). We assume that the parameter vector $\boldsymbol{\nu}$ is known. In practice, the optimum value of $\boldsymbol{\nu}$ can be determined using the profile likelihood and the Schwarz information criterion (see Lange et al. (1989) [23]).

Let $\mathbf{y} = (\mathbf{y}_1^\top, \ldots, \mathbf{y}_n^\top)^\top$, $\mathbf{u} = (u_1, \ldots, u_n)^\top$, and $\mathbf{t} = (t_1, \ldots, t_n)^\top$. Then, under the hierarchical representation (4.1)–(4.2), with $\boldsymbol{\Delta} = \boldsymbol{\Sigma}^{1/2}\boldsymbol{\delta}$ and $\boldsymbol{\Gamma} = \boldsymbol{\Sigma} - \boldsymbol{\Delta}\boldsymbol{\Delta}^\top$, it follows that the complete log-likelihood function associated with $\mathbf{y}_c = (\mathbf{y}^\top, \mathbf{u}^\top, \mathbf{t}^\top)^\top$ is given by

$$\ell_c(\boldsymbol{\theta}|\mathbf{y}_c) = c - \frac{n}{2}\log|\boldsymbol{\Gamma}| - \frac{1}{2}\sum_{i=1}^{n} u_i(\mathbf{y}_i - \boldsymbol{\mu} - \boldsymbol{\Delta}t_i)^\top\boldsymbol{\Gamma}^{-1}(\mathbf{y}_i - \boldsymbol{\mu} - \boldsymbol{\Delta}t_i),$$

where c is a constant independent of the parameter vector $\boldsymbol{\theta}$. By letting $\widehat{u}_i = \mathrm{E}[U_i|\boldsymbol{\theta} = \widehat{\boldsymbol{\theta}}, \mathbf{y}_i]$, $\widehat{ut_i} = \mathrm{E}[U_iT_i|\boldsymbol{\theta} = \widehat{\boldsymbol{\theta}}, \mathbf{y}_i]$, $\widehat{ut_i^2} = \mathrm{E}[U_iT_i^2|\boldsymbol{\theta} = \widehat{\boldsymbol{\theta}}, \mathbf{y}_i]$, and using known properties of conditional expectation we obtain

$$\widehat{ut_i} = \widehat{u}_i\widehat{\mu}_{Ti} + \widehat{M}_{T_i}\widehat{\tau}_i \tag{4.4}$$

$$\widehat{ut_i^2} = \widehat{u}_i\widehat{\mu}_{T_i}^2 + \widehat{M}_{T_i}^2 + \widehat{M}_{T_i}\widehat{\mu}_{T_i}\widehat{\tau}_i, \tag{4.5}$$

with $\widehat{\tau}_i = E[U_i^{1/2} W_{\Phi_1}(\frac{U_i^{1/2}\widehat{\mu}_{T_i}}{\widehat{M}_{T_i}})|\widehat{\boldsymbol{\theta}}, \mathbf{y}_i]$, $W_{\Phi_1}(x) = \phi_1(x)/\Phi_1(x)$, $\widehat{M}_T^2 = 1/(1+\widehat{\boldsymbol{\Delta}}^\top \widehat{\boldsymbol{\Gamma}}^{-1}\widehat{\boldsymbol{\Delta}})$ and $\widehat{\mu}_{T_i} = \widehat{M}_{T_i}^2 \widehat{\boldsymbol{\Delta}}^\top \widehat{\boldsymbol{\Gamma}}^{-1}(\mathbf{y}_i - \boldsymbol{\mu})$, $i = 1, \ldots, n$.

As we have $\frac{\mu_{T_i}}{M_{T_i}} = \boldsymbol{\lambda}^\top \boldsymbol{\Sigma}^{-1/2}(\mathbf{y}_i - \boldsymbol{\mu})$, the conditional expectations given in (4.4)–(4.5), specifically $\widehat{u}_i$ and $\widehat{\tau}_i$, can be easily derived from the results of Section 3.1. Thus, at least for the skew-t and skew-contaminated normal distributions of the SNI class we have closed-form expressions for the quantities $\widehat{u}_i$ and $\widehat{\tau}_i$. For the skew-slash case, Monte Carlo integration may be employed, which yield the so-called MC-EM algorithm.

It follows, after some simple algebra involving (4.4)–(4.5), that the conditional expectation of the complete log-likelihood function has the form

$$\begin{aligned} Q(\boldsymbol{\theta}|\widehat{\boldsymbol{\theta}}) &= \mathrm{E}[\ell_c(\boldsymbol{\theta}|\mathbf{y}_c)|\mathbf{y}, \widehat{\boldsymbol{\theta}}] \\ &= c - \frac{n}{2}\log|\boldsymbol{\Gamma}| - \frac{1}{2}\sum_{i=1}^n \widehat{u}_i(\mathbf{y}_i - \boldsymbol{\mu})^\top \boldsymbol{\Gamma}^{-1}(\mathbf{y}_i - \boldsymbol{\mu}) \\ &\quad + \sum_{i=1}^n \widehat{ut}_i(\mathbf{y}_i - \boldsymbol{\mu})^\top \boldsymbol{\Gamma}^{-1}\boldsymbol{\Delta} - \frac{1}{2}\sum_{i=1}^n \widehat{ut^2}_i \boldsymbol{\Delta}^\top \boldsymbol{\Gamma}^{-1}\boldsymbol{\Delta}. \end{aligned}$$

We then have the following EM-type algorithm.

E-step: Given $\boldsymbol{\theta} = \widehat{\boldsymbol{\theta}}$, compute $\widehat{ut_i^2}$, $\widehat{ut_i}$ and $\widehat{u}_i$ using (4.4)–(4.5).
M-step: Update $\widehat{\boldsymbol{\theta}}$ by maximizing $Q(\boldsymbol{\theta}|\widehat{\boldsymbol{\theta}})$ over $\boldsymbol{\theta}$, which leads us to the following closed-form expressions

$$\begin{aligned} \widehat{\boldsymbol{\mu}} &= \sum_{i=1}^n (\widehat{u}_i \mathbf{y}_i - \widehat{ut_i}\boldsymbol{\Delta})/(\sum_{i=1}^n \widehat{u}_i), \qquad (4.6) \\ \widehat{\boldsymbol{\Gamma}} &= \frac{1}{n}\sum_{i=1}^n \left[\widehat{u}_i(\mathbf{y}_i - \boldsymbol{\mu})(\mathbf{y}_i - \boldsymbol{\mu})^\top - 2\widehat{ut}_i \boldsymbol{\Delta}(\mathbf{y}_i - \boldsymbol{\mu})^\top + \widehat{ut_i^2}\boldsymbol{\Delta}\boldsymbol{\Delta}^\top\right], \\ \widehat{\boldsymbol{\Delta}} &= \frac{\sum_{i=1}^n \widehat{ut}_i(\mathbf{y}_i - \boldsymbol{\mu})}{\sum_{i=1}^n \widehat{ut_i^2}}. \end{aligned}$$

The skewness parameter vector and the unstructured scale matrix can be estimated by the equalities $\widehat{\boldsymbol{\Sigma}} = \widehat{\boldsymbol{\Gamma}} + \widehat{\boldsymbol{\Delta}}\widehat{\boldsymbol{\Delta}}^T$ and $\widehat{\boldsymbol{\lambda}} = \widehat{\boldsymbol{\Sigma}}^{-1/2}\widehat{\boldsymbol{\Delta}}/(1-\widehat{\boldsymbol{\Delta}}^\top\widehat{\boldsymbol{\Sigma}}^{-1}\widehat{\boldsymbol{\Delta}})^{1/2}$. It is clear that when $\boldsymbol{\lambda} = \mathbf{0}$ (or when $\boldsymbol{\Delta} = \mathbf{0}$), the M-step equations reduce to the equations obtained assuming normal/independent distribution. This algorithm clearly generalizes results found in Lachos et al. (2007) [20, Section 2] by taking $U_i = 1$, $i = 1, \ldots, n$. Useful starting values required to implement this algorithm are those obtained under the normality assumption, with the starting values for the skewness parameter vector set equal to $\mathbf{0}$. However, in order to ensure that the true ML estimate is identified, we recommend running the EM algorithm using a range of different starting values. The log-likelihood function for $\boldsymbol{\theta} = (\boldsymbol{\mu}^\top, \boldsymbol{\gamma}^\top, \boldsymbol{\lambda}^\top)^\top$, given the observed sample $\mathbf{y} = (\mathbf{y}_1^\top, \ldots, \mathbf{y}_n^\top)^\top$, is of the form

$$\ell(\boldsymbol{\theta}) = \sum_{i=1}^{n} \ell_i(\boldsymbol{\theta}), \tag{4.7}$$

where $\ell_i(\boldsymbol{\theta}) = \log 2 - \frac{p}{2} log 2\pi - \frac{1}{2}\log|\boldsymbol{\Sigma}| + \log K_i$, with

$$K_i = K_i(\boldsymbol{\theta}) = \int_0^\infty u_i^{p/2} \exp\{-\frac{1}{2}u_i d_i\}\Phi_1(u_i^{1/2}A_i)dH(u_i),$$

where $d_i = (\mathbf{y}_i - \boldsymbol{\mu})^\top\boldsymbol{\Sigma}^{-1}(\mathbf{y}_i - \boldsymbol{\mu})$ and $A_i = \boldsymbol{\lambda}^\top\boldsymbol{\Sigma}^{-1}(\mathbf{y}_i - \boldsymbol{\mu})$. Explicit expressions for the observed information matrix can be derived from the results presented in Section 5.

4.2 Multivariate measurement error model

In this section we further apply the SNI distribution to a multivariate measurement error model. Let n be the sample size, X_i the observed value in unit i of the covariate, y_{ij} the j-th observed response in unit i, and x_i the unobserved (true) covariate value for unit i; here i ranges from 1 to n, and j from 1 to r. Relating these variables we postulate as working model (see also Barnett (1969) [8] and Shyr and Gleser (1986) [28]) the equations

$$X_i = x_i + u_i, \tag{4.8}$$

and

$$\mathbf{Z}_i = \boldsymbol{\alpha} + \boldsymbol{\beta} x_i + \mathbf{e}_i, \tag{4.9}$$

where $\mathbf{Z}_i = (z_{i1}, \ldots, z_{ir})^\top$ is the vector of responses for the i-th experimental unit, $\mathbf{e}_i = (e_{i1}, \ldots, e_{ir})^\top$ is a random vector of measurement errors of dimension r, and $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_r)^\top, \boldsymbol{\beta} = (\beta_1, ..., \beta_r)^\top$ are parameter vectors of dimension r.

Set $\boldsymbol{\epsilon}_i = (u_i, \mathbf{e}_i^\top)^\top$ and $\mathbf{Y}_i = (X_i, \mathbf{Z}_i^\top)^\top$. Then the model defined by Equations (4.8)–(4.9) can be rewritten as

$$\mathbf{Y}_i = \mathbf{a} + \mathbf{b} x_i + \boldsymbol{\epsilon}_i, \tag{4.10}$$

where $\mathbf{a} = (0, \boldsymbol{\alpha}^\top)^\top$ and $\mathbf{b} = (1, \boldsymbol{\beta}^\top)^\top$ are $p \times 1$ vectors, with $p = r + 1$. We assume

$$\begin{pmatrix} x_i \\ \boldsymbol{\epsilon}_i \end{pmatrix} \overset{\text{iid}}{\sim} SNI_{p+1}\left(\begin{pmatrix} \mu_x \\ \mathbf{0} \end{pmatrix}, D(\phi_x, \boldsymbol{\phi}), \begin{pmatrix} \lambda_x \\ \mathbf{0} \end{pmatrix}; H \right), \tag{4.11}$$

where $D(\phi_x, \boldsymbol{\phi}) = diag(\phi_x, \phi_1, \ldots, \phi_p)^\top$, with $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_p)$, called **structural SNI-MMEM**. From (2.2), this formulation implies

$$\begin{pmatrix} x_i \\ \boldsymbol{\epsilon}_i \end{pmatrix} | U_i = u_i \sim SN_{p+1}\left(\begin{pmatrix} \mu_x \\ \mathbf{0} \end{pmatrix}, u_i^{-1} D(\phi_x, \boldsymbol{\phi}), \begin{pmatrix} \lambda_x \\ \mathbf{0} \end{pmatrix} \right), \tag{4.12}$$

$$U_i \sim h(u_i; \boldsymbol{\nu}). \tag{4.13}$$

From Corollary 3.5, marginally we get

$$\boldsymbol{\epsilon}_i \overset{\text{ind}}{\sim} NI_{m+1}(\mathbf{0}, \mathrm{D}(\boldsymbol{\phi}); H) \quad \text{and} \quad x_i \overset{\text{ind}}{\sim} SNI_1(\mu_x, \phi_x, \lambda_x; H). \tag{4.14}$$

The asymmetric parameter λ_x incorporates asymmetry in the latent variable x_i and consequently in the observed quantities $\mathbf{Y}_i$, which will be shown to have marginally multivariate SNI distributions. If $\lambda_x = 0$, then the asymmetric model reduces to the symmetric MMEM considering NI distributions. Under (4.11), it follows from (1.2) that the regression setup defined in (4.8)–(4.11) can be written hierarchically as

$$\begin{aligned} \mathbf{Y}_i \mid x_i, U_i = u_i &\stackrel{\text{ind}}{\sim} N_p(\mathbf{a} + \mathbf{b}x_i, u_i^{-1} D(\boldsymbol{\phi})), && (4.15)\\ x_i \mid T_i = t_i, U_i = u_i &\stackrel{\text{ind}}{\sim} N_1(\mu_x + \phi_x^{1/2}\delta_x t_i, u_i^{-1}\phi_x(1-\delta_x^2)), && (4.16)\\ T_i &\stackrel{\text{iid}}{\sim} HN_1(0, \frac{1}{u_i}), && (4.17)\\ U_i &\stackrel{\text{iid}}{\sim} h(u_i; \boldsymbol{\nu}), && (4.18) \end{aligned}$$

all independent, where $\delta_x = \lambda_x/(1+\lambda_x^2)^{1/2}$. As in Lange et al. (1989) [23], we assumed $\boldsymbol{\nu}$ to be known. Classical inference on the parameter vector $\boldsymbol{\theta} = (\boldsymbol{\alpha}^\top, \boldsymbol{\beta}^\top, \boldsymbol{\phi}^\top, \mu_x, \phi_x, \lambda_x)^\top$ in this type of model is based on the marginal distribution for $\mathbf{Y}_i$ given in the following proposition (see Bolfarine and Galea-Rojas (1995) [10]).

Proposition 4.1. *For the structural SNI-MMEM model (4.8)–(4.11), the marginal distribution of $\mathbf{Y}_i$ is given by*

$$f_{\mathbf{Y}_i}(\mathbf{y}_i|\boldsymbol{\theta}) = 2\int_0^\infty \phi_p(\mathbf{y}_i|\boldsymbol{\mu}, u_i^{-1}\boldsymbol{\Sigma})\Phi_1(u_i^{1/2}\bar{\boldsymbol{\lambda}}_x^\top \boldsymbol{\Sigma}^{-1/2}(\mathbf{y}_i - \boldsymbol{\mu}))dH(u_i) \qquad (4.19)$$

(i.e., by $\mathbf{Y}_i \stackrel{\text{iid}}{\sim} SNI_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \bar{\boldsymbol{\lambda}}_x; H)$, with $\boldsymbol{\mu} = \mathbf{a} + \mathbf{b}\mu_x$, $\boldsymbol{\Sigma} = \phi_x \mathbf{b}\mathbf{b}^\top + D(\boldsymbol{\phi})$, and $\bar{\boldsymbol{\lambda}}_x = \frac{\lambda_x\phi_x\boldsymbol{\Sigma}^{-1/2}\mathbf{b}}{\sqrt{\phi_x + \lambda_x^2\Lambda_x}}$; here $\Lambda_x = ({\phi_x}^{-1} + \mathbf{b}^\top D^{-1}(\boldsymbol{\phi})\mathbf{b})^{-1}$).

Proof. The proof is a direct consequence of Proposition 3.8 after some algebraic manipulations. □

It follows that the log-likelihood function for $\boldsymbol{\theta}$, given the observed sample $\mathbf{y} = (\mathbf{y}_1^\top, \ldots, \mathbf{y}_n^\top)^\top$, is

$$\ell(\boldsymbol{\theta}) = \sum_{i=1}^n \ell_i(\boldsymbol{\theta}), \qquad (4.20)$$

where $\ell_i(\boldsymbol{\theta}) = \log 2 - \frac{p}{2} log 2\pi - \frac{1}{2}\log|\boldsymbol{\Sigma}| + \log K_i$, with

$$K_i = K_i(\boldsymbol{\theta}) = \int_0^\infty u_i^{p/2}\exp\{-\frac{1}{2}u_i d_i\}\Phi_1(u_i^{1/2}A_i)\, dH(u_i),$$

and $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$, $\bar{\boldsymbol{\lambda}}_x$ as in Proposition 4.1. Here $d_i = (\mathbf{y}_i - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{y}_i - \boldsymbol{\mu})$ and $A_i = \bar{\boldsymbol{\lambda}}_x^\top \boldsymbol{\Sigma}^{-1/2}(\mathbf{y}_i - \boldsymbol{\mu}) = A_x a_i$ hold with $A_x = \frac{\lambda_x \Lambda_x}{\sqrt{\phi_x + \lambda_x^2 \Lambda_x}}$ and $a_i = (\mathbf{y}_i - \boldsymbol{\mu})^\top D^{-1}(\boldsymbol{\phi})\mathbf{b}$.

The ML estimators of the parameters in the model (4.10)–(4.11) can be found by direct maximization of the log-likelihood (4.20) which can be computed numerically using the *optim* routine in platform R or *fmincon* in Matlab. An oft-voiced complaint about these methods is that they may not converge unless good starting values are provided. The EM algorithm —which takes advantage of being insensitive to the stating values— is a tool that requires the construction of unobserved data, and has been well developed and has become a broadly applicable approach to the iterative computation of ML estimates. Thus, if we let $\mathbf{y} = (\mathbf{y}_1^\top, \ldots, \mathbf{y}_n^\top)^\top$, $\mathbf{x} = (x_1, \ldots, x_n)^\top$, $\mathbf{u} = (u_1, \ldots, u_n)^\top$, $\mathbf{t} = (t_1, \ldots, t_n)^\top$, $\nu_x^2 = \phi_x(1 - \delta_x^2)$ and $\tau_x = \phi_x^{1/2}\delta_x$, it follows that the complete log-likelihood function associated with $(\mathbf{y}, \mathbf{x}, \mathbf{t}, \mathbf{u})$ is given by

$$\ell_c(\boldsymbol{\theta}|\mathbf{y}, \mathbf{x}, \mathbf{t}, \mathbf{u}) \quad \propto \quad -\frac{n}{2}\log(|D(\boldsymbol{\phi})|) \tag{4.21}$$

$$-\frac{1}{2}\sum_{i=1}^{n} u_i(\mathbf{y}_i - \mathbf{a} - \mathbf{b}x_i)^\top D^{-1}(\boldsymbol{\phi})(\mathbf{y}_i - \mathbf{a} - \mathbf{b}x_i) \tag{4.22}$$

$$-\frac{n}{2}\log(\nu_x^2) - \frac{1}{2\nu_x^2}\sum_{i=1}^{n} u_i(x_i - \mu_x - \tau_x t_i)^2. \tag{4.23}$$

Letting $\widehat{u}_i = E[U_i|\widehat{\boldsymbol{\theta}}, \mathbf{y}_i]$, $\widehat{ut}_i = E[U_i t_i|\widehat{\boldsymbol{\theta}}, \mathbf{y}_i]$, $\widehat{t_i^2} = E[t_i^2|\widehat{\boldsymbol{\theta}}, \mathbf{y}_i]$, $\widehat{ux}_i = E[U_i x_i|\widehat{\boldsymbol{\theta}}, \mathbf{y}_i]$, $\widehat{ux_i^2} = E[U_i x_i^2|\widehat{\boldsymbol{\theta}}, \mathbf{y}_i]$, and $\widehat{utx}_i = E[U_i t_i x_i|\widehat{\boldsymbol{\theta}}, \mathbf{y}_i]$ we obtain

$$\widehat{ut}_i \quad = \quad \widehat{u}_i\widehat{\mu}_{Ti} + \widehat{M}_T E[U_i^{1/2} W_{\Phi_1}(\frac{U_i^{1/2}\widehat{\mu}_{T_i}}{\widehat{M}_T})|\widehat{\boldsymbol{\theta}}, \mathbf{y}_i], \tag{4.24}$$

$$\widehat{t_i^2} \quad = \quad \widehat{u}_i\widehat{\mu}_{T_i}^2 + \widehat{M}_T^2 + \widehat{M}_T\widehat{\mu}_{T_i} E[U_i^{1/2} W_{\Phi_1}(\frac{U_i^{1/2}\widehat{\mu}_{T_i}}{\widehat{M}_T})|\widehat{\boldsymbol{\theta}}, \mathbf{y}_i],$$

$$\widehat{ux}_i \quad = \quad \widehat{r}_i\ \widehat{u}_i + \widehat{s}\ \widehat{ut}_i, \quad \widehat{ux_i^2} = \widehat{T_x}^2 + \widehat{r}_i^2\ \widehat{u}_i + 2\widehat{r}_i\widehat{s}\ \widehat{ut}_i\ + \widehat{s}^2\ \widehat{t_i^2},$$

$$\widehat{utx}_i \quad = \quad \widehat{r}_i\ \widehat{ut}_i + \widehat{s}\ \widehat{t_i^2},$$

where $\widehat{M_T}^2 = [1 + \widehat{\tau_x}^2\widehat{\mathbf{b}}^\top(D(\widehat{\boldsymbol{\phi}}) + \widehat{\nu_x}^2\widehat{\mathbf{b}}\widehat{\mathbf{b}}^\top)^{-1}\widehat{\mathbf{b}}]^{-1}$, $\widehat{\mu}_{T_i} = \widehat{\tau_x}\widehat{M}_T^2\widehat{\mathbf{b}}^\top (D(\widehat{\boldsymbol{\phi}}) + \widehat{\nu_x}^2\widehat{\mathbf{b}}\widehat{\mathbf{b}}^\top)^{-1}(\mathbf{y}_i - \widehat{\mathbf{a}} - \widehat{\mathbf{b}}\widehat{\mu}_x)$, $\widehat{T_x}^2 = \widehat{\nu_x}^2[1 + \widehat{\nu_x}^2\widehat{\mathbf{b}}^\top D^{-1}(\widehat{\boldsymbol{\phi}})\widehat{\mathbf{b}}]^{-1}$,

$\widehat{r}_i = \widehat{\mu}_x + \widehat{T_x}^2 \widehat{\mathbf{b}}^\top D^{-1}(\widehat{\boldsymbol{\phi}})(\mathbf{y}_i - \widehat{\mathbf{a}} - \widehat{\mathbf{b}}\widehat{\mu}_x)$, and $\widehat{s} = \widehat{\tau}_x(1 - \widehat{T_x}^2 \widehat{\mathbf{b}}^\top D^{-1}(\widehat{\boldsymbol{\phi}})\widehat{\mathbf{b}})$. A closed-form expression for $E[U_i^{1/2} W_{\Phi_1}(\frac{U_i^{1/2}\widehat{\mu}_{T_i}}{\widehat{M}_T})|\widehat{\boldsymbol{\theta}}, \mathbf{y}_i]$ can be found from the results given in Section 3.1.

In this way we have the following EM type algorithm.

E-step: Given $\boldsymbol{\theta} = \widehat{\boldsymbol{\theta}}$, compute $\widehat{u}_i$, $\widehat{t_i^2}$, $\widehat{ut}_i$, $\widehat{ux}_i$, $\widehat{ux_i^2}$, and $\widehat{utx}_i$ using (4.24).

M-step: Update $\widehat{\boldsymbol{\theta}}$ by maximizing $E[\ell_c(\boldsymbol{\theta}|\mathbf{y}, \mathbf{x}, \mathbf{t}, \mathbf{u})|\mathbf{y}, \widehat{\boldsymbol{\theta}}]$ over $\boldsymbol{\theta}$; which leads to

$$\begin{aligned}
\widehat{\boldsymbol{\alpha}} &= \overline{\mathbf{z}}_u - \overline{x}_u \widehat{\boldsymbol{\beta}}, \\
\widehat{\boldsymbol{\beta}} &= \frac{\sum_{i=1}^n \widehat{ux}_i(\mathbf{z}_i - \overline{\mathbf{z}}_u)}{\sum_{i=1}^n \widehat{ux_i^2} - n\overline{\widehat{u}}\overline{x}_u^2}, \\
\widehat{\phi}_1 &= \frac{1}{n}\sum_{i=1}^n (\widehat{u}_i X_i^2 - 2\widehat{ux}_i X_i + \widehat{ux_i^2}), \\
\widehat{\phi}_{j+1} &= \frac{1}{n}\sum_{i=1}^n (\widehat{u}_i z_{ij}^2 + \widehat{u}_i \alpha_j^2 + \beta_j^2 \widehat{ux_i^2} - 2\widehat{u}_i \alpha_j z_{ij} - \\
&\quad 2y_{ij}\beta_j \widehat{ux}_i + 2\alpha_j \beta_j \widehat{ux}_i), \qquad j = 1, \ldots, r, \\
\widehat{\mu}_x &= \overline{x}_u - \widehat{\tau}_x \overline{t}_u, \quad \widehat{\nu}_x^2 = \frac{1}{n}\sum_{i=1}^n (\widehat{ux_i^2} - \widehat{\mu}_x \widehat{ux}_i) - \widehat{\tau}_x \frac{1}{n}\sum_{i=1}^n \widehat{utx}_i, \\
\widehat{\tau}_x &= \frac{\sum_{i=1}^n (\widehat{utx}_i - \overline{x}_u \widehat{ut}_i)}{\sum_{i=1}^n (\widehat{t_i^2} - \overline{t}_u \widehat{ut}_i)},
\end{aligned}$$

where $\overline{\mathbf{z}}_u = \frac{\sum_{i=1}^n \widehat{u}_i \mathbf{z}_i}{\sum_{i=1}^n \widehat{u}_i}$, $\overline{x}_u = \frac{\sum_{i=1}^n \widehat{ux}_i}{\sum_{i=1}^n \widehat{u}_i}$, $\overline{t}_u = \frac{\sum_{i=1}^n \widehat{ut}_i}{\sum_{i=1}^n \widehat{u}_i}$, and $\overline{\widehat{u}} = \frac{1}{n}\sum_{i=1}^n \widehat{u}_i$.

When $U_i = 1$, the M-step equations reduce to the equations obtained by Lachos et al. (2005) [21] under the skew-normal distribution. When $\lambda_x = 0$ (or when $\tau_x = 0$), the M-step equations become the equations by Bolfarine and Galea-Rojas (1995) [10]. Moreover, when $U \sim Gamma(\nu/2, \nu/2)$ and $\lambda_x = 0$, the M-step reduces to equations obtained by Bolfarine and Galea-Rojas (1996) [11]. The shape and scale parameters of the latent variable x can be estimated by noting the equalities $\tau_x/\nu_x = \lambda_x$ and $\phi_x = \tau_x^2 + \nu_x^2$.

We now consider an empirical Bayes inference for the latent variable that is useful for estimating the x_i quantities. Models (4.10) and (4.14) imply $\mathbf{Y}_i|x_i \sim NI_p(\mathbf{a}+\mathbf{b}x_i, D(\boldsymbol{\phi}); H)$ and $x_i \sim SNI_1(\mu_x, \sigma_x^2, \lambda_x; H)$. The conditional density of x_i, given $\mathbf{y}_i, u_i$, is

$$f(x_i|\mathbf{y}_i, u_i) = \phi_q(x_i|\mu_x + \Lambda_x a_i, u_i^{-1}\Lambda_x)\frac{\Phi_1(u_i^{1/2}\frac{\lambda_x(x_i-\mu_x)}{\sigma_x})}{\Phi_1(u_i^{1/2}A_i)},$$

where Λ_x and a_i, A_i are as in Proposition 4.1 and Equation (4.20), respectively. It follows from Lemma 8.1 in the appendix that we have

$$E[x_i|\mathbf{y}_i, u_i] = \mu_x + \Lambda_x a_i + u_i^{-1/2}\frac{\Lambda_x\lambda_x}{\sqrt{1+\lambda_x^2\Lambda_x}}W_{\Phi_1}(u_i^{1/2}A_i),$$

and as $E[x|\mathbf{y}] = E_U[E[x|\mathbf{y}, U]|\mathbf{y}]$ holds, we conclude that the minimum mean-square error (MSE) estimator of x_i obtained by the conditional mean of x_i, given $\mathbf{y}_i$, is

$$\widehat{x}_i = E[x_i|\mathbf{y}_i] = \mu_x + \Lambda_x a_i + \frac{\Lambda_x\lambda_x}{\sqrt{1+\lambda_x^2\Lambda_x}}E[U_i^{-1/2}\frac{\phi_1(U_i^{1/2}A_i)}{\Phi_1(U_i^{1/2}A_i)}|\mathbf{y}_i], \quad (4.25)$$

If $\mathbf{Y}_i$ has distribution $ST_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \bar{\boldsymbol{\lambda}}_x, \nu)$ or $SCN_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \bar{\boldsymbol{\lambda}}_x, \nu, \gamma)$, then we obtain closed-form expressions for the expected values given in (4.25) from the results exhibited in Section 3.1. In practice the Bayes estimator of x_i, namely $\widehat{x}_i$, can be obtained by substituting the ML estimate $\boldsymbol{\theta}$ into (4.25).

5. The observed information matrix

In this section we develop the observed information matrix in a general form. Suppose that we have observations on n independent individuals $\wedge_1, \ldots, \wedge_n$, where $\wedge_i \sim SNI_{n_i}(\boldsymbol{\mu}_i(\boldsymbol{\beta}), \boldsymbol{\Sigma}_i(\boldsymbol{\gamma}), \boldsymbol{\lambda}_i(\boldsymbol{\lambda}); H)$. Then the log-likelihood function for $\boldsymbol{\theta} = (\boldsymbol{\beta}^\top, \boldsymbol{\gamma}^\top, \boldsymbol{\lambda}^\top)^\top \in \mathbb{R}^q$, given the observed sample $\mathbf{y} = (\mathbf{y}_1^\top, \ldots, \mathbf{y}_n^\top)^\top$, is of the form

$$\ell(\boldsymbol{\theta}) = \sum_{i=1}^{n} \ell_i(\boldsymbol{\theta}), \quad (5.1)$$

where $\ell_i(\boldsymbol{\theta}) = \log 2 - \dfrac{n_i}{2} log 2\pi - \dfrac{1}{2}\log|\boldsymbol{\Sigma}_i| + \log K_i$, with

$$K_i = K_i(\boldsymbol{\theta}) = \int_0^\infty u_i^{n_i/2}\exp\{-\frac{1}{2}u_i d_i\}\Phi_1(u_i^{1/2}A_i)dH(u_i),$$

and $d_i = (\mathbf{y}_i - \boldsymbol{\mu}_i)^\top \boldsymbol{\Sigma}_i^{-1}(\mathbf{y}_i - \boldsymbol{\mu}_i)$, $A_i = \boldsymbol{\lambda}_i^\top \boldsymbol{\Sigma}_i^{-1}(\mathbf{y}_i - \boldsymbol{\mu}_i)$. Using the notation

$$\begin{aligned} I_i^\Phi(w) &= \int_0^\infty u_i^w \exp\{-\frac{1}{2}u_i d_i\}\Phi_1(u_i^{1/2}A_i)dH(u_i), \\ I_i^\phi(w) &= \int_0^\infty u_i^w \exp\{-\frac{1}{2}u_i d_i\}\phi_1(u_i^{1/2}A_i|0,1)dH(u_i), \end{aligned}$$

(so that $K_i(\boldsymbol{\theta})$ can be expressed as $K_i(\boldsymbol{\theta}) = I_i^\Phi(\frac{n_i}{2}))$, it follows that the matrix of second derivatives with respect to $\boldsymbol{\theta}$ is just

$$\begin{aligned} \mathbf{L} &= \sum_{i=1}^n \frac{\partial^2 \ell_i(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^\top} && (5.2) \\ &= -\frac{1}{2}\sum_{i=1}^n \frac{\partial^2 \log|\boldsymbol{\Sigma}_i|}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^\top} - \sum_{i=1}^n \frac{1}{K_i^2}\frac{\partial K_i}{\partial\boldsymbol{\theta}}\frac{\partial K_i}{\partial\boldsymbol{\theta}^\top} + \sum_{i=1}^n \frac{1}{K_i}\frac{\partial^2 K_i}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^\top}, && (5.3) \end{aligned}$$

where here we have $\frac{\partial K_i}{\partial\theta} = I_i^\phi(\frac{n_i+1}{2})\frac{\partial A_i}{\partial\theta} - \frac{1}{2}I_i^\Phi(\frac{n_i+2}{2})\frac{\partial d_i}{\partial\theta}$ and $\frac{\partial^2 K_i}{\partial\theta\partial\theta^\top} = \frac{1}{4}I_i^\Phi(\frac{n_i+4}{2})\frac{\partial d_i}{\partial\theta}\frac{\partial d_i}{\partial\theta^\top} - \frac{1}{2}I_i^\Phi(\frac{n_i+2}{2})\frac{\partial^2 d_i}{\partial\theta\partial\theta^\top} - \frac{1}{2}I_i^\phi(\frac{n_i+3}{2})(\frac{\partial A_i}{\partial\theta}\frac{\partial d_i}{\partial\theta^\top} + \frac{\partial d_i}{\partial\theta}\frac{\partial A_i}{\partial\theta^\top}) - I_i^\phi(\frac{n_i+3}{2})A_i\frac{\partial A_i}{\partial\theta}\frac{\partial A_i}{\partial\theta^\top} + I_i^\Phi(\frac{n_i+1}{2})\frac{\partial^2 A_i}{\partial\theta\partial\theta^\top}$.

From Propositions 3.13, 3.15, and 3.17 we have that for each distribution considered in this work, the integrates can be written as follows.

- *Skew-t:*

$$\begin{aligned} I_i^\Phi(w) &= \frac{2^w \nu^{\nu/2}\Gamma(w+\nu/2)}{\Gamma(\nu/2)(\nu+d_i)^{\nu/2+w}} T_1\left(\frac{A_i}{(d_i+\nu)^{1/2}}\sqrt{2w+\nu}|0,1,2w+\nu\right), \\ I_i^\phi(w) &= \frac{2^w \nu^{\nu/2}}{\sqrt{2\pi}\Gamma(\nu/2)}\left(\frac{1}{d_i+A_i^2+\nu}\right)^{\frac{\nu+2w}{2}}\Gamma(\frac{\nu+2w}{2}). \end{aligned}$$

- *Skew-slash:*

$$
\begin{aligned}
I_i^{\Phi}(w) &= \frac{2^{w+\nu}\Gamma(w+\nu)}{d_i^{w+\nu}} P_1(w+\nu, \frac{d_i}{2}) E[\Phi(S_i^{1/2} A_i)],\\
I_i^{\phi}(w) &= \frac{\nu 2^{w+\nu}\Gamma(w+\nu)}{\sqrt{2\pi}(d_i + A_i^2)^{w+\nu}} P_1(w+\nu, \frac{d_i + A_i^2}{2}),
\end{aligned}
$$

where $S_i \sim Gamma(w+\nu, \frac{d_i}{2})\mathbb{I}_{(0,1)}$.

- *Contaminated skew–normal:*

$$
\begin{aligned}
I_i^{\Phi}(w) &= \sqrt{2\pi}\{\nu\gamma^{w-1/2}\phi_1(d_i|0, \frac{1}{\gamma})\Phi(\gamma^{1/2}A_i) + (1-\nu)\phi_1(d_i|0,1)\Phi(A_i)\}\\
I_i^{\phi}(w) &= \nu\gamma^{w-1/2}\phi_1(d_i + A_i^2|0, \frac{1}{\gamma}) + (1-\nu)\phi_1(d_i + A_i^2).
\end{aligned}
$$

In many situations the derivatives of $\log \mathbf{\Sigma}_i$, d_i, and A_i involve complicated algebraic manipulation. For SNI-MEM, the derivatives of $\log \mathbf{\Sigma}$, d_i, and A_i can be found in Lachos et al. (2007) [20]. Asymptotic confidence intervals and test on the maximun likelihood estimators can be obtained using this matrix. Thus, if $\mathbf{J} = -\mathbf{L}$ denotes the observed information matrix for the marginal log-likelihood $\ell(\boldsymbol{\theta})$, then asymptotic confidence intervals and hypotheses tests for the parameter $\boldsymbol{\theta} \in \mathbb{R}^q$ are obtained once we assume the MLE $\widehat{\boldsymbol{\theta}}$ has approximately a $N_q(\boldsymbol{\theta}, \mathbf{J}^{-1})$ distribution. In practice, $\mathbf{J}$ is usually unknown and has to be replaced by its maximun likelihood estimation $\widehat{\mathbf{J}}$, that is, the matrix $\widehat{\mathbf{J}}$ evaluated at $\hat{\boldsymbol{\theta}}$. More generally speaking, for models as those in Proposition 3.7, the observed information matrix can be derived from the results given here.

6. Some examples

We illustrate the usefulness of the proposed class of distributions by applying them to two real data sets. The first example is an application of the methodology for univariate SNI responses, while the second is an application of SNI-MEM with $p = 5$.

6.1 Fiber-glass data set

In this section we apply four specific distributions of the skew normal/independent class, specifically, the univariate skew-normal, skew-t, skew-slash, and skew-contaminated normal, to fit the data on the breaking strength of 1.5cm long glass fiber, consisting of 63 observations. Jones and Faddy (2003) [19] and Wang and Genton (2006) [31] had previously analyzed this data with a skew-t and a skew-slash distribution, respectively. They both reported a strong presence of skewness on the left as well as a heavy-tailed behavior of the data, as depicted in Figure 4. We compare in the sequel the skew-normal (SN), skew-t (ST), contaminated skew-normal (SCN), and skew-slash (SSL) fitting for this data set. The resulting parameter estimates for the four models is given in Table 1. As suggested by Lange et al. (1989) [23], for each model the Schwarz information criterion was used for choosing the value of $\boldsymbol{\nu}$. This strategy is illustrated in Figure 5. Figure 4 shows the histogram of the fiber data superimposed with the fitted curves of the densities from the four considered models. We observe that the contaminated skew-normal fits the fiber data better than the other three distributions, especially at the peak part of the histogram. This conclusion is also supported by the log-likelihoods given in Table 1. Replacing the ML estimates of $\boldsymbol{\theta}$ in the Mahalanobis distance $d_i = (y_i - \mu)^2/\sigma^2$, we present Q-Q plots and envelopes in Figure 6 (lines represent the 5th percentile, the mean, and the 95th percentile of 100 simulated points for each observation). Plots in Figure 6 again provide strong evidence that the SNI distributions provides a better fit than the skew-normal distribution.

6.2 Chipkevitch et al. (1996) [12] data set

In this application, the multivariate skew-normal, skew-t, skew-slash, and skew-contaminated normal distributions are applied to fit the data studied by Chipkevitch et al. (1996) [12], where measurements of the testicular volume of 42 adolescents were converted to certain sequences by five different techniques: ultrasound (US), a graphical method proposed

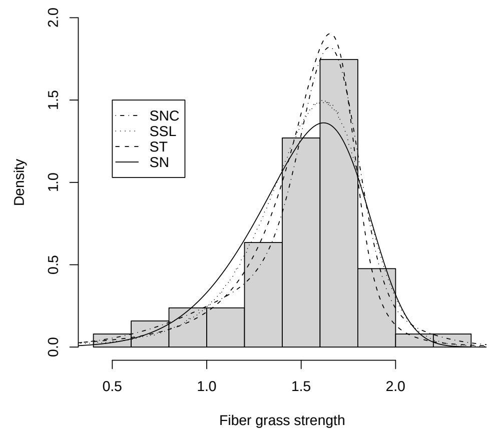

Figure 4: The histogram of the fiber grass strength superimposed with the fitted densities curves of the four distributions.

distribution	$\widehat{\mu}$	$\widehat{\sigma}$	$\widehat{\lambda}$	ν	γ	Log-likelihood
SN	1.8503	0.4705	-2.6789	-	-	
	(0.0468)	(0.0464)	(0.7513)			-13.9571
ST	1.7549	0.2725	-1.6196	3	-	
	(0.045)	(0.0353)	(0.6523)			-11.7546
SCN	1.7241	0.1615	-1.2940	0.5	0.10	
	(0.0393)	(0.0187)	(0.5080)			-9.1928
SSL	1.805591	0.2989667	-1.870298	1.7	-	
	(0.0461)	(0.0366)	(0.6320)			-12.9367

Table 1: MLE of the four fitted models on the fiber grass strength data set. Standard errors are based on the observed information matrix of Section 5.

(a) **(b)**

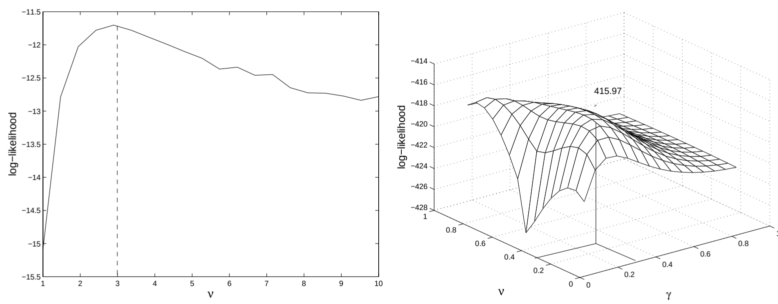

Figure 5: Plot of the profile log-likelihood for fitting (a) a skew-t model for fiber grass strength, (b) a contaminated skew-normal model for Chipkevitch data.

by the authors (I), dimensional measurement (II), Prader orchidometer (III), and ring orchidometer (IV). The ultrasound approach is assumed to be the reference measurement device. A histogram of the measurements (see Figure 7c) shows a certain asymmetry in the data set so that Galea-Rojas et al. (2002) [14] recommended a cubic root transformation to achieve better normality. Resulting parameter estimates for the four models are given in Table 2. The AIC criterion was used for choosing among some values of $\boldsymbol{\nu}$. For the ST model we found $\nu = 6$, for the SSL $\nu = 3$, and for the SCN $\nu = 0.3$, $\gamma = 0.3$. Therefore, for the three models a heavy-tailed distribution will be assumed. We can note from Table 2 that the intercept and slope estimates are similar among the four fitted models. However, the standard errors of the SNI distributions are smaller than the ones for the skew-normal model, indicating that the three models with longer than skew-normal tails seem to produce more accurate maximum likelihood estimates. The estimates for the variance components are not comparable since they are in a different scale. Note also that the log-likelihood values, shown at the bottom of Table 2, favor the SNI models. Particularly, we can see that the skew-t distribution fits the data better than the other three. The plots in Figure 7 provide even

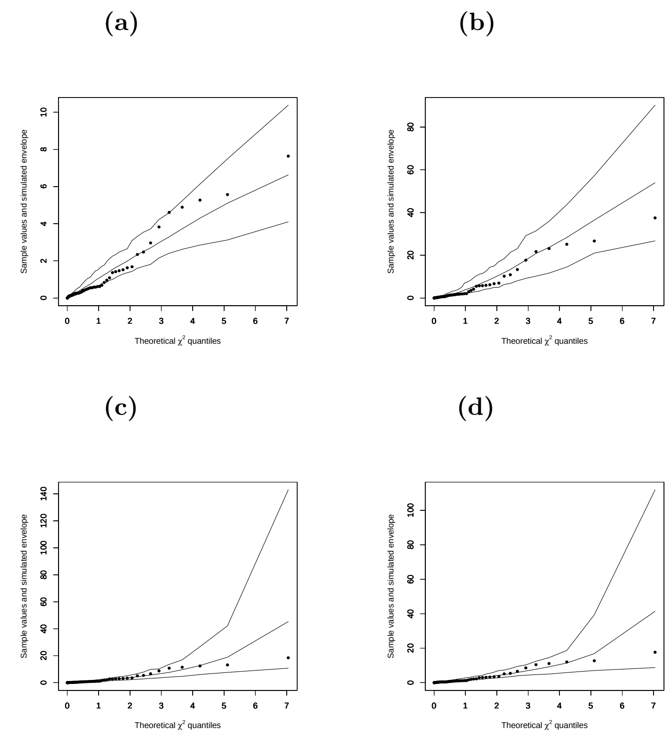

Figure 6: Fiber grass strength data set. Q-Q plots and simulated envelopes: (a) skew-normal model, (b) contaminated skew-normal model, (c) skew-t model, and (d) skew-slash model.

stronger evidence that the ST distribution allows a better fit to the data than the SN distribution.

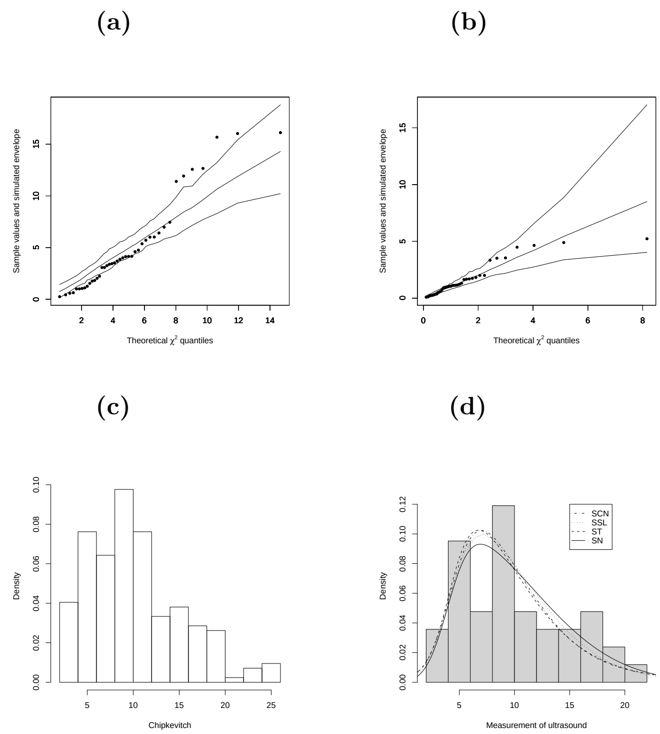

Figure 7: The Chipkevitch data set: (a) Q-Q plots and simulated envelopes for skew-normal model, (b) skew-t model, (c) histogram of the observed measurement, (d) histogram of the reference device measurement with superimposed fitted SNI densities.

	SN		ST		SCN		SSL	
Parameter	Estimate	SE	Estimate	SE	Estimate	SE	Estimate	SE
α_1	0.1022	0.5655	0.0426	0.4940	0.1559	0.5017	0.1226	0.5317
α_2	-0.0096	0.6216	-0.2472	0.5261	-0.1527	0.5227	-0.1716	0.5777
α_3	0.0483	0.6277	0.1110	0.5674	0.1464	0.5643	0.1027	0.5999
α_4	1.5391	0.6337	1.5444	0.5605	1.6404	0.5729	1.5784	0.5978
β_1	0.8838	0.0509	0.8990	0.0513	0.8911	0.0511	0.8887	0.0517
β_2	0.9495	0.0559	0.9866	0.0565	0.9782	0.0547	0.9754	0.0577
β_3	1.1419	0.0565	1.1537	0.0586	1.1540	0.0574	1.1466	0.0583
β_4	1.0826	0.0570	1.0957	0.0579	1.0885	0.0584	1.0864	0.0578
ϕ_1	1.3384	0.3714	0.9291	0.2893	0.7467	0.2240	0.8345	0.2508
ϕ_2	1.3284	0.3480	0.9538	0.2841	0.7972	0.2225	0.8405	0.2369
ϕ_3	1.6736	0.4322	0.9028	0.2960	0.7294	0.2105	0.8938	0.2849
ϕ_4	1.1578	0.3710	0.9481	0.3160	0.7845	0.2537	0.7998	0.2581
ϕ_5	1.4105	0.3994	1.0196	0.3385	0.9119	0.2761	0.9080	0.2783
μ_x	3.9952	1.3958	4.1688	1.0890	4.2784	1.3844	4.1830	1.3096
σ_x^2	59.2857	21.5487	38.1512	14.3057	30.9914	13.6063	35.0036	13.3506
λ_x	4.7842	4.7925	3.4300	2.4361	3.2404	2.8759	3.7562	3.2021
ν	-	-	6	-	0.3	-	3	-
γ	-	-	-	-	0.3	-	-	-
log-likelihood	-422.1628		-416.7776		-415.9791		-419.3461	

Table 2: Results of fitting skew-normal and SNI-MEM to the Chipkevitch data. Standard errors are based on the observed information matrix of Section 5.

7. Conclusion

In this work we have defined a new family of asymmetric models by extending the symmetric normal/independent family. Our proposal generalizes results by Azzalini and Capitanio (2003) [6], Gupta (2003) [17], and Wang and Genton (2006) [31]. In addition, we have developed a general method based on the EM algorithm for estimating the parameters of the skew-normal/independent distributions. Closed-form expressions were derived for the iterative estimation processes based on the fact that the proposed distributions possess a stochastic representation that can be used to represent them hierarchically. This stochastic representation also allows us to study many of its properties easily. We believe that the approaches proposed here can be applied to other asymmetric multivariate models like those proposed by Branco and Dey (2001) [9, Section 3]. The assessment of influence of data and model assumption on the result of the statistical analysis is a key aspect of any new class of distribution. We are currently exploring the local influence and residual analysis to address this issue.

8. Appendix: some lemmas

Now we take care of some technical lemmas needed in Section 3.

Lemma 8.1. *Let* $\mathbf{Y} \sim SN_p(\boldsymbol{\lambda})$. *Then for any fixed* p*-dimensional vector* $\mathbf{b}$ *and a* $p \times p$ *matrix* $\mathbf{A}$ *we have*

$$E[\mathbf{Y}^\top \mathbf{A}\mathbf{Y}\mathbf{b}^\top \mathbf{Y}] = -\sqrt{\frac{2}{\pi}}[(\boldsymbol{\delta}^\top \mathbf{A}\boldsymbol{\delta} + tr(\mathbf{A}))\mathbf{b}^\top \boldsymbol{\delta} + 2\boldsymbol{\delta}^\top \mathbf{A}\mathbf{b}],$$

where $\boldsymbol{\delta}$ *is as in (3.3).*

Proof. The proof follows by the stochastic representation of $\mathbf{Y}$ given in (1.2) and the calculation of the moments $E[|X_0|]$ and $E[|X_0|^3]$, when $X_0 \sim N(0,1)$. ☐

Lemma 8.2. *Let* $\mathbf{Y} \in \mathbb{R}^p$ *be a random vector with*

$$f(\mathbf{y}|u) = k^{-1}(u)\phi_p(\mathbf{y}|\boldsymbol{\mu}, u^{-1}\boldsymbol{\Sigma})\Phi_1(u^{1/2}A + u^{1/2}\mathbf{B}^\top\mathbf{y})$$

as probability density function, with u *a positive constant,* $A \in \mathbb{R}$*,* $\mathbf{B}$ *a* p*-dimensional vector, and* $k(u) = \Phi_1(u^{1/2}\dfrac{A + \mathbf{B}^\top\boldsymbol{\mu}}{\sqrt{1 + \mathbf{B}^\top\boldsymbol{\Sigma}\mathbf{B}}})$ *a standardized constant. Then we have*

$$E[\mathbf{Y}|u] = \boldsymbol{\mu} + u^{-1/2}\frac{\boldsymbol{\Sigma}\mathbf{B}}{\sqrt{1 + \mathbf{B}^\top\boldsymbol{\Sigma}\mathbf{B}}}W_{\Phi_1}(u^{1/2}\frac{A + \mathbf{B}^\top\boldsymbol{\mu}}{\sqrt{1 + \mathbf{B}^\top\boldsymbol{\Sigma}\mathbf{B}}}).$$

Proof. If we notice, by using Lemma 2 from Arellano-Valle et al. (2005) [2], that

$$\begin{aligned} E[\mathbf{Y}|u] &= k^{-1}(u)\int_{\mathbb{R}}\int_0^\infty \mathbf{y}\phi_1(t|u^{1/2}A + u^{1/2}\mathbf{B}^\top\mathbf{y}, 1)\phi(\mathbf{y}|\boldsymbol{\mu}, u^{-1}\boldsymbol{\Sigma})dtd\mathbf{y} \\ &= k^{-1}(u)\int_0^\infty \phi_1(t|u^{1/2}A + u^{1/2}\mathbf{B}^\top\boldsymbol{\mu}, 1 + \mathbf{B}^\top\boldsymbol{\Sigma}\mathbf{B})E_{\mathbf{Y}|t}[\mathbf{Y}]dt \end{aligned}$$

holds, where $\mathbf{Y}|t \sim N_p(\boldsymbol{\mu} - \boldsymbol{\Lambda}\mathbf{B}(A + \mathbf{B}^\top\boldsymbol{\mu}) + u^{-1/2}\boldsymbol{\Lambda}\mathbf{B}t, u^{-1}\boldsymbol{\Lambda})$, with $\boldsymbol{\Lambda} = (\boldsymbol{\Sigma}^{-1} + \mathbf{B}\mathbf{B}^\top)^{-1}$, then the proof follows from well known properties of the truncated normal distribution (compare Johnson et al. (1994) [18, Section 10.1]). □

Lemma 8.3. *Let* $\mathbf{Y} \sim Gamma(\alpha, \beta)$*. Then for any* $a \in \mathbb{R}$ *we have*

$$E[\Phi_1(a\sqrt(Y))] = T_1(a\sqrt{\frac{\alpha}{\beta}}|0, 1, 2\alpha).$$

Proof. See Azallini and Capitanio (2003) [6, Lemma 1]. □

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References

[1] Andrews, D. F. and Mallows, C. L.; *Scale mixtures of normal distributions.* Journal of the Royal Statistical Society, Series B, **36**, 99-102, 1974.

[2] Arellano-Valle, R. B., Bolfarine, H. and Lachos, V. H.; *Skew-normal linear mixed models.* Journal of Data Science, **3**, 415-438, 2005.

[3] Arnold, B. C. and Beaver, R. J.; *Skewed multivariate models related to hidden truncation and/or selective reporting.* Test, **11**, 7-54, 2002.

[4] Azzalini, A.; *A class of distributions which includes the normal ones.* Scandinavian Journal of Statistics, **12**, 171-178, 1985.

[5] Azzalini, A., Capitanio, A.; *Statistical applications of the multivariate skew normal distribution.* Journal of the Royal Statistical Society ,Series B, **61**, 579-602, 1999.

[6] Azzalini, A., Capitanio, A.; *Distributions generated y perturbation of symmetry with emphasis on the multivariate skew t-distribution.* Journal of the Royal Statistical Society, Series B, **61**, 367-389, 2003.

[7] Azzalini, A. and Dalla-Valle, A.; *The multivariate skew-normal distribution.* Biometrika, **83**, 715-726, 1996.

[8] Barnett, V.D.; *Simultaneous pairwise linear structural relationships.* Biometrics, 25, 129-142, 1969.

[9] Branco, M. and Dey, D.; *A general class of multivariate skew-elliptical distribution.* Journal of Multivariate Analysis, **79**, 93-113, 2001.

[10] Bolfarine, H. and Galea-Rojas, M.; *Maximum likelihood estimation of simultaneous pairwise linear structural relationships.* Biometrical Journal, 37, 673-689, 1995.

[11] Bolfarine, H. and Galea-Rojas, M.; *On structural comparative calibration under a t-model.* Computational Statistics, 11, 63-85, 1996.

[12] Chipkevitch, E., Nishimura, R., Tu, D. and Galea-Rojas, M.; *Clinical measurement of testicular volume in adolescents: Comparison of the reliability of 5 methods.* Journal of Urology, 156, 2050-2053, 1996.

[13] Diciccio, T. and Monti, A. C.; *Inferential aspects of the skew exponential power distribution.* Journal of the American Statistical Association, 99, 439-450, 2004.

[14] Galea-Rojas, M., H. and de Castro, M. *Influence diagnostics for the structural errors-in-variables model under the Student-t distribution.* Journal of Applied Statistics, 29, 1191-1204, 2002.

[15] Genton, M. G., He, L. and Liu, X.; *Moments of skew-normM random vectors and their quadratic forms.* Statistics and Probability Letters, 51,319-325, 2001.

[16] Genton, M. G., Loperfido, N.; *Generalized skew-elliptical distributions and their quadratic forms.* Annals of the Institute of Statistical Mathematics, **57**, 389-401, 2005.

[17] Gupta A. K.; *Multivariate skew t-distributions.* Statistics, **37(4)**, 359-363, 2003.

[18] Johnson, N. L., Kotz, S., Balakrishnan, N.; *Continuous Univariate Distributions, vol. 1.* New York: John Wiley, 1994.

[19] Jones, M. C. and Faddy, M. J.; *A skew extension of the s distribution, with applications.* Journal of the Royal Statistical Society, Series B, **65**, 159-174, 2003.

[20] Lachos, V. H., Bolfarine, H., Arellano-Valle, R. B. and Montenegro, L. C.; *Likelihood based inference for multivariate skew-normal regression models.* Communications in Statistics: Theory and Methods, **36**, 1769-1786, 2007.

[21] Lachos, V. H., Bolfarine, H., Vilca, L. F., Galea–Rojas, M.; *Estimation and influence diagnostic for structural comparative calibration models under the skew-normal distribution.* Technical report, RT-MAE 2005-12, IME-USP, 2005.

[22] Lange, K. and Sinsheimer.; *Normal/independent distributions and their applications in robust regression.* Journal of Computational and Graphical Statistics, **2**, 175-198, 1993.

[23] Lange, K. L., Little, J. A. and Taylor, M. G. J.; *Robust statistical modeling using the t distribution.* Journal of the American Statistical Association, **84**, 881-896, 1989.

[24] Little, R. J. A.; *Robust Estimation of the mean and covariance matrix from data with missing values.* Applied Statistics, **37**, 23-38, 1988.

[25] Liu, C.; *Bayesian robust linear regression with incomplete data.* Journal of the American Statistical Association, **91**, 1219-1227, 1996.

[26] Mardia, K. V.; *Applications of some measures of multivariate skewness and kurtosis in testing normality and robustness studies.* Sankhyá B, 36(2), 115-128, 1974.

[27] Sahu, S. K., Dey, D. K. and Branco, M. D.; *A new class of multivariate skew distributions with applications to Bayesian regression models.* The Canadian Journal of Statistics, **31**, 129-150, 2003.

[28] Shyr, I. and Gleser, L.; *Inference about comparative precision in linear structural relationships.* Journal of Statistical Planning and Inference, 14, 339-358, 1986.

[29] Tan, F. and Peng, H.; *The slash and the skew-slash Student t distributions.* Preprint, 2006.

[30] Wang, J., Boyer, J. and Genton, M.; *A skew-symmetric representation of multivariate distributions.* Statistica Sinica, **14**, 1259-1270, 2004.

[31] Wang, J. and Genton, M.; *The multivariate skew-slash distribution.* Journal of Statistical Planning and Inference, **136**, 209-220, 2006.

[32] Zhu, H. and Lee S.; *Local influence for incomplete-data models.* Journal of the Royal Statistical Society, Series B, **63**, 111-126., 2001.

Resumen

Liu (1996) discute una clase de distribuciones robustas a las que apela como normal/independiente, y que contiene un grupo de distribuciones de colas pesadas. En este artículo desarrollamos una versión asimétrica de tales distribuciones en un escenario multivariado, a las que llamaremos distruciones normales asimétricas independientes multivariadas. Para tales distribuciones derivamos algunas propiedades. La principal virtud de los miembros de esta familia es que son fáciles de simular y se prestan a un algoritmo de tipo EM para realizar estimaciones de máxima verosimilitud de sus parámetros. Para dos modelos multivariados de interés práctico se discute el algoritmo EM con énfasis en las distribuciones t-asimétrica, slash asimétrica y normal asimétrica contaminada. Los resultados obtenidos a partir de simulaciones y de dos conjuntos de datos reales son reportados.

Palabras clave: Algoritmo EM, normal/independiente.

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Poincaré duality in equivariant intersection theory

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October, 2014

Abstract

We study the Poincaré duality map from equivariant Chow cohomology to equivariant Chow groups in the case of torus actions on complete, possibly singular, varieties with isolated fixed points. Our main results yield criteria for the Poincaré duality map to become an isomorphism in this setting. The methods rely on the localization theorem for equivariant Chow cohomology and the notion of algebraic rational cell. We apply our results to complete spherical varieties and their generalizations.

MSC(2010): 14C15, 14L30, 14M27, 55N91.

Keywords: *Chow groups, torus actions, cell decompositions, Poincaré duality, spherical varieties.*

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1. Introduction

Let T be a complex algebraic torus of dimension d. Let X be a compact complex algebraic variety where T acts with isolated fixed points. If X has no rational cohomology in odd degrees (i.e., if X is *equivariantly formal* [16]), then the equivariant cohomology ring $H_T^*(X)$ with rational coefficients is a commutative positively graded algebra; it is a free module of finite rank over the equivariant cohomology ring of a point (the latter is a polynomial ring in d variables). Examples include Schubert varieties and rationally smooth projective varieties where a complex reductive group acts with finitely many orbits (a complex variety of pure dimension n is rationally smooth if the local cohomology at any point is the same as the local cohomology of $\mathbb{C}^n$). In [6], Brion has shown that several topological invariants of a T-variety X can be read off $H_T^*(X)$. For instance, if X is equivariantly formal and $n = \dim X$, then equivariant Kronecker duality holds, i.e., the dualizing module of $H_T^*(X)$ is the equivariant homology $H_*^T(X)$. Furthermore, the following conditions are equivalent: (i) Poincaré duality, (ii) the algebra $H_T^*(X)$ is Gorenstein, (iii) the Betti numbers of X satisfy $b_q(X) = b_{2n-q}(X)$, for $0 \leq q \leq n$, and all equivariant multiplicities are nonzero (these are certain functionals on $H_*^T(X)$). See [6, Theorem 4.1]. Finally, Brion obtains some Morse inequalities for the Betti numbers of X, assuming all equivariant multiplicities are nonzero [6, Theorem 4.2].

The purpose of this article is to generalize Brion's results to the purely algebraic and more delicate setting of equivariant Chow groups and equivariant operational Chow groups (or Chow cohomology); we refer to Sections 2 and 3 for appropriate definitions and notation. In order to achieve our goal, first we find a suitable class of varieties that resemble equivariantly formal varieties from the viewpoint of equivariant intersection theory. This is done by combining two classes of varieties considered in previous work, namely T-linear varieties [13] and $\mathbb{Q}$-filtrable varieties [14]. Let us quickly mention some of their relevant features. In [13] we show that projective T-linear varieties satisfy equivariant Kronecker duality. This property is rather strong, and does not hold for arbitrary T-varieties. On the other hand, in [14] we introduced the class of $\mathbb{Q}$-filtrable varieties (Definition 2.12). A remarkable property of these schemes is that their equivariant Chow groups are free modules of finite rank over the equivariant Chow ring of a point. Hence, it is quite natural to consider the class of $\mathbb{Q}$-filtrable T-linear schemes as a suitable replacement for the notion of equivariant formality in equivariant intersection theory (cf. Theorem 3.5, Theorem 3.7). This expectation is confirmed in Section 4, where we obtain criteria for Poincaré duality on projective $\mathbb{Q}$-filtrable T-linear varieties. Our main results (Theorems 4.1, 4.3, and 4.5, and Corollary 4.7) yield the sought-after generalizations of Brion's results, and open the way for further work in this direction.

2. Definitions and basic properties

2.1 Conventions and notation

Throughout this paper, we fix an algebraically closed field $\Bbbk$ of characteristic zero. All schemes and algebraic groups are assumed to be defined over $\Bbbk$. By a scheme we mean a separated scheme of finite type. A variety is a reduced scheme. Observe that varieties need not be irreducible. A subvariety is a closed subscheme which is a variety. A point on a scheme will always be a closed point.

We denote by T an algebraic torus. A scheme X provided with an algebraic action of T is called a **T-scheme**. For a T-scheme X, we denote by X^T the fixed point subscheme and by $i_T : X^T \to X$ the natural inclusion. If H is a closed subgroup of T, we similarly denote by $i_H : X^H \to X$ the inclusion of the fixed point subscheme. When comparing X^T and X^H we write $i_{T,H} : X^T \to X^H$ for the natural (T-equivariant) inclusion. We denote by Δ the character group of T, and by S the symmetric algebra over $\mathbb{Q}$ of the abelian group Δ. The quotient field of S is denoted by $\mathcal{Q}$.

In this paper, torus actions are assumed to be **locally linear**, i.e., the schemes we consider are covered by invariant affine open subsets. This assumption is fulfilled for instance by T-stable subschemes of normal T-schemes [30]. A T-scheme is called **T-quasiprojective** if it has an ample T-linearized invertible sheaf. This assumption is fulfilled, among others, by T-stable subschemes of normal quasiprojective T-schemes [30].

Let G be a connected reductive group. Recall that a normal G-variety X is called **spherical** if a Borel subgroup B of G has a dense orbit in X. Then it is known that G and B have finitely many orbits in X. It follows that X contains only finitely many fixed points of a maximal torus $T \subset B$, see for example [31].

Equivariant Chow groups and equivariant operational Chow groups are considered with rational coefficients.

2.2 The Bialynicki-Birula decomposition

The material in this subsection is due to Bialynicki-Birula [2], [3] (in the smooth case) and Konarski [20] (in the general case).

Let X be a T-scheme with isolated fixed points. Then X^T is finite. We write $X^T = \{x_1, \dots, x_m\}$. A one-parameter subgroup $\lambda : \mathbb{G}_m \to T$ is called **generic** if $X^{\mathbb{G}_m} = X^T$, where $\mathbb{G}_m$ acts on X via λ. Generic one-parameter subgroups always exist due to the local linearity of the action. Fix a generic $\lambda : \mathbb{G}_m \to T$. For each i, define the subset

$$X_+(x_i, \lambda) = \{x \in X \mid \lim_{t \to 0} \lambda(t) \cdot x = x_i\}.$$

Then $X_+(x_i, \lambda)$ is a locally closed T-invariant subscheme of X. The (disjoint) union of the $X_+(x_i, \lambda)$'s might not cover all of X, but when it does (e.g., when X is complete), the decomposition $\{X_+(x_i, \lambda)\}_{i=1}^m$ is called the Bialynicki-Birula decomposition, or **BB-decomposition**, of X associated to λ. Each $X_+(x_i, \lambda)$ is called a **cell** of the decomposition.

Definition 2.1. Let X be a T-scheme with finitely many fixed points. Let $\{X_+(x_i, \lambda)\}_{i=1}^m$ be the BB-decomposition associated to some generic $\lambda : \mathbb{G}_m \to T$. The decomposition $\{X_+(x_i, \lambda)\}$ is said to be **filtrable** if

there is a finite increasing sequence $\Sigma_0 \subset \Sigma_1 \subset \ldots \subset \Sigma_m$ of T-invariant closed subschemes of X such that:
a) $\Sigma_0 = \emptyset$, $\Sigma_m = X$,
b) $\Sigma_j \setminus \Sigma_{j-1}$ is a cell of the decomposition $\{X_+(x_i, \lambda)\}$, for each $j = 1, \ldots, m$.

In this context, it is common to say that X is **filtrable**, and refer to Σ_j as the **j-th filtered piece** of X. If, moreover, the cells $X_+(x_i, \lambda)$ are isomorphic to affine spaces $\mathbb{A}^{n_i}$, then X is called **T-cellular**.

Theorem 2.2 ([2], [3])**.** *Let X be a complete T-scheme with isolated fixed points, and let λ be a generic one-parameter subgroup. If X admits an ample T-linearized invertible sheaf, then the associated BB-decomposition $\{X_+(x_i, \lambda)\}$ is filtrable. Furthermore, if X is smooth, then X is T-cellular.* □

2.3 Equivariant Chow groups for torus actions

Let X be a T-scheme of dimension n (not necessarily equidimensional). Let V be a finite dimensional T-module, and let $U \subset V$ be an invariant open subset so that a principal bundle quotient $U \to U/T$ exists. Then T acts freely on $X \times U$ and the quotient scheme $X_T = (X \times U)/T$ exists. Following Edidin and Graham [8], we define the i-th **equivariant Chow group** $A_i^T(X)$ by $A_i^T(X) = A_{i+\dim U - \dim T}(X)$ if $V \setminus U$ has codimension more than $n - i$. The definition is independent of the choice of (V, U), see [8] for details. Set $A_*^T(X) = \oplus_i A_i^T(X)$. If X is a T-scheme and $Y \subset X$ is a T-stable closed subscheme, then Y defines a class $[Y]$ in $A_*^T(X)$. If X is smooth, then so is X_T, and $A_*^T(X)$ admits an intersection pairing; in this case, denote by $A_T^*(X)$ the corresponding ring graded by codimension. The equivariant Chow ring $A_T^*(pt)$ is isomorphic to S, and $A_*^T(X)$ is a S-module, where Δ acts on $A_*^T(X)$ by homogeneous maps of degree -1. This module structure is induced by pullback through the flat map $p_{X,T} : X_T \to U/G$. Restriction to a fiber of $p_{X,T}$ gives $i^* : A_*^T(X) \to A_*(X)$. If X is complete, we denote by $\int_X(\alpha) \in S$ the proper pushforward to a point of a class $\alpha \in A_*^T(X)$.

Next we state Brion's description [4] of the equivariant Chow groups in terms of invariant cycles. It also shows how to recover the usual Chow groups from equivariant ones.

Theorem 2.3. *Let X be a T-scheme. Then the S-module $A_*^T(X)$ is defined by generators $[Y]$, where Y is an invariant irreducible subvariety of X, and relations $[\mathrm{div}_Y(f)] - \chi[Y]$, where f is a rational function on Y which is an eigenvector of T of weight χ. Furthermore, the map $A_*^T(X) \to A_*(X)$ vanishes on $\Delta A_*^T(X)$, and it induces an isomorphism $A_*^T(X)/\Delta A_*^T(X) \to A_*(X)$.* □

The following is a slightly more general version of the localization theorem for equivariant Chow groups [4, Corollary 2.3.2]. For a proof, see [13, Proposition 2.15].

Theorem 2.4. *Let X be a T-scheme, let $H \subset T$ be a closed subgroup, and let $i_H : X^H \to X$ be the inclusion of the fixed point subscheme. Then the induced morphism of equivariant Chow groups*

$$i_{H*} : A_*^T(X^H) \to A_*^T(X)$$

becomes an isomorphism after inverting finitely many characters of T that restrict non-trivially to H. □

2.4 T-linear schemes

We introduce our main class of testing spaces.

Definition 2.5. Let T be an algebraic torus and let X be a T-scheme.

1. We say that X is **T-equivariantly 0-linear** if it is either empty or isomorphic to $\mathrm{Spec}\,(\mathrm{Sym}(V^*))$, where V is a finite-dimensional rational representation of T.

2. For a positive integer n, we say that X is **T-equivariantly n-linear** if either one of the following conditions hold.

(i) There is a T-scheme Y, which contains X as a T-invariant open subscheme, so that Y and $Z = Y \setminus X$ are T-equivariantly $(n-1)$-linear.

(ii) There exists a T-invariant closed subscheme $Z \subseteq X$, with complement U, so that Z and U are T-equivariantly $(n-1)$-linear.

3. We say that X is **T-equivariantly linear** (or simply, **T-linear**) if it is T-equivariantly n-linear for some $n \geq 0$. T-linear varieties are varieties that are T-linear schemes.

Clearly, if $T \to T'$ is a morphism of algebraic tori, then every T'-linear scheme is also T-linear. On the other hand, if X is T-equivariantly n-linear, then the fixed point subscheme X^H of any subtorus $H \subset T$ is T-equivariantly n-linear. Observe that T-linear schemes are *linear schemes* in the sense of [17], [32], and [18].

It is known that if X is a T-linear scheme, then $A_*^T(X)$ is a finitely generated S-module and $A_*(X)$ is a finitely generated abelian group (see e.g. [13, Lemma 2.7]). The next theorem provides some concrete examples. For a proof of items (i)-(ii) see [19, Proposition 3.6], for item (iii) see [13, Theorem 2.5].

Theorem 2.6. *Let T be an algebraic torus. Then the following holds.*

(i) A T-cellular scheme is T-linear.

(ii) Every T-scheme with finitely many T-orbits is T-linear. In particular, a toric variety with dense torus T is T-linear.

(iii) Let B be a connected solvable linear algebraic group with maximal torus T. Let X be a B-scheme. If B acts on X with finitely many orbits, then X is T-linear. In particular, spherical varieties are T-linear. □

2.5 Equivariant multiplicities at nondegenerate fixed points

Let X be a T-scheme. A fixed point $x \in X$ is called **nondegenerate** if all weights of T in the tangent space T_xX are non-zero. A fixed point in a nonsingular T-variety is nondegenerate if and only if it is isolated. To study singular schemes, Brion [4] developed a notion of equivariant multiplicity at nondegenerate fixed points. The main features of this concept are outlined below, for details see [4, Section 4].

Theorem 2.7. *Let X be a T-scheme with an action of T, let $x \in X$ be a nondegenerate fixed point and let $\chi_1, \dots, \chi_n$ be the weights of T_xX (counted with multiplicity).*

(i) There exists a unique S-linear map

$$e_{x,X} : A_*^T(X) \longrightarrow \frac{1}{\chi_1 \cdots \chi_n} S$$

such that $e_{x,X}[x] = 1$ and that $e_{x,X}[Y] = 0$ for any T-invariant irreducible subvariety $Y \subset X$ which does not contain x.

(ii) For any T-invariant irreducible subvariety $Y \subset X$, the rational function $e_{x,X}[Y]$ is homogeneous of degree $-\dim(Y)$ and coincides with $e_{x,Y}[Y]$.

(iii) The point x is nonsingular in X when $e_x[X] = \dfrac{1}{\chi_1 \cdots \chi_n}$. □

For a T-stable irreducible subvariety $Y \subset X$, set $e_{x,X}[Y] = e_x[Y]$, and call $e_x[Y]$ the **equivariant multiplicity of Y at x**.

Proposition 2.8. *Let X be a T-scheme such that all fixed points in X are nondegenerate, and let $\alpha \in A_*^T(X)$. Then, in $A_*^T(X) \otimes_S \mathcal{Q}$, we have $\alpha = \sum_{x \in X^T} e_x(\alpha)[x]$.* □

Next we describe a special class of nondegenerate fixed points. Let X be a T-variety. Call a fixed point $x \in X$ **attractive** if all weights in the tangent space T_xX are contained in an open half-space of $\Delta_{\mathbb{R}} = \Delta \otimes_{\mathbb{Z}} \mathbb{R}$.

Theorem 2.9. *Let X be a T-variety with a fixed point x. The following conditions are equivalent.*

(i) The point x is attractive.

(ii) There exists a one-parameter subgroup $\lambda : \mathbb{G}_m \to T$ such that, for all y in a neighborhood of x, we have $\lim_{t\to 0} \lambda(t)y = x$.

If (i) or (ii) holds, then x admits a unique open affine T-stable neighborhood in X, denoted X_x, and X_x admits a closed equivariant embedding into T_xX. Moreover, $e_x[X]$ is non-zero. □

2.6 $\mathbb{Q}$-filtrable varieties and equivariant Chow groups

Here we recall some of the main results from [14].

Definition 2.10. Let X be an affine T-variety with an attractive fixed point x, and let $n = \dim X$. We say that (X, x), or simply X, is an **algebraic rational cell** when it satisfies

$$A_k(X) = \begin{cases} \mathbb{Q} & \text{if} \quad k = n \\ 0 & \text{if} \quad k \neq n. \end{cases}$$

In particular, if (X, x) is an algebraic rational cell, then it is irreducible.

Let X be an affine T-variety with an attractive fixed point x. Then there exists a generic one-parameter subgroup $\lambda : \mathbb{G}_m \to T$ for which $X = X_+(x, \lambda)$ and X admits a closed T-equivariant embedding into T_xX (Theorem 2.9). Since all the weights of the $\mathbb{G}_m$-action on T_xX (via λ) are positive, the geometric quotient $\mathbb{P}_\lambda(X) = (X \setminus \{x\})/\mathbb{G}_m$ exists and is a projective variety. In fact, it is a closed subvariety of the weighted projective space $\mathbb{P}_\lambda(T_xX)$. Remarkably, X is an algebraic rational cell if and only if

$$A_k(\mathbb{P}_\lambda(X)) = \begin{cases} \mathbb{Q} & \text{if} \quad 0 \leq k \leq n-1, \\ 0 & \text{otherwise} \end{cases}$$

holds. See [14] for details.

Example 2.11. Let $\Bbbk = \mathbb{C}$. Algebraic rational cells are naturally found on rationally smooth spherical varieties. Indeed, let X be a G-spherical variety with an attractive fixed point $x \in X$. Let X_x be the unique open affine T-stable neighborhood of x. If X is rationally smooth at x, then (X_x, x) is an algebraic rational cell [14, Theorem 7.2].

Using algebraic rational cells as building blocks, one can study the global geometry of T-varieties equipped with a paving by algebraic rational cells.

Definition 2.12. Let X be a T-variety. We say that X is **$\mathbb{Q}$-filtrable** if the following hold:

1. the fixed point set X^T is finite, and
2. there exists a generic one-parameter subgroup $\lambda : \mathbb{G}_m \to T$ for which the associated BB-decomposition of X is filtrable (Definition 2.1) and consists of T-invariant **algebraic rational cells**.

In particular, we have $X = \bigsqcup_j X_+(x_j, \lambda)$. Also, observe that the fixed points $x_j \in X^T$ need not be attractive in X, but they are so in their corresponding algebraic rational cells $X_+(x_j, \lambda)$. The key result is stated below.

Theorem 2.13 ([14, Theorem 4.4])**.** *Let X be a $\mathbb{Q}$-filtrable T-variety. Then the T-equivariant Chow group of X is a free S-module of rank $|X^T|$. In fact, it is freely generated by the classes of the closures of the cells $X_+(x_i, \lambda)$. Consequently, $A_*(X)$ is also freely generated by the classes of the cell closures $\overline{X_+(x_i, \lambda)}$.* □

Next we compute equivariant multiplicities of algebraic rational cells and $\mathbb{Q}$-filtrable varieties. Recall that a primitive character χ of T is called **singular** if it satisfies $X^{\ker(\chi)} \neq X^T$.

Theorem 2.14 ([14, Corollary 3.8])**.** *Let X be an irreducible T-variety with attractive fixed point x. Let X_x be the unique open affine T-stable neighborhood of x. If (X_x, x) is an algebraic rational cell, then the following hold.*

(i) $e_x[X]$ *is the inverse of a polynomial. In fact, we have*

$$e_x[X] = \frac{d}{\chi_1 \cdots \chi_r},$$

here the χ_i*'s are singular characters,* $r = \dim X$*, and* d *is a positive rational number.*

(ii) Moreover, if the number of closed irreducible T*-stable curves through* x *is finite, say* $\ell(x)$*, then* $\dim X = \ell(x)$*. Furthermore, we may take for* $\chi_1, \ldots, \chi_r$ *the characters associated to these curves.* □

In general, if X is an affine T-variety with an attractive fixed point x, and $\ell(x)$ as above is finite, then $\dim X \leq \ell(x)$ [5, Corollary 1.4.2].

Example 2.15. Let $\Bbbk = \mathbb{C}$. Let X be a G-spherical variety. Recall that X^T is finite. If there is a generic $\lambda : \mathbb{G}_m \to T$ so that $\{X_+(x_i, \lambda)\}$ is a filtrable BB-decomposition and each cell is rationally smooth, then X is $\mathbb{Q}$-filtrable and both the equivariant and non-equivariant cycle maps are isomorphisms [14, Theorem 7.3]. In particular, this holds for rationally smooth projective group embeddings [14, Corollary 5.10]; see below for a definition of this important class of spherical varieties.

2.7 Applications to group embeddings

We recall some results and notation on group embeddings that will be used freely throughout the paper. Here G denotes a connected reductive group with Borel subgroup B and maximal torus $T \subset B$.

An irreducible algebraic variety is called an **embedding** of G, or a **group embedding**, if it is a normal $G \times G$-variety containing an open orbit isomorphic to G itself, where $G \times G$ acts on G by left and right multiplication. When G is a torus, we recover the notion of toric varieties. Group embeddings are spherical $G \times G$-varieties (by the Bruhat decomposition). Affine embeddings of G are nothing but reductive monoids having G as group of units [29]. Recall that an **algebraic monoid** is an algebraic variety equipped with an associative product map, which is a morphism of varieties and admits an identity element. An affine algebraic monoid is called **reductive** if it is irreducible, normal, and its unit group is a reductive algebraic group.

Let M be a reductive monoid with zero and unit group G. Then there exists a central one-parameter subgroup $\epsilon : \mathbb{G}_m \to T$ that satisfies $\lim_{t\to 0} \epsilon(t) = 0$, see [7, Lemma 1.1.1]. Moreover, the quotient space $\mathbb{P}_\epsilon(M) = (M \setminus \{0\})/\epsilon(\mathbb{G}_m)$ is a projective embedding of the quotient group $G/\epsilon(\mathbb{G}_m)$. In fact, projective embeddings of connected reductive groups are exactly the projectivizations of reductive monoids [24].

Let M be a reductive monoid with zero and unit group G. It is worth noting that 0 is the unique attractive $T \times T$-fixed point of M (see e.g., [7, Lemma 1.1.1]). Let $\overline{T} \subset M$ be the Zariski closure of T in M. Then $\overline{T}$ is a normal affine toric variety [25, Theorem 5.4]. We denote by $E(M)$ the idempotent set of M, that is, $E(M) = \{e \in M \,|\, e^2 = e\}$. Likewise, $E(\overline{T})$ denotes the idempotent set of $\overline{T}$. One defines a partial order on $E(M)$ (and thus on $E(\overline{T})$) by declaring $f \le e$ if and only if $fe = f = ef$. Write W for the Weyl group of (G, T), and denote by $\mathfrak{S}$ its set of simple reflections. Then W acts on $E(\overline{T})$ by conjugation, and the corresponding set of W-conjugacy classes can be identified with $\Lambda = \{e \in E(\overline{T}) \,|\, Be = eBe\}$. See [25] for details. The lattice Λ is called the **cross-section lattice** of M. Notably, Λ can also be identified with the (finite) set $G \times G$-orbits of M [25, Theorem 4.5]. For $e \in E(M)$, set $M_e = \overline{\{g \in G \,|\, ge = eg = e\}}$. Then M_e is a reductive monoid with e as its zero element [7]. Finally, let $\Lambda_k = \{x \in \Lambda \,|\, \dim Tx = k\,\}$ be the set of elements of rank k in Λ.

Definition 2.16. A reductive monoid M with zero element is called **quasismooth** if, for any minimal non-zero idempotent $e \in E(M)$, the submonoid M_e is an algebraic rational cell.

This definition agrees with that of [26]. For details, we refer to [14, Definition 5.8]. Over the complex numbers, M is quasismooth if and only if $\mathbb{P}_\epsilon(M)$ is rationally smooth [28, Theorem 2.5]. For a combinatorial classification of quasismooth monoids, see [28]. The main result in this context is stated next. See [14, Theorem 5.9] for a proof.

Theorem 2.17. *If M is a quasismooth monoid, then the projective group embedding $\mathbb{P}_\epsilon(M)$ is $\mathbb{Q}$-filtrable.* □

3. Equivariant Chow cohomology: localization and equivariant Kronecker duality

Let X be a T-scheme. The i-th **T-equivariant operational Chow group** of X, denoted $\mathrm{op}A^i_T(X)$, is defined as follows: an element $c \in \mathrm{op}A^i_T(X)$ is a collection of homomorphisms $c_f^{(m)} : A^T_m(Y) \to A^T_{m-i}(Y)$, written $z \mapsto f^*c \cap z$, for every T-map $f : Y \to X$ and all integers m. (The underlying category is the category of T-schemes.) These homomorphisms must satisfy certain compatibility conditions, see [9, Chapter 17] and [8] for details. For any X, the ring structure on $\mathrm{op}A^*_T(X) = \oplus_i \mathrm{op}A^i_T(X)$ is given by composition of such homomorphisms. The ring $\mathrm{op}A^*_T(X)$ is graded, and $\mathrm{op}A^i_T(X)$ can be non-zero for any $i \geq 0$. The basic properties we need are summarized below.

(i) The cup product $\mathrm{op}A^p_T(X) \otimes \mathrm{op}A^q_T(X) \to \mathrm{op}A^{p+q}_T(X)$, $a \otimes b \mapsto a \cup b$, is well defined and makes $\mathrm{op}A^*_T(X)$ into a graded associative ring. Note that this ring is commutative since $\mathrm{char}(\Bbbk) = 0$, and so all T-schemes admit equivariant resolution of singularities.

(ii) There are contravariant graded maps $f^* : \mathrm{op}A^i_T(X) \to \mathrm{op}A^i_T(Y)$, for arbitrary equivariant morphisms $f : Y \to X$.

(iii) The cap product $\mathrm{op}A^i_T(X) \otimes A^T_m(X) \to A^T_{m-i}(X)$, $c \otimes z \mapsto c \cap z$, is well defined and makes $A^T_*(X)$ into an $\mathrm{op}A^*_T(X)$-module satisfying the projection formula.

(iv) For any T-scheme X of pure dimension n, there is an **equivariant Poincaré duality map:**

$$\mathcal{P}_T : \mathrm{op}A^k_T(X) \to A^T_{n-k}(X), \quad z \mapsto z \cap [X].$$

If X is nonsingular, then $\mathcal{P}_T$ is an isomorphism, and the ring structure on $\mathrm{op}A^*_T(X)$ is that determined by intersection products of cycles on the mixed spaces X_T. In particular, by (iii) and (iv), $\mathrm{op}A^*_T(X)$ is a graded S-algebra. We say that X **satisfies equivariant Poincaré duality** if $\mathcal{P}_T$ is an isomorphism. Similar remarks apply to the non-equivariant Poincaré duality map (denoted $\mathcal{P}$).

Now we state the localization theorem for equivariant Chow cohomology. It is applicable to possibly singular complete T-schemes, regardless of whether $\mathrm{op}A^*_T(X)$ is a free S-module or not.

Theorem 3.1 ([13, Theorem A.6]). *Let X be a complete T-scheme and let $i_T : X^T \to X$ be the inclusion of the fixed point subscheme. Then the pull-back map*

$$i_T^* : \mathrm{op}A^*_T(X) \to \mathrm{op}A^*_T(X^T)$$

is injective, and its image is exactly the intersection of the images of

$$i_{T,H}^* : \mathrm{op}A^*_T(X^H) \to \mathrm{op}A^*_T(X^T),$$

where H runs over all subtori of codimension one in T. □

Theorem 3.1 makes equivariant Chow cohomology more computable. For instance, a version of GKM theory also holds [13, Theorem A.9], and there is a description of the equivariant operational Chow groups of spherical varieties [13, Section 4], which generalizes [4, Theorem 7.3].

Let X be a T-scheme, and let (V, U) be as in Subsection 2.2. By [8, Corollary 2], there is an isomorphism $\mathrm{op}A^j_T(X) \simeq \mathrm{op}A^j(X \times U/T)$, provided $V \setminus U$ has codimension more than j. Thus there is a canonical map $i^* : \mathrm{op}A^*_T(X) \to \mathrm{op}A^*(X)$ induced by restriction to a fiber of $p_{X,T} : X_T \to U/T$. But, unlike the case of equivariant Chow groups, this map is not surjective in general, and its kernel is not necessarily generated in degree one, not even for toric varieties [22]. This becomes an issue when trying to translate results from equivariant to non-equivariant Chow cohomology. Nevertheless, for certain $\mathbb{Q}$-filtrable varieties the map i^* is surjective. Before studying them, let us recall a definition from [13].

Definition 3.2. Let X be a complete T-scheme. We say that X **satisfies T-equivariant Kronecker duality** if the following conditions hold.

(i) The S-module $A_*^T(X)$ is finitely generated.

(ii) The equivariant Kronecker duality map

$$\mathcal{K}_T : \mathrm{op}A_T^*(X) \longrightarrow \mathrm{Hom}_S(A_*^T(X), S), \quad \alpha \mapsto (\beta \mapsto \int_X (\beta \cap \alpha)).$$

is an isomorphism of S-modules.

If, in addition, the ordinary Kronecker duality map $\mathcal{K}$ is also an isomorphism, then we say that X satisfies the **strong** T-equivariant Kronecker duality.

Example 3.3. By [13, Lemma 3.3], a nonsingular projective T-variety satisfies the T-equivariant Kronecker duality if and only if it satisfies ordinary Kronecker duality. In particular, a projective smooth curve of positive genus (with any T-action) does not satisfy T-equivariant Kronecker duality, for the kernel of $\mathcal{K}$ in degree one is the Jacobian of the curve [10].

The main result on equivariant Kronecker duality needed here is a consequence of [10], [32], and [13, Theorem 3.6].

Theorem 3.4. *Let X be a complete T-linear scheme. If X has an ample T-linearized invertible sheaf (e.g., if X is a nonsingular projective T-variety with isolated fixed points or X is a possibly singular projective spherical variety), then X satisfies the strong T-equivariant Kronecker duality.* □

The next result makes $\mathbb{Q}$-filtrations relevant to the study of the equivariant Chow cohomology of T-schemes. The proof is an easy adaptation of [13, Corollary 3.9].

Theorem 3.5. *Let X be a complete T-scheme. If X satisfies the strong T-equivariant Kronecker duality and $A_*^T(X)$ is S-free, then the S-module $\mathrm{op}A_T^*(X)$ is free, and the map*

$$\mathrm{op}A_T^*(X)/\Delta\mathrm{op}A_T^*(X) \to \mathrm{op}A^*(X)$$

is an isomorphism. □

Corollary 3.6. *Let X be a complete T-linear variety having an ample T-linearized invertible sheaf. If X is $\mathbb{Q}$-filtrable, then the S-module $\mathrm{op}A_T^*(X)$ is free, and we have $\mathrm{op}A_T^*(X)/\Delta\mathrm{op}A_T^*(X) \simeq \mathrm{op}A^*(X)$.* □

To conclude this section, we present two results which motivate our quest for conditions guaranteeing (equivariant) Poincaré duality.

Theorem 3.7. *Let G be a complex connected reductive group with maximal torus T. Let X be a projective complex G-spherical variety. If X is equivariantly formal, then there is a natural isomorphism $\mathrm{op}A_T^*(X) \simeq H_T^*(X)$ of S-algebras, and we get $\mathrm{op}A_T^*(X)/\Delta\mathrm{op}A_T^*(X) \simeq H^*(X)$. In particular, the S-module $\mathrm{op}A_T^*(X)$ is free. If, moreover, X is $\mathbb{Q}$-filtrable, then we get $\mathrm{op}A^*(X) \simeq H^*(X)$.*

Proof. By Theorem 3.1, the pullback $i_T^* : \mathrm{op}A_T^*(X) \to \mathrm{op}A_T^*(X^T)$ is injective, and its image $\mathrm{im}(i_T^*)$ is described explicitly in [13, Theorem 4.8]. On the other hand, because X^T is finite, we get a canonical identification $\mathrm{op}A_T^*(X^T) \simeq H_T^*(X^T)$. Since X is equivariantly formal, the pullback $\widetilde{i_T}^* : H_T^*(X) \to H_T^*(X^T)$ is also injective. Moreover, X^H is equivariantly formal for any codimension-one subtorus $H \subset T$. Hence, using [13, Subsection 4.2] and the localization theorem for equivariant cohomology [16], one easily checks the equality $\mathrm{im}(i_T^*) = \mathrm{im}(\widetilde{i_T}^*)$. Consequently, we obtain $\mathrm{op}A_T^*(X) \simeq H_T^*(X)$. This, together with equivariant formality, yields $\mathrm{op}A_T^*(X)/\Delta\mathrm{op}A_T^*(X) \simeq H_T^*(X)/\Delta H_T^*(X) \simeq H^*(X)$. Finally, the last assertion follows from Theorem 3.5. □

Proposition 3.8. *Let M be a reductive monoid with zero. If M is quasismooth (Definition 2.16), then $\mathrm{op}A_{T\times T}^*(\mathbb{P}_\epsilon(M))$ is a free S-module,* *and it is isomorphic, as an S-algebra, to the ring of piecewise polynomial functions $PP_{T\times T}(\mathbb{P}_\epsilon(M))$ associated to the GKM graph of $\mathbb{P}_\epsilon(M)$. Furthermore, if $\Bbbk = \mathbb{C}$, then we have $\mathrm{op}A^*_{T\times T}(\mathbb{P}_\epsilon(M)) \simeq H^*_{T\times T}(\mathbb{P}_\epsilon(M))$ and thus also $\mathrm{op}A^*(X) \simeq H^*(X)$.*

Proof. The first part follows from Theorem 2.17, Corollary 3.6 and [13, Theorem A.9]. For the second one, use Theorem 3.7. □

For a description of the GKM graph of $\mathbb{P}_\epsilon(M)$ see [12].

4. Equivariant Poincaré duality and Chow homology Betti numbers

The goal of this section is to show that $\mathbb{Q}$-filtrable T-linear varieties are analogues of the equivariantly formal spaces of Goresky, Kottwitz, and MacPherson [16] from the viewpoint of equivariant operational Chow groups.

Let X be a projective T-variety of pure dimension n. Suppose that X is $\mathbb{Q}$-filtrable. Then, by Theorem 2.13, $A^T_*(X)$ is a free S-module of finite rank and $A_*(X)$ is a free $\mathbb{Q}$-vector space of finite dimension. Now set $b_k = \dim_{\mathbb{Q}} A_k(X)$, and call it the k-th **Chow homology Betti number** of X. It follows from Theorem 2.13 that b_k equals the number of k-dimensional algebraic rational cells. When X is smooth, these cells are actually affine spaces, and we get $b_k = b_{n-k}$ [2, Corollary 1]. Moreover, Poincaré duality holds, and all equivariant multiplicities are non-zero (Theorem 2.7). In the singular case this is not necessarily true, and our motivation for this section is to determine in which cases the identity $b_k = b_{n-k}$ holds. Is this equivalent to Poincaré duality for the Chow cohomology of X? Could it be studied via equivariant multiplicities? Notice that these multiplicities played a fundamental role in Section 2. In equivariant cohomology and for equivariantly formal varieties these questions have been answered in [6]. Below we provide some analogues of the results of [6] in equivariant Chow cohomology. Our methods rely on Theorem 3.1, Theorem 3.5, and the notion of algebraic rational cells. No comparison via the cycle map is needed.

A first approximation to Poincaré duality via equivariant multiplicities is given next. For the corresponding statement in equivariant cohomology, see [6, Theorem 4.1].

Theorem 4.1. *Let X be a complete equidimensional T-scheme with only finitely many fixed points. If all equivariant multiplicities are non-zero, then the equivariant Poincaré duality map is injective.*

Proof. In view of Theorem 3.1, the argument is the same as that of [6, Theorem 4.1]. We include it for convenience. Let $\alpha \in \mathrm{op}A_T^*(X)$ and suppose that $\alpha \cap [X] = 0$. Then we have

$$\int_X (\alpha \cup \beta) \cap [X] = 0$$

for all $\beta \in A_T^*(X)$. Thus, in $\mathcal{Q}$, we obtain

$$\sum_{x \in X^T} \alpha_x \beta_x e_T(x, X) = 0.$$

By the localization theorem, the identity holds for all sequences $(\beta_x)_{x \in X^T}$ in $\mathcal{Q}$. Since, by assumption, no $e_x[X]$ vanishes, we must have $\alpha_x = 0$ for all $x \in X^T$. Thus we get $\alpha = 0$ (for the map $i_T^* : \mathrm{op}A_T^*(X) \to \mathrm{op}A_T^*(X^T)$ is injective). □

Remark 4.2. Theorem 4.1 applies to: (i) projective nonsingular T-varieties with isolated fixed points, for then the equivariant multiplicities are all inverses of polynomials (Theorem 2.7); (ii) Schubert varieties and toric varieties, as they have only attractive fixed points, so Theorem 2.9 implies that the corresponding equivariant multiplicities are non-zero; (iii) simple projective embeddings of a connected reductive group G, as they have only one closed $G \times G$-orbit, and $W \times W$ acts transitively on the $T \times T$-fixed points (at least one of these is attractive, hence so are all of them).

We now combine our previous results to produce a criterion for Poincaré duality. For equivariantly formal varieties and equivariant cohomology this was done in [6, Theorem 4.1].

Theorem 4.3. *Let X be a complete equidimensional T-variety with isolated fixed points. Suppose*

(a) X is $\mathbb{Q}$-filtrable and

(b) X satisfies the strong T-equivariant Kronecker duality.

Then the following conditions are equivalent.

(i) X satisfies Poincaré duality.

(ii) X satisfies T-equivariant Poincaré duality.

(iii) The Chow homology Betti numbers of X satisfy $b_q(X) = b_{n-q}(X)$, for $0 \leq q \leq n$, and all equivariant multiplicities are nonzero.

If any of these conditions holds, then all equivariant multiplicities are inverses of polynomial functions.

Proof. Assumptions (a) and (b) imply that the S-modules $A_*^T(X)$ and $\mathrm{op}A_T^*(X)$ are free. So the equivalence of (i) and (ii) follows readily from Theorem 3.5 and the graded Nakayama lemma.

We prove that (ii) implies (iii). It only remains to show that all the equivariant multiplicities are nonzero. For this, let $\{[\overline{W_1}], \ldots, [\overline{W_m}]\}$ be the basis of $A_*^T(X)$ consisting of the closures of the algebraic rational cells. Fix $j \in \{1, \ldots, m\}$, and let x_j be the unique attractive fixed point of W_j. By (ii) there is a unique $\alpha \in \mathrm{op}A_T^*(X)$ for which we have

$$\alpha \cap [X] - [\overline{W_j}] = 0.$$

But then, arguing as in the proof of Theorem 4.1, the identity

$$\sum_{x_i \in X^T} \beta_{x_i}(\alpha_{x_i} e_{x_i}[X] - e_{x_i}[\overline{W_j}]) = 0$$

holds for all sequences $(\beta_{x_i})_{x_i \in X^T}$ in $\mathcal{Q}$. In particular, we have

$$\alpha_{x_j} e_{x_j}[X] - e_{x_j}[\overline{W_j}] = 0.$$

Since x_j is an attractive fixed point of $\overline{W_j}$, we get $e_{x_j}[\overline{W_j}] \neq 0$. This yields $\alpha_{x_j} \neq 0$ and $e_{x_j}[X] \neq 0$, so that $e_{x_j}[X]$ is the inverse of a polynomial. Indeed, we have $e_{x_j}[\overline{W_j}] = \frac{d}{\prod_{s=1}^{\ell} \chi_s}$ (Theorem 2.14) and $\alpha_{x_j} \in S$.

Now is the turn for (iii) implies (i). In view of Theorem 4.1, it remains to show that

$$\mathcal{P}_T : \mathrm{op}A_T^q(X) \to A_{n-q}^T(X)$$

is surjective for all $q \in \mathbb{Z}$. For this, it suffices to show that the dimension of $\mathrm{op}A_T^q(X)$ matches that of $A_{n-q}^T(X)$. But this follows from the assumption on the Chow homology Betti numbers combined with the isomorphisms

$$\mathrm{op}A_T^*(X) \simeq \mathrm{op}A^*(X) \otimes_{\mathbb{Q}} S \quad \text{and} \quad A_*^T(X) \simeq A_*(X) \otimes_{\mathbb{Q}} S,$$

where the first one is granted by Theorem 3.5. □

Remark 4.4. It is worth noting that Kronecker duality does not imply Poincaré duality. For instance, consider the following example from [10, page 184]. Let X be the closure of a generic torus orbit in the Grassmannian $G(2,4)$. Then X is a toric variety with Chow homology groups $\mathbb{Q}$, $\mathbb{Q}$, $\mathbb{Q}^5$, and $\mathbb{Q}$ in dimensions 0, 1, 2, and 3. By Kronecker duality, the Chow cohomology groups are $\mathbb{Q}$, $\mathbb{Q}$, $\mathbb{Q}^5$, and $\mathbb{Q}$ in codimensions 0, 1, 2, and 3. Clearly, the Poincaré duality maps $A^k \to A_{3-k}$ are not isomorphisms.

It stems from Theorem 4.3 that the class of $\mathbb{Q}$-filtrable varieties satisfying the strong T-equivariant Kronecker duality indeed resembles that of equivariantly formal spaces [16]. To push the analogy even further, here is a version of the Morse inequalities for these varieties. For the analogous result in equivariant cohomology, see [6, Theorem 4.2].

Theorem 4.5. *Let X be a T-quasiprojective T-linear variety of pure dimension n with isolated fixed points. If X is complete, $\mathbb{Q}$-filtrable, and all equivariant multiplicities are nonzero, then the following inequalities hold for the Chow homology Betti numbers:*

$$b_q + b_{q-1} + \ldots + b_0 \leq b_{n-q} + b_{n-q+1} + \ldots + b_n,$$

for $0 \leq q \leq n$, and

$$2b_1 + 4b_2 + \ldots + 2nb_n \geq n\chi(X),$$

where $\chi(X) = b_0 + b_1 + \ldots + b_n$ is the Euler characteristic, i.e., the number of algebraic rational cells of X. In fact, we get $\chi(X) = |X^T|$. Furthermore, X satisfies Poincaré duality if and only if

$$2b_1 + 4b_2 + \ldots + 2nb_n = n\chi(X).$$

Proof. The proof is an easy adaptation of [6, Theorem 4.2], with a few changes. First note that, as X is T-linear, it is also $\mathbb{G}_m$-linear, where $\mathbb{G}_m$ acts on X via the generic one-parameter subgroup $\lambda : \mathbb{G}_m \to T$ chosen to obtain the $\mathbb{Q}$-filtration. Secondly, since X has an ample T-linearized invertible sheaf, and this sheaf is clearly $\mathbb{G}_m$-linearized, then Theorem 3.4 implies that X satisfies the strong $\mathbb{G}_m$-equivariant Kronecker duality. Thus, by Theorem 3.5, we have $\mathrm{op}A^*_{\mathbb{G}_m}(X) \simeq \mathrm{op}A^*(X) \otimes_{\mathbb{Q}} \mathbb{Q}[t]$ and $A_*^{\mathbb{G}_m}(X) \simeq A_*(X) \otimes_{\mathbb{Q}} \mathbb{Q}[t]$, as graded vector spaces, where t is an indeterminate of degree 1. On the other hand, since the $e_x[X]$ are nonzero, the same holds for $e'_x[X]$, the $\mathbb{G}_m$-equivariant multiplicity at x, by [4, Lemma 4.5]. It follows that the map

$$\mathcal{P}_{\mathbb{G}_m} : opA^q_{\mathbb{G}_m}(X) \to A^{\mathbb{G}_m}_{n-q}(X), \quad z \mapsto z \cap [X],$$

is injective for all $q \in \mathbb{Z}$. In view of these results, Brion's argument from [6, Proof of Theorem 4.2] applies verbatim, yielding the result. □

Example 4.6. Let M be a quasismooth monoid, and consider the associated projective embedding $X = \mathbb{P}_\epsilon(M)$. Suppose that X has a unique closed $G\times G$-orbit (i.e., X is **simple**). By the calculations of [26] we get $b_k = b_{n-k}$. Since all the $T\times T$-fixed points in X are attractive, then by Theorem 4.3, X satisfies Poincaré duality for Chow cohomology. Over the complex numbers this reflects the fact that X is rationally smooth. Once more, we point out that the cycle map was not needed in our arguments.

Corollary 4.7. *Let X be a projective G-spherical variety. Let* $\mathrm{Pic}(X)$ *(respectively* $\mathrm{Cl}(X)$*) denote the Picard group (respectively Class group) of X with rational coefficients. If X is $\mathbb{Q}$-filtrable and satisfies Poincaré duality, then* $\mathrm{Pic}(X)\simeq\mathrm{Cl}(X)$.

Proof. Because X is normal, the natural map $\mathrm{Pic}(X)\to\mathrm{Cl}(X)$ is injective. But both, source and target, are finite dimensional vector spaces, so in order to obtain the result it suffices to show that they have the same dimension. By [10], we have $\mathrm{Pic}(X)\simeq \mathrm{op}A^1(X)\simeq \mathrm{Hom}(A_1(X),\mathbb{Q})$. The dimension of the latter vector space is b_1, which equals b_{n-1}, by Poincaré duality. Since $A_{n-1}(X)\simeq\mathrm{Cl}(X)$, the proof is complete. □

Remark 4.8. If X is a complex variety with rational singularities, then we have $Pic(X)\simeq \mathrm{op}A^1(X)$, by [21, Prop. 12.1.4]. So, the conditions of Corollary 4.7 could be slightly relaxed in that case.

Corollary 4.7 admits a combinatorial interpretation in the case of simple group embeddings. Let $X=\mathbb{P}_\epsilon(M)$ be a projective group embedding. Recall that $\Lambda\setminus\{0\}$ indexes the $G\times G$-orbits of X (see Subsection 2.7 for notation). If X is simple, then the unique closed orbit of X is a projective homogeneous variety $G/P_J\times G/P_J^-$, where $J\subset\mathfrak{S}$, P_J is a standard parabolic subgroup, and P_J^- is its opposite (see e.g. [24]). Remarkably, Λ is completely determined by J and the Dynkin diagram of G [25, Section 7.3]. For instance, we have $\Lambda_2\simeq\mathfrak{S}\setminus J$. On the other hand, notice that the number of $G\times G$-stable divisors of X is $|\Lambda_{d-1}|$, where $d=\dim T$.

Next we give a qualitative relation between Λ_{d-1} and J.

Corollary 4.9. *Let M be a quasismooth monoid. If $X = \mathbb{P}_\epsilon(M)$ is simple, then $|\Lambda_{d-1}| = |\mathfrak{S} \setminus J|$.*

Proof. By Theorem 2.17 and Corollary 4.7 we have $\mathrm{Pic}(X) \simeq \mathrm{Cl}(X)$. Since $\mathrm{Cl}(X)$ is freely generated by the $G \times G$-stable divisors of X (since $Cl(G)_{\mathbb{Q}} = 0$), we get $\dim_{\mathbb{Q}} \mathrm{Cl}(X) = |\Lambda_{d-1}|$. Finally, by a result of Brion (see e.g. [23]) the Picard group of X is freely generated by those $B \times B$-stable irreducible divisors which do not contain $G/P_J \times G/P_J^-$. But these correspond to $\mathfrak{S} \setminus J$, by [25, Theorem 5.1]. □

For a complete list of all J's that yield quasismooth monoids M, see [27]. Corollary 4.9 states that Poincaré duality is reflected on the poset structure of the $G \times G$-orbits.

Final remarks

Let X be a complete equidimensional T-variety with isolated fixed points. If all equivariant multiplicities are nonzero (e.g., all fixed points are attractive), then due to Theorem 4.1 the equivariant Poincaré duality map is injective. Thus we get $\mathrm{op}A_T^*(X) \subseteq A_*^T(X)$. An interesting open problem is to describe $\mathrm{op}A_T^*(X)$ as a subgroup of $A_*^T(X)$ in terms of T-invariant cycles. Notice that $\mathrm{op}A_T^*(X)$ carries an additional ring structure. A related task is to assess the effect of this "abstract" product on the associated (geometric) cycles. Solutions to these problems will yield a geometric interpretation of operational Chow groups, at least in the cases of Example 4.2 and those where Poincaré duality holds (Theorem 4.3). Applications to equivariant operational K-theory ([1], [15]) are also envisioned. Notice that all the analysis can be carried out intrinsically using the tools developed in Section 4 and the rich structure of equivariant Chow groups (there is no need for comparing with equivariant cohomology). This will be pursued elsewhere.

References

[1] Anderson, D. Payne, S.; *Operational K-theory.* arXiv:1301.0425v2.

[2] Bialynicki-Birula, A.; *Some theorems on actions of algebraic groups.* The Annals of Mathematics, 2nd Ser., Vol 98, No. 3, Nov. 1973, pp. 480-497.

[3] Bialynicki-Birula, A.; *Some properties of the decompositions of algebraic varieties determined by actions of a torus.* Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 24 (1976), no. 9, 667-674.

[4] Brion, M.; *Equivariant Chow groups for torus actions.* Transf. Groups, vol. 2, No. 3, 1997, pp. 225-267.

[5] Brion, M.; *Rational smoothness and fixed points of torus actions.* Transf. Groups, Vol. 4, No. 2-3, 1999, pp. 127-156.

[6] Brion, M.; *Poincaré duality and equivariant cohomology.* Michigan Math. J. 48, 2000, pp. 77-92.

[7] Brion, M.; *Local structure of algebraic monoids.* Mosc. Math. J. 8 (2008), no. 4, 647-666, 846.

[8] Edidin, D., Graham, W.; *Equivariant Intersection Theory.* Invent. Math. 131, 595-634 (1998).

[9] Fulton, W.; *Intersection Theory.* Springer-Verlag, Berlin, 1984.

[10] Fulton, W., MacPherson, R., Sottile, F., Sturmfels, B. *Intersection theory on spherical varieties.* J. of Algebraic Geometry, 4 (1994), 181-193.

[11] Gonzales, R.; *Rational smoothness, cellular decompositions and GKM theory.* Geometry and Topology, Vol. 18, No. 1 (2014), 291-326.

[12] Gonzales, R.; *Equivariant cohomology of rationally smooth group embeddings*. Submitted. arxiv:1112.1365.

[13] Gonzales, R.; *Equivariant operational Chow rings of T-linear schemes*. Submitted. arXiv:1304.3645v4.

[14] Gonzales, R.; *Algebraic rational cells, equivariant intersection theory, and Poincaré duality*. Max-Planck-Institut für Mathematik Preprint, 2014-21. arXiv:1404.2486.

[15] Gonzales, R.; *Localization in equivariant operational K-theory and the Chang-Skjelbred property*. Max-Planck-Institut für Mathematik Preprint, 2014-20. arXiv:1403.4412.

[16] Goresky, M., Kottwitz, R., MacPherson, R.; *Equivariant cohomology, Koszul duality, and the localization theorem.* Invent. math. 131, 25-83 (1998)

[17] Jannsen, U.; *Mixed motives and algebraic K-theory.* Lecture Notes in Mathematics 1400, Springer-Verlag, New York, 1990.

[18] Joshua, R.; *Algebraic K-theory and higher Chow groups of linear varieties*. Math. Proc. Cambridge Phil. Soc., 130, pp. 37-60.

[19] Joshua, R. Krishna, A.; *Higher K-theory of toric stacks.* http://arxiv.org/abs/1210.1057.

[20] Konarski, J.; *Decompositions of normal algebraic varieties determined by an action of a one-dimensional torus.* Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 26 (1978), no. 4, 295-300.

[21] Kollár J.; Mori, S.; *Classification of three-dimensional flips*, J. AMS 5 (1992), 533-703.

[22] Katz, E.; Payne, S.; *Piecewise polynomials, Minkowski weights, and localization on toric varieties.* Algebra Number Theory 2 (2008), no. 2, 135-155.

[23] Perrin, N.; *On the geometry of spherical varieties.* Transform. Groups 19 (2014), no. 1, 171-223.

[24] Renner, L.; *Classification of semisimple varieties.* J. of Algebra, vol. 122, No. 2, 1989, 275-287.

[25] Renner, L.; *Linear algebraic monoids.* Encyclopedia of Math. Sciences, vol. 134, Springer, Berlin (2005).

[26] Renner, L.; *The H-polynomial of a semisimple monoid.* J. Alg. 319 (2008), 360-376.

[27] Renner, L. *Descent systems for Bruhat posets.* J. of Alg. Combinatorics, 29 (2009), 413-435.

[28] Renner, L.; *Rationally smooth algebraic monoids.* Semigroup Forum 78 (2009), 384-395.

[29] Rittatore, A.; *Algebraic monoids and group embeddings.* Transformation Groups, Vol. 3, No. 4, 1998, pp. 375-396.

[30] Sumihiro, H.; *Equivariant completion.* J. Math. Kyoto University 14 (1974), 1-28.

[31] Timashev, D.; *Homogeneous spaces and equivariant embeddings.* Encyc. Math. Sci. 138, Springer.

[32] Totaro, B.; *Chow groups, Chow cohomology and linear varieties.* Forum of Mathematics, Sigma (2014), Vol. 2, e17 (25 pages).

Resumen

En este artículo estudiamos el homomorfismo de dualidad de Poincaré, el cual relaciona cohomología de Chow equivariante y grupos de Chow equivariante en aquellos casos donde un toro algebraico actúa sobre una variedad singular compacta y con puntos fijos aislados. Nuestros resultados proporcionan criterios bajo los cuales el homomorfismo de dualidad de Poincaré es un isomorfismo. Para ello, usamos el teorema de localización en cohomología de Chow equivariante y la noción de célula algebraica racional. Aplicamos nuestros resultados a las variedades esféricas compactas y sus generalizaciones.

Palabras clave: Grupos de Chow, acciones tóricas, descomposiciones celulares, dualidad de Poincaré, variedades esféricas.

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Duality on 5-dimensional S^1-Seifert bundles

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Abstract

We describe a correspondence between two different links associated to the same K3 orbifold. This duality is produced when two elements, one inside and the other on the boundary of the Kähler cone, are identified. We call this correspondence ∂-*duality.* We also discuss the consequences of ∂-duality at the level of metrics.

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1. Introduction

Orbifolds are geometric objects that can be realized as quotients of Riemannian manifolds by isometries. However, in doing so some finite proper subgroup of the isometries may fix some points. Sometimes they come about as space forms, others in more subtle situations, for instance, as compactifications of moduli spaces. Usually one would like to construct fiber bundles over orbifolds in such a manner that the total space desingularizes geometric data arising from the orbifold. Obviously, additional conditions on both the singularities and the geometric structure on the orbifold should bring additional benefits. In recent years, an specific type of fibration, namely *Riemannian orbifold submersions*, has been intensively studied to establish the existence of Einstein or η-Einstein metrics on manifolds that do not admit special holonomy groups. More precisely, in [4, 6, 7, 8], attention is given to S^1-principal orbibundles over orbifold Kähler surfaces. These bundles can be viewed as a type of real resolution of the singularities of the corresponding orbifold. Under certain conditions, this sort of resolution ends up being the link of the associated Milnor fiber. On the other hand, there is an algebraic geometric procedure to resolve the singularities, namely, minimal models associated to an algebraic variety. The affinity between the resolution and the Milnor fiber, through the link, is well understood in complex dimension 1, where the complex algebraic curve is studied via its 2-dimensional affine cone. This interplay has produced one of the most important relations in singularity theory for surfaces. The extension of this picture to complex dimension 2 brought some remarkable results from Orlik and Wagreich [26]. However, there is no precise description of the relationship of these two types of resolutions at this level.

In this article, we describe a correspondence between two different links associated to the same K3 orbifold. This duality is produced when two elements, one in the interior and the other on the boundary of the Kähler cone, are identified. We call this correspondence ∂-*duality.* We also discuss the consequences of ∂-duality at the level of metrics. It will be interesting to determine whether this duality can be interpreted in terms of the Milnor fiber and the minimal model of a K3 surface. This possible connection is encrypted in a variation of the Dynkin diagram of the resolution and its associated equivariant plumbing manifold. This questions –and others– are part of joint work in progress with R. Gonzales.

Here is a concise outline of the paper. In Section 2 some preliminaries on Riemannian orbifolds are presented, avoiding singularities of codimension 1 (due to a significant result of Kollar [18]). We also explain the Hopf map in the context of orbifold Riemannian submersions (a snapshot that will be recurrent on this article). In Section 3, we give a brief introduction to projective Kähler orbifolds, we also exhibit some canonical examples and review some important recent results of Boyer and Galicki relating Kähler geometry and Sasakian geometry. In Section 4, we introduce K3 surfaces and some of their properties, and discuss S^1-orbibundles over K3 surfaces. We conclude the paper in Section 5 explaining ∂-duality and discussing some consequences for the Riemannian metrics associated to the dual pairs.

2. Orbifolds and orbifold Riemannian submersions.

The notion of an orbifold was introduced by Satake in [29] under the name V-manifold. The symbol 'V' in that context indicated the cone-like singularity he was dealing with. In the late seventies Thurston rediscovered the concept of V-manifolds under the name orbifold in his study of the geometry of 3-manifolds [34].

In the sequel $\mathbb{K}$ denotes either $\mathbb{C}$ or $\mathbb{R}$. Let $O(m) = O(m, \mathbb{K})$ be the orthogonal group, and let B_r be the open ball of radius r in $\mathbb{K}^m$ centered at the origin. If G is a subgroup of $O(m)$, then G acts by isometries on B_r; let B_r/G be the associated quotient space.

A compact metric space M is said to be an **orbifold** if every point $p \in M$ has an open neighborhood U_p which is homeomorphic to $B_{r(p)}/G_p$ for some $r(p) > 0$ and some finite subgroup $G_p \subset O(m)$; the groups G_p are called **local uniformizing groups.** Set $\tilde{U}_p = B_{r(p)}$ and let

$$\rho_p : \tilde{U}_p \to \tilde{U}_p/G_p = U_p$$

be the natural projection. A point q of $\tilde{U}_p$ whose isotropy subgroup $\Gamma_q \subset G_p$ is non-trivial is called a **singular** point of $\tilde{U}_p$. The set of all singular points of $\tilde{U}_p$ is denoted by Σ_p. Then the map

$$\rho_p : \tilde{U}_p \backslash \Sigma_p \to U_p \backslash \rho_p(\Sigma_p)$$

is a covering projection. The **singular set** or **orbifold singular locus** of M is defined to be

$$\Sigma(M) = \bigcup_{p \in M} \{\rho_p(\Sigma_p)\}.$$

Note that Σ_p is the union of a finite number of linear subspaces of $\tilde{U}_p$. In this paper the on going assumption is that these subspaces all have codimension at least 2.

We will say that M is a **smooth (or complex) orbifold** if $M \backslash \Sigma(M)$ has the structure of a smooth (or complex) manifold and the maps ρ_p from $\tilde{U}_p \backslash \Sigma_p$ to $U_p \backslash \rho_p(\Sigma_p)$ are local diffeomorphisms (or local biholomorphisms). The orbifold is a **smooth (or complex) manifold** if $\Sigma(M)$ is empty or, equivalently, if $G_p = \{e\}$ for every $p \in M$.

A **Riemannian metric** g on an orbifold M is a Riemannian metric g_p on every $\tilde{U}_p$ that is invariant under the action of G_p and such that each ρ_p is a local isometry from $\tilde{U}_p \backslash \Sigma_p$ to $U_p \backslash \rho_p(\Sigma_p)$. Similarly, one defines a **Hermitian metric** h for complex orbifolds, but this time one requires the maps ρ_p to be local Hermitian isometries.

Remark 2.1. In general one talks about tensors on orbifolds, defining them on the complement of the singular set of the orbifold X (assumed to be of codimension at least 2): a **tensor** θ on an orbifold X is a tensor θ_{ns} on $X\backslash\Sigma(X)$ such that for every chart $\rho_p : \tilde{U}_p \to \tilde{U}_p/G_p = U_p$ the pullback $\rho_p{}^*(\theta_{ns})$ extends to a tensor on $\tilde{U}_p$. Therefore the notions of curvature, Kähler metrics, and Kähler-Einstein metrics on orbifolds make sense (see [7] for details).

The following proposition is a slight variation of the partition of unity argument (see [24] for details).

Proposition 2.2. *Every orbifold admits a Riemannian metric, and every complex orbifold admits a Hermitian metric.* □

Remark 2.3. The definition of smooth (or complex) orbifold given here avoids several technical subtleties that will not affect the subsequent arguments; the skeptical reader is encouraged to seek out other sources (*e.g.* [25]).

Recall that a **submersion** in the smooth setting is a (smooth) map $\pi : M \to B$ of closed Riemannian manifolds (M, g) and (B, g_B) with maximal rank. It follows that for $z \in M$ the tangent space T_zM splits as $V_z \bigoplus H_z$, where

$$V_z = \ker(\pi_{*z}) \quad \text{and} \quad H_z = V_z^{\perp}$$

are the **vertical** and **horizontal** spaces, respectively. If, additionally, π_* is an isometry from H_z to $T_{\pi(z)}B$, one says that π is a **Riemannian submersion**.

Now we briefly explain how to extend this notion to the singular setting. First, consider the extension of charts given previously and consider the (usual) action of G_p on $\tilde{U} \times F$, where F is a closed smooth G_p-manifold, given by

$$\gamma.(\tilde{u}, x) = (\gamma\tilde{u}, \gamma(\tilde{u})x) \text{ for } \gamma \in G_p. \tag{2.1}$$

This action is referred to as $\tilde{U} \times_{G_p} F$. We use this notation in the next definition.

Definition 2.4. Let X and Y orbifolds and let F be a smooth manifold. One says that $\pi : Y \to X$ is a **fiber orbibundle** with fiber F if one can choose charts $\tilde{U}_x/G_x$ in X and charts $\tilde{U}_x \times_{H_x} F$ over Y such that the following diagram

$$\begin{array}{ccc} \tilde{U}_x \times F & \xrightarrow{\rho_x^Y} & \tilde{U}_x \times_{H_x} F \\ \downarrow \tilde{\pi} & & \downarrow \pi \\ \tilde{U}_x & \xrightarrow{\rho_x} & U_x = \tilde{U}_x/G_x \end{array} \tag{2.2}$$

commutes. Here H_x is a subgroup of G_x, $\tilde{\pi}$ and π are projections onto the first factor, and the projections ρ_x^Y and ρ_x are the associated quotient maps. We say that π is a **Riemannian orbifold submersion** if additionally $\tilde{\pi}$ is a Riemannian submersion.

In general the Riemannian metric on $\tilde{U}_x \times F$ is not a product metric, and the decomposition under discussion works only locally.

Remark 2.5. It must be understood that the orbifold fibration is not a fibration in the usual sense. However one can think of this object as a fibration rationally, that is, such that certain tensor power of the fiber is indeed a conventional fiber.

If the fiber F is a vector space of dimension r and all the uniformizing groups act on F as linear transformations, then the orbibundle is called a **vector orbibundle** of **rank** r. If the rank of the vector orbibundle equals 1, we talk about **line orbibundles**. Similarly, if F is a Lie group G, then the orbibundle is called a **principal orbibundle.** Of particular interest is the case when all the uniformizing groups G_x are subgroups of the Lie group G that act freely on the fiber, in which case the total space ends up being a smooth manifold.

Now, let us briefly recall the Hopf fibration in order to generate some examples of orbifold Riemannian submersions. Consider the sphere S^{2n+1} in $\mathbb{C}^{n+1}$ centered at the origin, and let z be the unit outward normal. Let J be the natural almost complex structure. Then Jz defines an integral distribution on S^{2n+1} with totally geodesic leaves. Identifying the leaves as points, one obtains the complex projective space $\mathbb{CP}^n$. The horizontal distribution can be taken to be the orthogonal complement to Jz in the tangent bundle TS^{2n+1}, and one can turn this into a Riemannian submersion, known as the **Hopf fibration** $h : S^{2n+1} \to \mathbb{CP}^n$ with fibers given by great circles.

We explain this fibration with some detail for $n = 1$. The unit sphere S^k is given the standard metric g_k of constant sectional curvature $+1$. We regard $S^3 \subset \mathbb{C}^2$ and $S^2 \subset \mathbb{C} \oplus \mathbb{R}$. The Hopf fibration $H : S^3 \to S^2$ is defined via the rule

$$H(z_1, z_2) = (2z_1\bar{z}_2, |z_1|^2 - |z_2|^2).$$

It is not difficult to show that if two points (z_1, z_2) and (z_3, z_4) in the sphere project to the same point, then there is an $r \in S^1$ such that $(z_1, z_2) = r(z_3, z_4)$. Then the fibers are **Hopf circles**, that is, are orbits of points of S^3 under the S^1 action $(z, w) \mapsto (e^{i\theta}z, e^{i\theta}w)$. The O'Neill formulae [20] show that S^2 is a half-radius sphere of constant sectional curvature equal to 4. Hence $H : (S^3, g_3) \to (S^2, \frac{1}{4}g_2)$ is a Riemannian submersion.

Example 2.6. Let $\mathbb{Z}/n\mathbb{Z}$ be the group of n roots of unity. Consider the actions

$$\begin{array}{rcl} \rho_2 : \mathbb{Z}/n\mathbb{Z} \times S^2 & \longrightarrow & S^2 \\ (\gamma, w, t) & \longmapsto & (\gamma w, t) \end{array}$$

and

$$\begin{array}{rcl} \rho_3 : \mathbb{Z}/n\mathbb{Z} \times S^3 & \longrightarrow & S^3 \\ (\gamma, z_1, z_2) & \longmapsto & (\gamma z_1, z_2). \end{array}$$

These two actions (performing as isometries) turn $S^2/\rho_2(\mathbb{Z}/n\mathbb{Z})$ and $S^3/\rho_3(\mathbb{Z}/n\mathbb{Z})$ into Riemannian orbifolds. Moreover, as the group actions are compatible with the Hopf fibration H, one obtains the following commutative diagram:

$$\begin{array}{ccc} S^3 & \xrightarrow{\pi_3} & S^3/\rho_3(\mathbb{Z}/n\mathbb{Z}) \\ \Big\downarrow H & & \Big\downarrow \tilde{H} \\ S^2 & \xrightarrow{\pi_2} & S^2/\rho_2(\mathbb{Z}/n\mathbb{Z}). \end{array} \tag{2.3}$$

Here the induced Hopf map $\tilde{H}$ is the Riemannian orbifold submersion.

Example 2.7. In a similar fashion, one can define different actions with different orbifold structures. For instance, consider $p, q \in \mathbb{Z}$ such that $\gcd(p, q) = 1$, and let $n = pq$. Let a, b be integers subject to $ap - bq = 1$. Here we do not modify the action ρ_2 of $\mathbb{Z}/n\mathbb{Z}$ on S^2 given in the previous example. However, this time the action ρ_3 of $\mathbb{Z}/n\mathbb{Z}$ on S^3 is taken to be

$$(\gamma, z_1, z_2) \mapsto (\gamma^{ap} z_1, \gamma^{bq} z_2).$$

It is clear that these actions are compatible with the Hopf map, and we obtain a commutative diagram similar to Diagram (2.3). Please do not overlook the following interesting fact: the isotropy groups are different over different components of the singular set. For instance, if one considers the north pole $(0, 1)$ and the south pole $(0, -1)$ of S^2, it follows that the action of $\mathbb{Z}/n\mathbb{Z}$ on $H^{-1}((0, 1)) = (z_1, 0)$ and $H^{-1}((0, -1)) = (0, z_2)$ is not faithful. The isotropy groups are $\mathbb{Z}/q\mathbb{Z}$ and $\mathbb{Z}/p\mathbb{Z}$. This construction is going to be revisited from another point of view in Section 3.

To conclude this section, we state a theorem due to Molino [23] for foliations with compact leaves. The setup of this theorem is the prototype of Riemannian orbifold submersion that will be used in the sequel.

Theorem 2.8. *Let M be a manifold with a p-dimensional Riemannian foliation $\mathcal{F}$ with compact leaves. Then the space of leaves $M/\mathcal{F}$ admits the structure of an orbifold of codimension p. Moreover, the canonical projection $\pi : M \to M/\mathcal{F}$ is an orbifold Riemannian submersion.* □

3. Projective Kähler orbifolds

It is well known that a complex manifold (M, J) always admits a Hermitian metric h that can be written as

$$h = g - i\omega,$$

where g is a Riemannian metric and ω is a 2-form called the **Kähler form** which is of type $(1,1)$ for the almost complex structure J (as follows from the invariance of h under J). If, in addition, we have $d\omega = 0$, then we say that the manifold is **Kähler** and that g is a **Kähler metric**. Sometimes, by abuse of language, one even says that ω is a Kähler metric. There are several characterizations of Kähler metrics: see [35] for a thorough treatment on the matter. From Remark 2.1 (see also [7]), these notions carry over easily to the level of orbifolds.

Formally, a **positive line orbibundle over a compact orbifold** X is a holomorphic orbibundle that carries a Hermitian metric whose associated curvature form Ω with respect to the Hermitian connection is positive, or in other words, $\frac{i}{2\pi}\Omega$ is a closed Kähler form. One can reinterpret this stiffness of style by saying that certain power $L^{\otimes\nu}$ of this line orbibundle is a positive line bundle on X in the usual sense (ν is just the least common multiple of the orders of the isotropy groups). Of course, this is equivalent to saying that $L^{\otimes\nu}$ admits enough holomorphic sections to provide an embedding of X into some complex projective space. This colloquialism is adopted –or tolerated– due to the following version of Kodaira's embedding theorem as proven by Baily [1].

Theorem 3.1. *Let X be a compact complex orbifold that admits a positive orbibundle. Then X is a projective algebraic variety.* □

Remark 3.2. At the level of cohomology, line orbibundles on X can be considered as rational elements and, as such, line orbibundles lie in $H^2(X, \mathbb{Q})$.

An interesting family of examples of projective Kähler orbifolds is given by weighted projective spaces and weighted complete intersections.

Let us consider an affine variety $V \simeq \mathbb{C}^n$. As a vector space, the grading of $V = \bigoplus_k V^k$ is equivalent to saying that V is endowed with a $\mathbb{C}^*$-action acting on the eigen-spaces (or weight-spaces) V^k with weight k. This is equivalent to a $\mathbb{Z}$-grading of the coordinate ring $\mathbb{C}[V]$ (in this case a $\mathbb{Z}$-grading of the ring of polynomials $\mathbb{C}[x_1, \dots x_n]$).

The weights are taken to be strictly positive. Afterwards, one considers the quotient

$$\mathbb{P}(V) = (V \backslash \{0\})/\mathbb{C}^*.$$

This space, usually denoted $\mathbb{CP}(w_0, \dots, w_n)$, is called the **weighted projective space**. Here we have $n = \dim_{\mathbb{C}} V$, and the w_k are the **weights**. It is always assumed $\gcd(w_0, \dots, w_n) = 1$.

Let x_j, for $j = 1, \dots, n$, be coordinates on V such that x_j has weight w_j. Then $\mathbb{CP}(w_0, \dots, w_n)$ is covered by charts

$$\rho : \{x_j = 1\} \simeq \mathbb{C}^{n-1} \longrightarrow \mathbb{CP}(w_0, \dots, w_n).$$

The w_j-th roots of unity in $\mathbb{C}^*$ act trivially on the x_j coordinate and hence preserve the slice $\mathbb{C}^{n-1}$ displayed above. The map ρ is the quotient by $\mathbb{Z}/w_j\mathbb{Z} \subset \mathbb{C}^*$, explicitly given by $(x_l) \mapsto (\exp(2\pi i w_l/w_j)x_l)$.

The orbifold points of $\mathbb{CP}(w_0, \dots, w_n)$ are determined on each stratum. For instance, each vertex $P_i = [0, \dots, 1, \dots, 0]$ is of type

$$\frac{1}{w_i}(w_1, \dots, \widehat{w}_i, \dots, w_n).$$

The general points along the line P_iP_j are orbifold points of type

$$\frac{1}{\gcd(w_i, w_j)}(w_1, \dots, \widehat{w}_i, \dots, \widehat{w}_j, \dots, w_n),$$

with similar orbifold types for higher dimensional strata. We will always assume that $d_i = \gcd(w_0, \dots, w_{i-1}, w_{i+1}, \dots, w_n)$ equals 1 for all $i = 0, \dots, n$. This assumption will exclude the case where the singularities have codimension 1.

As in the smooth case, the **tautological line orbibundle** $\mathcal{O}_{\mathbb{P}(V)}(-1)$ over the weighted projective space is the orbibundle over $\mathbb{CP}(w_0, \ldots, w_n)$ whose fiber over $[v]$ is the union of the orbit $\mathbb{C}^*.v$ and $0 \in V$. The structure of vector space is given as follows: any two elements on the fiber $\mathcal{O}_{[v]}(-1)$ can be written $u_i = t_i.v$ for $t_i \in \mathbb{C}$, $i = 1, 2$, so the linear structure is defined via $au_1 + bu_2 = (at_1 + bt_2).v$. However, this linear structure is not necessarily the one arising from the vector space structure of V, and hence in general $\mathcal{O}_{[v]}(-1) \subset V$ is not a linear subspace.

By definition, weighted projective spaces are projective and hence admit (orbifold) Kähler metrics. One can argue in a similar way as it is done for the smooth case: since the dual $\mathcal{O}_{\mathbb{P}(V)}(1)$ of the tautological line orbibundle ends up being ample, the Kähler metric, associated to it, is the curvature associated to the hermitian metric on $\mathcal{O}_{\mathbb{P}(V)}(1)$. The interested reader can find many details on this type of metric in [28] for instance (where the authors even allowed the orbifolds to have singularities of codimension 1).

A polynomial $f \in \mathbb{C}[z_0, \ldots z_n]$ is a **weighted homogeneous polynomial of degree** d and **weight** $\mathbf{w} = (w_0, \ldots w_n) \in \mathbb{Z}^{n+1}$ if for any $\lambda \in \mathbb{C}^*$ we have

$$f(\lambda \mathbf{z}) = f(\lambda^{w_0} z_0, \ldots, \lambda^{w_n} z_n) = \lambda^d f(z_0, z_1, \ldots, z_n) = \lambda^d f(\mathbf{z}).$$

The space of weighted homogeneous polynomials of degree d is a λ^d-eigenspace of the induced $\mathbb{C}^*(\mathbf{w})$-action on $\mathbb{C}[z_0, \ldots z_n]$. A **weighted hypersurface** X_f is the zero locus in $\mathbb{CP}(\mathbf{w})$ of a single weighted homogeneous polynomial f. A weighted variety is called a **weighted complete intersection** if the number of polynomials in the collection equals the codimension of X. We denote by $X_d \subset \mathbb{CP}(\mathbf{w})$ the weighted hypersurface of degree d and by $X_{d_1, \ldots, d_c} \subset \mathbb{CP}(\mathbf{w})$ the weighted complete intersection of multidegree d_i (here c denotes the codimension of the variety). We say that the weighted variety is **quasi-smooth** if its affine cone is smooth outside the origin $\mathbf{0}$. Under the quasi-smoothness hypothesis, it is not difficult to see that the orbifold structure on $\mathbb{CP}(\mathbf{w})$

induces a locally cyclic orbifold structure on $X_{d_1,\ldots,d_c}$. Clearly, the resulting orbifolds are of Kähler type.

For example, let x, y, z, u and w be the homogeneous coordinates on $\mathbb{CP}(1,1,1,2,2)$ of weights $1,1,1,2$ and 2 respectively. Let $f = x^3 + y^3 + z^3 + ux + wy$ and $g = x^4 + y^4 + z^4 + u^2 + w^2$ be polynomials of homogeneous degree 3 and 4 respectively. Then the intersection locus of f and g defines an orbifold $X_{3,4} \subset \mathbb{CP}(1,1,1,2,2)$ with only two 2 cyclic singularities, both modeled on $\mathbb{C}^2/(\mathbb{Z}/2\mathbb{Z})$.

Now we discuss one of many incarnations of the Hopf map: locally free actions on odd spheres. First, let us consider the lowest possible dimension, say $S^3 = \{(z_0, z_1) \in \mathbb{C}^2 : |z_0|^2 + |z_1|^2 = 1\}$. This time we reinterpret the Hopf map in terms of the vector field ξ on $\mathbb{C}^2$ given by

$$\xi = i\left(w_0 z_0 \frac{\partial}{\partial z_0} + w_1 z_1 \frac{\partial}{\partial z_1}\right),$$

where w_0, w_1 are non-zero real numbers such that quotient w_1/w_0 is rational. Restricted to S^3 this vector field is everywhere tangent to S^3 and defines a nowhere vanishing vector field on S^3. Hence ξ generates a 1-dimensional foliation $\mathcal{F}_\xi$ on the sphere, and this time the associated action is given by the rule

$$(z_0, z_1) \mapsto (e^{2\pi i w_0 t} z_0, e^{2\pi i w_1 t} z_1).$$

We will assume that w_0, w_1 are coprime positive integers (were it not the case, we reparametrize and use complex conjugation if necessary to achieve this assumption). Hence, the leaves of this foliation are all circles. According to Theorem 2.8, the space of leaves $S^3/\mathcal{F}_\xi$ is an orbifold: the weighted projective space $\mathbb{CP}(w_0, w_1)$. We also obtain an orbifold Riemannian submersion

$$\pi : (S^3, \bar{g}_{\mathbf{w}}) \to (\mathbb{CP}(w_0, w_1), g_{\mathbf{w}}).$$

Here the metric $\bar{g}_{\mathbf{w}}$ on the sphere is of **Sasaki** type (see [9]), thus, the metric determines a contact distribution $\mathcal{D} \subset TS^3$ that admits an integrable CR-structure (this structure is inherited from the standard complex structure on $\mathbb{C}^2$). Furthermore, $\bar{g}_{\mathbf{w}}$ defines a Kähler orbifold metric $g_{\mathbf{w}}$ on the weighed projective space.

Of course, one can generalize this procedure to higher dimensions in exactly the same way, and one obtains an orbifold Riemannian submersion from the sphere to the weighted projective space

$$\pi : (S^{2n+1}, \bar{g}_{\mathbf{w}}) \to (\mathbb{CP}(\mathbf{w}), g_{\mathbf{w}}), \tag{3.1}$$

for $\mathbf{w} = (w_0, \ldots w_n)$ positive integers satisfying the expected condition: $\gcd(w_0, \ldots w_n) = 1$. In recent years, fibrations of this type have been used to establish, due to the intimate relationship between Kähler structures on weighted projective spaces and Sasakian structures on the corresponding total space, existence of Einstein metrics on exotic spheres (see [7] and the references therein). Let us see how to extend this correspondence to weighted complete intersections.

The notion of links of hypersurface singularities was introduced by Milnor in [22]. In [12], Hamm generalized this idea to complete intersections: they are defined as a p-tuple of linearly independent weighted homogeneous polynomials $\mathbf{f} = (f_1, \ldots, f_p) \in (\mathbb{C}[z_0, \ldots, z_n])^p$ of degrees $d_1, \ldots, d_p$ respectively, and weight vector $\mathbf{w}$. Consider the **weighted affine cone** $C_{\mathbf{f}} = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} | \, \mathbf{f}(z_0, \ldots, z_n) = 0\}$, which has dimension $n + 1 - p$. Let us assume that the origin in $\mathbb{C}^{n+1}$ is the only singularity, and it is isolated. Then we define the link

$$L_{\mathbf{f}} = C_{\mathbf{f}} \cap S^{2n+1},$$

which is smooth, of real dimension $2(n-p)+1$, and $(n-p-1)$-connected (cf. [12, 19]).

It is clear that the link admits a locally free S^1-action inherited from the weighted circle action on the sphere S^{2n+1}. Furthermore, the Riemannian submersion given in Equation (3.1) endows $L_{\mathbf{f}}$ with a Sasakian structure. Thus, one obtains the commutative diagram

$$\begin{array}{ccc} L_{\mathbf{f}} & \longrightarrow & S^{2n+1} \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow \\ \mathcal{Z}_{\mathbf{f}} & \longrightarrow & \mathbb{CP}(\mathbf{w}), \end{array} \tag{3.2}$$

where the horizontal arrows are Sasakian and Kählerian embeddings, respectively, while $\pi : L_{\mathbf{f}} \to \mathcal{Z}_{\mathbf{f}}$ is an orbifold Riemannian submersion, and the algebraic variety $\mathcal{Z}_{\mathbf{f}}$ is a weighted complete intersection in $\mathbb{CP}(\mathbf{w})$ (see [5] for a proof of this result).

Of course, the S^1-actions with the qualities described above are the associated actions coming from the proper analytic actions of $\mathbb{C}^*$ on the associated weighted affine cone $C_{\mathbf{f}}$ to $Z_{\mathbf{f}}$. It is well known that these actions are determined by the Picard group of the variety $Z_{\mathbf{f}}$. In this case, the action is induced by the natural action of the transition functions of the line orbibundle on the trivializations (see [26] and [18] for generalizations of this procedure in an algebro-geometric setting). In that direction, Boyer and Galicki showed an interesting result (cf. [9]), of which we present a simplified version good enough for our purposes.

Theorem 3.3. *Let (Z, ω) be a polarized Kähler orbifold with rational Kähler form ω, that is, such that $[\omega] \in H^2(Z, \mathbb{Q})$. Then the associated principal S^1-orbibundle $\pi : M_{[\omega]} \longrightarrow Z$ defined by $[\omega]$ determines a Sasakian structure on the total space $M_{[\omega]}$. The curvature two-form on $M_{[\omega]}$ is given by the pullback $\pi^*\omega$ of the Kähler form defining this fibration. Moreover, if the orbifold is locally cyclic (that is, if it has an orbifold atlas all of whose local uniformizing groups are cyclic groups) then M is a manifold.* □

4. K3 orbifolds and S^1-orbibundles

In this section we briefly describe certain results in [10] on circle orbibundles over K3 surfaces. In particular, we discuss the associated

Sasakian metrics on the total space of this bundles coming from pull-backs of orbifold Ricci-flat metrics on the K3 surface. We also give an explicit description of the Kähler cone for a K3 surface of low rank. This calculation will come handy for the last section.

First, we present some facts about K3 surfaces (see [3] or [30] for proofs of the results stated here). A **K3 surface** X is a compact Kähler surface with only du Val singularities such that $H^1(X, \mathcal{O}_x) = 0$ and whose dualizing sheaf ω_X is trivial in the sense that it satisfies $\omega_X = \mathcal{O}_X$.

If X is a K3 surface and $\rho : \tilde{X} \to X$ is a minimal resolution, then ρ induces an isomorphism between $H^1(X, \mathcal{O})$ and $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$, and also satisfies $\omega_{\tilde{X}} = \rho^* \omega_X = \mathcal{O}_{\tilde{X}}$. So $\tilde{X}$ turns out to be a smooth K3 surface.

Non-singular rational curves on a K3 surface X can be blown down to yield rational double points. On the other hand, under resolution of singularities, a rational singularity determines an exceptional locus consisting of smooth rational curves that intersect transversally. In terms of intersection theory, the arrangement of the curves can be viewed as a configuration of a Dynkin diagram of one of the following types: A_n, D_n, E_6, E_7, E_8.

An important feature of smooth K3 surfaces is that any smooth curve C is rational if and only if it satisfies $C.C = -2$ (this follows from a direct application of the adjunction formula). Moreover, any irreducible curve on a smooth K3 surface has self-intersection 0 (mod 2).

If X is non-singular, the dualizing sheaf becomes the line bundle associated to the canonical divisor K_X. In that case $H^2(X, \mathbb{Z})$ is torsion free of rank $b^2(X) = 22$. By means of the intersection form, $H_2(X, \mathbb{Z})$ is endowed with the structure of a lattice L_Λ which is isomorphic to $-E_8 \oplus -E_8 \oplus H \oplus H \oplus H$. Here and for future reference H is the indefinite rank 2 lattice with intersection $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $-E_8$ is the root-lattice associated to the Dynkin diagram E_8, which is even, unimodular and negative definite. It follows from Poincaré duality that in $H^2(X, \mathbb{Z})$ the cup-product is even, unimodular, and indefinite with signature $(b^2_+, b^2_-) = (3, 19)$.

Consider the exact sequence

$$H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X{}^*) = \mathrm{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \tag{4.1}$$

induced by the exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X{}^* \longrightarrow 0.$$

From Lefschetz $(1,1)$-theorem and $H^1(X, \mathcal{O}_X) = 0$, the map c_1 sends isomorphically $\mathrm{Pic}(X)$ onto the Picard lattice $H^{1,1}(X) \cap H^2(X, \mathbb{Z})$. We will denote by ρ the rank of the Picard lattice. From the Hodge index theorem, the signature of $\mathrm{Pic}(X)$ equals $(1, \rho - 1)$.

Example 4.1. A **Kummer surface** is defined as the minimal resolution S of the 16 singularities of type A_1 of Z/ι, the quotient of a complex torus Z of dimension 2 by an involution ι on Z which is induced by multiplication by -1 on $\mathbb{C}^2$. The quotient Z/ι is simply-connected and $H^k(Z/\iota, \mathbb{C})$ is the ι-invariant part of $H^k(Z, \mathbb{C})$. Thus, the second Betti number $b^2(Z/\iota)$ equals 3. The blow-up replaces each singular point with a copy of $\mathbb{CP}^1$ with self-intersection -2. This leaves π_1 and b^2_+ invariant but adds 1 to b^2_- for each of the 16 singular points. Hence S is simply-connected with signature $(3, 19)$. Thus S is a smooth K3 surface. Notice that Kummer surfaces are not necessarily projective, however, they admit Kähler metrics: Siu (cf. [32]) showed that every K3 surface is Kähler.

Example 4.2. Consider the **Fermat quartic**

$$X_4 = \{[z_0, \ldots, z_3] \in \mathbb{CP}^3 : z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\}.$$

From the adjuntion formula, it follows that X_4 has trivial canonical bundle. From Lefschetz hyperplane theorem, X_4 is connected and simply-connected, hence $H^1(X_4, \mathcal{O}) = 0$. Thus X_4 is a smooth K3 surface. Next, we consider a principal circle bundle L_f over X_4. One can always choose a line bundle on $\mathrm{Pic}(X_4)$ such that L_f is simply-connected and spin. From work of Smale on the classification of simply-connected 5-dimensional spin manifolds (cf. [31]), one concludes $L_f = 21\#(S^2 \times S^3)$.

It follows from Theorem 3.3 that L_f admits a Sasakian metric coming from the corresponding Riemannian submersion. Here the corresponding metric on X_4 is of **Calabi-Yau** type, that is, a Ricci-flat Kähler metric. Sasakian metrics satisfying this property are called **null Sasaki-η-Einstein metrics**, and all of them have scalar curvature equal to -4 (see [8]).

Example 4.3. Weighted K3 surfaces of codimension one and two. A classification of quasi-smooth weighted surface complete intersections of codimension 1 and 2 was given by Reid [27] and Iano-Fletcher [13]. All these surfaces are defined in terms of weighted affine cones that are smooth outside the origin, fact that translates into the quasi-smoothness of the corresponding weighted surface. Thanks to the extension of the adjunction formula to weighted surfaces given in [13], it is straightforward to detect the members of these two families that end up being K3 surfaces with at worst rational doble points (compare Tables 1 and 2). In [10] the results of the previous example are generalized to these two lists where we established the existence of Sasakian metrics of constant scalar curvatures on manifolds diffeomorphic to $\#k(S^2 \times S^3)$, where k is the second Betti number of the link, and k ranges from 3 to 21 inclusive. The projections of these metrics, via the Riemannian submersions, on the corresponding weighted K3 surfaces are orbifold metrics of Calabi-Yau type. Actually, in [10] is given a complete classification of null Sasaki η-Einstein metrics in 5-manifolds. Here we present a simplified version of this theorem.

Theorem 4.4. *Let $\pi : L \longrightarrow X$ be a S^1-orbibudle with L a smooth simply-connected 5-manifold and let X be a Calabi-Yau orbifold. Then L admits a null Sasaki η-Einstein structure if L is diffeomorphic to $\#k(S^2 \times S^3)$ for $3 \leqslant k \leqslant 21$.* □

Next, we will calculate the space of Kähler classes of a particular non-singular K3 surface. It will be useful to review the description of the Kähler cone for a smooth K3 surface.

Recall that in a compact complex manifold (X, J) admitting a Kähler metric g, the Kähler form ω of g defines a de Rham cohomology class $[\omega] \in H^2(X, \mathbb{R})$, called the **Kähler class** of g. Since ω is also a $(1,1)$-form, $[\omega]$ lies in the intersection of $H^{1,1}(X, \mathbb{C})$ with $H^2(X, \mathbb{R})$. The **Kähler cone** of X is the set of Kähler classes of X.

On a smooth K3 surface X, the description of the Kähler cone can be made more precise. Let us consider the set

$$\mathcal{C} = \{x \in \mathrm{Pic}(X) \otimes \mathbb{R} \text{ with } x.x > 0\}.$$

Due to the signature $(1, \rho-1)$ of the Picard lattice, the condition $x.x > 0$ determines two disjoint connected cones $\mathcal{C}^+$ and $\mathcal{C}^-$, and since the Kähler classes form a convex subcone of $\mathcal{C}^+ \cup \mathcal{C}^-$, they all belong to one of them, say $\mathcal{C}^+$. The Kähler cone of a K3 (cf. [3] or [30]) is the convex subcone of $\mathcal{C}^+$ given by

$$\mathcal{K}(X) = \{y \in \mathcal{C}^+ : y.d > 0 \text{ for all } d \in \Delta\},$$

where $\Delta = \{d \in \mathrm{Pic}(X) : d.d = -2 \text{ and } d \text{ effective}\}$.

As an example consider number 2 in Table 1: $X_5 \subset \mathbb{CP}(1,1,1,2)$. This weighted K3 surface has only one cyclic singularity, of type A_1. The surface X_5 can be given by different polynomials, $f_1(x,y,z,w) = x^4+y^4+z^4+w^2$, $f_2(x,y,z,w) = x^2y^2+y^2w+z^4$, $f_3(x,y,z,w) = xyx+z^4$, etcetera. Nevertheless, in [2] it is shown that one can find a polynomial f such that the **orbifold Picard group** $\mathrm{Pic}(X) \otimes \mathbb{Q}$ of any weighted K3 surface of Table 1 has rank one. We will assume this type of polynomial as the one defining X_5. (Notice that the second Betti number of the link, and therefore the number of connected sums of $S^2 \times S^3$, is only determined by the type of singularities and not by how many elements in the orbifold Picard group one started with.) After resolving the singularity one obtains a smooth K3 surface with quadratic form determined by the hyperplane bundle and the exceptional divisor arising from A_1. This quadratic form is represented by the matrix $D = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$.

Let us determine the set of effective classes of (-2)-curves, that is, the set Δ. It is given by integral solutions of the equation

$$2x^2 + 2xy - 2y^2 = -2,$$

which is equivalent to

$$(x - \rho y)(x - \bar{\rho} y) = -1,$$

with $\rho = \dfrac{-1+\sqrt{5}}{2}$. Hence, one obtains

$$\Delta = \{v \in \mathbb{Z}[\rho] \,|v\bar{v} = -1\}.$$

It is not difficult to see that the solutions are generated by odd powers of $\alpha = \left(\frac{1+\sqrt{5}}{2}\right)$ and $\bar{\alpha} = \left(\frac{1-\sqrt{5}}{2}\right)$. These two numbers satisfy the relation $\alpha^3 = 3\alpha + \bar{\alpha}$. One obtains, by induction, $\alpha^{2k+1} = a\alpha + b\bar{\alpha}$ with a, b positive integers. Since the point $(0, 1)$ satisfies the relation given above, owing to $\bar{\alpha} = 1 - \alpha$, the point $(1, -1)$ is also a solution of this last equation. It is clear then that the Kähler cone of $\widetilde{X}_5$ is given by

$$\mathcal{K}(\widetilde{X}_5) = \{(x, y) : x - 2y > 0 \text{ and } x + 3y > 0\}.$$

5. Duality in connected sums of $S^2 \times S^3$

In this section we explain certain duality between k connected sums of $S^2 \times S^3$ for k an integer ranging from 3 to 21. This correspondence comes into sight if one prescribes the Riemannian structure on these 5-manifolds. They will be given in terms of a map which can be thought of as a transversely birational mapping. The aforementioned duality is a consequence of considering the transverse space, in general a K3 orbifold, as a smooth K3 surface with fixed exceptional lattice.

First, let us recall some notions from algebraic geometry. (See e.g., [17], for details).

Let X be a complex surface with a holomorphic line bundle L. What follows can be presented in more generality, the interested reader should consult [17].

Let us denote by $H^0(X,L)$ the vector space of holomorphic sections of L over X, which is known to be finite dimensional over $\mathbb{C}$, of dimension $\ell+1$, say. Let us denote by $|L|$ the complete linear system of L. The **base locus** $Bs(|L|) \subset X$ of $|L|$ is the set of points at which all sections of $H^0(X,L)$ vanish. One says that $|L|$ is **free** or **base point-free** (or simply that L is **globally generated**) if its base locus is empty. This is equivalent to obtaining, for each $x \in X$, a section $s \in H^0(X,L)$ subject to $s(x) \neq 0$.

Let us choose a basis $s_0, \ldots, s_\ell$ for $H^0(X,L)$. Then one has a natural map

$$\varphi_{|L|} : X - Bs(|L|) \longrightarrow \mathbb{P}H^0(X,L)$$

with rule $\varphi_{|L|}(x) = [s_0(x), \ldots, s_\ell(x)]$. It is costumary to ignore the base locus and construe $\varphi_{|L|}$ as a rational mapping $\varphi_{|L|} : X \dashrightarrow \mathbb{P}H^0(X,L)$. When L is globally generated one obtains a globally defined morphism

$$\varphi_{|L|} : X \longrightarrow \mathbb{P}H^0(X,L).$$

This map is finite, or equivalently satisfies $L.C > 0$ for any irreducible curve C in X, if and only if L is ample. A line bundle is **big** if the map

$$\varphi_{|mL|} : X \dashrightarrow \mathbb{P}H^0(X, L^{\otimes m})$$

is birational onto its image for some m. In this situation L is not, in general, globally generated. A divisor D is **numerically effective** (or **nef** for short) if it satisfies $D.C \geqslant 0$ for any irreducible curve $C \in X$. The notion of amplitud (or ampleness) can also be given numerically: a line bundle L on a smooth surface is ample if and only if satisfies $c_1(L)^2 > 0$ and $L.D > 0$ for every effective divisor on the surface. This characterization is known as **Nakai's criterion** (cf. [3, Corollary 5.4]).

Now, we explain the duality or correspondence between two supposedly different S^1-Seifert bundles. We will always consider elements $X_{\mathbf{w}}$ of one of the two familes of quasi-smooth K3 surfaces depicted in Tables 1 and 2.

Let $\widetilde{X}$ be the minimal resolution $f : \widetilde{X} \to X_{\mathbf{w}}$. By Theorem 4.5, one has for $(X_{\mathbf{w}}, [\omega])$, as a (polarized) projective orbifold, an associated S^1-orbibundle

$$\pi_1 : (b_2(X_{\mathbf{w}}) - 1)\#(S^2 \times S^3) \to X_{\mathbf{w}}$$

defined by $[\omega]$. On the other hand, recall (Section 4) that the resolution $\widetilde{X}$ is a smooth projective $K3$ surface, and then, as a consequence of Kodaira's embedding theorem, admits an integral Kähler class $[\omega_{\widetilde{X}}] \in H^2(\widetilde{X}, \mathbb{Z})$ associated to certain ample line bundle A on $\widetilde{X}$ subject to $c_1(A) = [\omega_{\widetilde{X}}]$. Again, we appeal to Theorem 4.5 to conclude the existence of a 5-manifold diffeomorphic, this time, to $21\#(S^2 \times S^3)$. As we mentioned previously, it is important to bear in mind that both $(b_2(X_{\mathbf{w}}) - 1)\#(S^2 \times S^3)$ and $21\#(S^2 \times S^3)$ have scalar curvature -4.

Now let us denote by L_1 the pullback, via f, of L, where $c_1(L) = [\omega]$ with L a positive orbibundle (or ample in the orbifold sense). Notice that L_1 is a big and nef line bundle (almost by definition) that cannot be ample, otherwise the null locus $\mathrm{Null}(L_1)$ of L_1, that is, the set of divisors D such that $L_1.D = 0$, ends up consisting of the exceptional divisors, contradicting Nakai's criterion for ampleness.

We would rather reinterpret the previous paragraph at the level of Kähler classes. Notice that the pullback $f^*[\omega]$ of $[\omega]$ lies on the boundary $\partial\mathcal{K}(\widetilde{X})$ of the Kähler cone of $\widetilde{X}$. From the discussion given above, it follows that the orbifold Riemannian submersion

$$\pi_1 : (b_2(X_{\mathbf{w}}) - 1)\#(S^2 \times S^3) \to (X_{\mathbf{w}}, [\omega])$$

induces a natural map

$$\widetilde{\pi}_1 : (b_2(X_{\mathbf{w}}) - 1)\#(S^2 \times S^3) \to (\widetilde{X}\backslash\Delta, f^*[\omega]),$$

where Δ denotes de exceptional divisor coming from the resolution of the orbifold. Of course, at the level of the total spaces one obtains the map

$$\hat{f} : 21\#(S^2 \times S^3) \longrightarrow (b_2(X_{\mathbf{w}}) - 1)\#(S^2 \times S^3).$$

As an example, consider number 2 in Table 1: $X_5 \subset \mathbb{P}(1,1,1,2)$ with singularity A_1 and Picard number 1. In [10], it is proven that the corresponding S^1-Seifert bundle is diffeomorphic to $20\#(S^2 \times S^3)$.

In the previous section we computed the Kähler cone of the resolution $\widetilde{X}_5$ of X_5, which is given by the set

$$\mathcal{K}(\widetilde{X}_5) = \{(x,y) : x - 2y > 0 \text{ and } x + 3y > 0\}.$$

Observe that a big and nef class that is not ample lies on the boundary of the Kähler cone, that is, in one of the lines $x-2y=0$ or $x+3y=0$. From the previous analysis, it is known that at least there is one element that corresponds to an ample class that is integral, in the orbifold sense, in X_5, and the duality between $21\#(S^2\times S^3)$ and $20\#(S^2\times S^3)$ shows up when an integral class inside $\mathcal{K}(\widetilde{X}_5)$ is related to a class in the boundary, that is, a class lying on either the line $x-2y=0$ or the line $x+3y=0$. Actually, it is always possible to find an element that provides this duality. We explain this in the next paragraph.

Even though it may no be true that a big and nef line bundle L comes from the pullback of a Hodge orbifold class, in [33], Tosatti showed that this is always the case for projective smooth K3 surfaces: it is enough to apply the basepoint free theorem (see [14, Theorem 6.1]) together with the fact that one is dealing with a Calabi-Yau manifold to conclude that mL is globally generated for m sufficiently large. Thus, any irreducible $D \in \mathrm{Null}(L)$ has negative self-intersection (this follows from the Hodge index theorem, since we obtain $(mL)^2 > 0$). As we mentioned before, in a smooth K3 surface, self-intersections are even, and since $D^2 \geqslant -2$ (see [3, Chapter VIII, Proposition 3.6]) then $D^2 = -2$. By the adjuntion formula one concludes that D is a smooth rational curve. Then the map $\varphi_{|mL|}$ contracts D to a rational double point while X contracts to a projective K3 orbifold with ample line bundle L_1 (in the orbifold sense) and one can assume that its pullback is mL. So, in general, pulling-back an ample class is not enough, as also "sliding" along the boundary may be necessary. The m under discussion is nothing more than the least common denominator of the orders of the cyclic

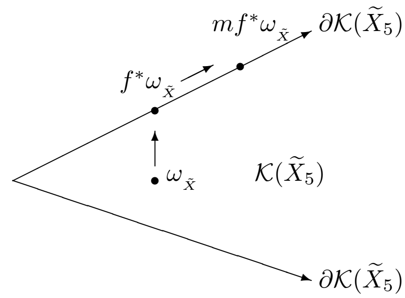
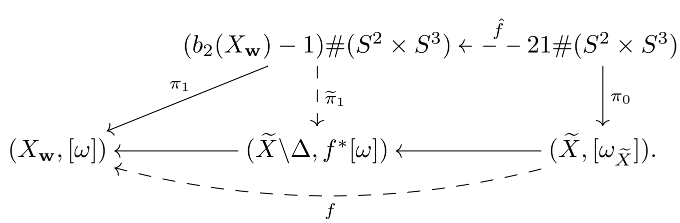


Figure 1: The Kähler cone of $\widetilde{X}_5$: even if $f^*\omega$ does not provide the ∂-duality, a multiple $mf^*\omega_{\tilde{X}}$ will achieve this goal.

singularities one encounters in the orbifold, so there is a globally defined line bundle on the orbifold. In the example given above $m = 2$ will suffice. For $X_{24,30} \subset \mathbb{CP}(8, 9, 10, 12, 15)$, number 84 in Table 2, it is enough to take $m = 180$.

Since a link is the boundary of a Milnor fiber, we will refer to this duality as **∂-duality**. Let us put the discussion given above in theorem form.

Theorem 5.1. *There is ∂-duality, in the sense explained above, between k connected sums of $S^2 \times S^3$ for any $k \in \{3, \dots 20\}$ and $21\#(S^2 \times S^3)$. The following diagram summarizes the relations among these maps*



Thus, the duality occurs when one considers simultaneously the integral class $[\omega_{\widetilde{X}}]$ *in the Kähler cone* $\mathcal{K}(\widetilde{X})$ *and a class lying on the closure of the Kähler cone that is big but not ample, i.e., a class* $[\omega_\partial] \in \overline{\mathcal{K}(\widetilde{X})} \backslash \mathcal{K}(\widetilde{X})$ *that we must take equal to* $f^*[\omega]$*.* □

A natural question will be whether one can extend this ∂-duality among k connected sums of $S^2 \times S^3$ with $3 \leqslant k \leqslant 20$. We consider a projective smooth K3 surface X with enough smooth rational curves that intersect each other transversally (this type of K3 surface always can be found, e.g., the Fermat quartic). Let us choose a subset $\mathcal{C} = \{C_1, \ldots, C_s, \ldots, C_l\}$ of rational curves with this sort of intersection. We will perform two blow-downs, the first one contracting all the curves of $\mathcal{C}$, and the other one contracting only $l - s$ curves from $\mathcal{C}$. These contractions manufacture two different K3 orbifolds $X_{\mathbf{w_1}}$ and $X_{\mathbf{w_2}}$, which are projective and hence admit ample line orbibundles L_1 and L_2, respectively. With abuse of notation, we will denote also by L_1 and L_2 the corresponding pullbacks of these two line orbibundles. These two line bundles end up being big and nef on X, so both are globally generated with corresponding maps

$$\varphi_{|L_1|} : X \to \mathbb{P}H^0(X, L_1) \text{ and } \varphi_{|L_2|} : X \to \mathbb{P}H^0(X, L_2).$$

If one pursues ∂-duality on the corresponding links, the null spaces of these line bundles must satisfy $\text{Null}(L_1) = \Delta = \cup_{i=1}^{l} D_i$ and $\text{Null}(L_2) = \Delta \backslash \cup_{i=1}^{s} D_i$. Here each D_i denotes disjoint components of the base locus Δ.

This is not necessarily the case in general. Take for instance $X_7 \subset \mathbb{P}(1, 1, 2, 3)$, number 5 in Table 1, with singularities A_1, A_2. Here the minimal resolution X has null space $\text{Null}(L_1) = D_1 \cup D_2$, where D_1 is just the rational curve created when the singularity of type A_1 is resolved and D_2 consists of the union of two smooth rational curves E_1 and E_2 with $E_1.E_2 = 1$, created when the singularity of type A_2 is resolved. Then the existence of duality between the link $18\#(S^2 \times S^3)$ associated to L_1 and some other connected sum of $S^2 \times S^3$ (with $k \neq 0$) forces the existence of another big and nef line bundle L_2 satisfying $\mathrm{Null}(L_2) = D_1$ (in this case our choice for the ∂-dual is $20\#(S^2 \times S^3)$, the other possibility is taking $\mathrm{Null}(L_2) = D_2$ with ∂-dual $19\#(S^2 \times S^3)$, but the argument works identically). Thus, condition $L_2.D_2 > 0$ is necessary, otherwise the map $\varphi_{|L_2|} : X \to \mathbb{P}H^0(X, L_2)$ will contract D_2 to a point and hence D_2 would belong to the null locus of L_2.

In general, in order to have duality one needs to verify the condition

$$L_2. \cup_{i=1}^{s} D_i > 0. \tag{5.1}$$

With the notation from the last two paragraphs, we have the following result.

Theorem 5.2. *Let $X_{\mathbf{w_1}}$ and $X_{\mathbf{w_2}}$ two K3 orbifolds from either Table 1 or 2. Let $K_1 = (21 - l)\#(S^2 \times S^3)$ and $K_2 = (21 - l + s)\#(S^2 \times S^3)$ be the corresponding links determined by the line orbibundles L_1 and L_2 as explained in the previous paragraphs. Then K_1 is ∂-dual to K_2 if and only if the condition stated on (5.1) is satisfied.* □

Let us interpret this result at the level of metrics. When one considers two classes, first an integral class inside the Kähler cone and then another big and nef class, both in the same smooth K3 surface, the first one has a corresponding Ricci-flat metric on the K3 surface (the celebrated Yau's theorem), while the second one gives rise to a Ricci-flat orbifold metric on the orbifold, obtained from contracting the rational curves belonging to the null space of this big and nef class (see [15]). Moreover, in [16] (see also [33] for a more general statement) it is shown that this metric is smooth Ricci-flat on $X \backslash E$, with E the corresponding set of exceptional divisors, object that can be extended to a closed positive current on the whole K3 surface. The translation of this fact to the corresponding five dimensional Seifert bundle is the existence of one smooth null Sasaki η-Einstein metric on $21\#(S^2 \times S^3)$ (with scalar curvature equal to -4) and a pseudometric on $21\#(S^2 \times S^3)$ that degenerates into l copies of $S^2 \times S^3$ (of course, here l corresponds in a natural way to the number of linearly independent rational curves that determine the null space of the big and nef class). The last metric turns into a smooth null Sasaki η-Einstein metric on $(21 - l)\#(S^2 \times S^3)$ also with constant scalar curvature equal -4.

Remark 5.3. From the previous paragraph, it would be tempting to conclude that all manifolds diffeomorphic to $21\#(S^2 \times S^3)$ admit a metric with scalar curvature -4 collapsing on l connected sums of $S^2 \times S^3$ for every $0 \leqslant l \leqslant 18$; however this is not the case. As indicated before, the ∂-duality is determined by the map $\hat{f}$ which exists only in the *transversely birational* sense, that is, only on the space of leaves determined by the action of S^1. Indeed, one cannot use this correspondence to state that the two null Sasakian metrics in display (both with constant scalar curvature -4) can be considered to exist in the same ambient space, that is, in the same differential structure. Actually, the open neighborhood $V_{f^{-1}(x)}$ of $f^{-1}(x)$ is not necessarily diffeomorphic to the open set $\widetilde{U}_x \subset \mathbb{C}^2$ coming from the local orbifold chart $\widetilde{U}_x/\Gamma_x$ (with uniformizing group Γ_x). In fact, $V_{f^{-1}(x)}$ is diffeomorphic to the corresponding Milnor fiber (see [11, page 148]). Whether there exists a natural differential geometric minimal model that contains these two seemingly unrelated Riemannian structures and, therefore, a common place where one can consider both metrics as defining a unique metric on this model amounts to establish a more precise partnership between the minimal resolution of a K3 surface and the link of its affine cone. The existence of a minimal model for S^1-Seifert bundles and its possible applications is part of a joint work in progress with R. Gonzales.

	$X_{\mathbf{w}}$	singularities	$b_2(X_{\mathbf{w}})$
No.1	$X_4 \subset \mathbb{P}(1,1,1,1)$		22
No.2	$X_5 \subset \mathbb{P}(1,1,1,2)$	A_1	21
No.3	$X_6 \subset \mathbb{P}(1,1,1,3)$		22
No.4	$X_6 \subset \mathbb{P}(1,1,2,2)$	$3 \times A_1$	19
No.5	$X_7 \subset \mathbb{P}(1,1,2,3)$	A_1, A_2	19
No.6	$X_8 \subset \mathbb{P}(1,1,2,4)$	$2 \times A_1$	20
No.7	$X_8 \subset \mathbb{P}(1,2,2,3)$	$4 \times A_1, A_2$	16
No.8	$X_9 \subset \mathbb{P}(1,1,3,4)$	A_3	19
No.9	$X_9 \subset \mathbb{P}(1,2,3,3)$	$A_1, 3 \times A_2$	15
No.10	$X_{10} \subset \mathbb{P}(1,1,3,5)$	A_2	20
No.11	$X_{10} \subset \mathbb{P}(1,2,2,5)$	$5 \times A_1$	17
No.12	$X_{10} \subset \mathbb{P}(1,2,3,4)$	$2 \times A_1, A_2, A_3$	15
No.13	$X_{11} \subset \mathbb{P}(1,2,3,5)$	A_1, A_2, A_5	15
No.14	$X_{12} \subset \mathbb{P}(1,1,4,6)$	A_1	21
No.15	$X_{12} \subset \mathbb{P}(1,2,3,6)$	$2 \times A_1, 2 \times A_2$	14
No.16	$X_{12} \subset \mathbb{P}(1,2,4,5)$	$3 \times A_1, A_4$	15
No.17	$X_{12} \subset \mathbb{P}(1,3,4,4)$	$3 \times A_3$	13
No.18	$X_{12} \subset \mathbb{P}(2,2,3,5)$	$6 \times A_1, A_4$	12
No.19	$X_{12} \subset \mathbb{P}(2,3,3,4)$	$3 \times A_1, 4 \times A_2$	11
No.20	$X_{13} \subset \mathbb{P}(1,3,4,5)$	A_2, A_3, A_4	13
No.21	$X_{14} \subset \mathbb{P}(1,2,4,7)$	$3 \times A_1, A_3$	16
No.22	$X_{14} \subset \mathbb{P}(2,2,3,7)$	$7 \times A_1, A_2$	13
No.23	$X_{14} \subset \mathbb{P}(2,3,4,5)$	$3 \times A_1, A_2, A_3, A_4$	10
No.24	$X_{15} \subset \mathbb{P}(1,2,5,7)$	A_1, A_6	15
No.25	$X_{15} \subset \mathbb{P}(1,3,4,7)$	A_3, A_6	13
No.26	$X_{15} \subset \mathbb{P}(1,3,5,6)$	$2 \times A_2, A_5$	13
No.27	$X_{15} \subset \mathbb{P}(2,3,5,5)$	$A_1, 3 \times A_4$	9
No.28	$X_{15} \subset \mathbb{P}(3,3,4,5)$	$5 \times A_2, A_3$	9
No.29	$X_{16} \subset \mathbb{P}(1,2,5,8)$	$2 \times A_1, A_4$	16

Table 1. Reid's List of 95 Codimension 1 Weighted K3 Surfaces.

	$X_{\mathbf{w}}$	singularities	$b_2(X_{\mathbf{w}})$
No.30	$X_{16} \subset \mathbb{P}(1,3,4,8)$	$A_2, 2 \times A_3$	14
No.31	$X_{16} \subset \mathbb{P}(1,4,5,6)$	A_1, A_4, A_5	12
No.32	$X_{16} \subset \mathbb{P}(2,3,4,7)$	$4 \times A_1, A_2, A_6$	10
No.33	$X_{17} \subset \mathbb{P}(2,3,5,7)$	A_1, A_2, A_4, A_6	9
No.34	$X_{18} \subset \mathbb{P}(1,2,6,9)$	$3 \times A_1, A_2$	15
No.35	$X_{18} \subset \mathbb{P}(1,3,5,9)$	$2 \times A_2, A_4$	14
No.36	$X_{18} \subset \mathbb{P}(1,4,6,7)$	A_3, A_1, A_6	12
No.37	$X_{18} \subset \mathbb{P}(2,3,4,9)$	$4 \times A_1, 2 \times A_2, A_3$	11
No.38	$X_{18} \subset \mathbb{P}(2,3,5,8)$	$2 \times A_1, A_4, A_7$	9
No.39	$X_{18} \subset \mathbb{P}(3,4,5,6)$	$3 \times A_2, A_3, A_1, A_4$	8
No.40	$X_{19} \subset \mathbb{P}(3,4,5,7)$	A_2, A_3, A_4, A_6	7
No.41	$X_{20} \subset \mathbb{P}(1,4,5,10)$	$A_1, 2 \times A_4$	13
No.42	$X_{20} \subset \mathbb{P}(2,3,5,10)$	$2 \times A_1, A_2, 2 \times A_4$	10
No.43	$X_{20} \subset \mathbb{P}(2,4,5,9)$	$5 \times A_1, A_8$	9
No.44	$X_{20} \subset \mathbb{P}(2,5,6,7)$	$3 \times A_1, A_5, A_6$	8
No.45	$X_{20} \subset \mathbb{P}(3,4,5,8)$	$A_2, 2 \times A_3, A_7$	7
No.46	$X_{21} \subset \mathbb{P}(1,3,7,10)$	A_9	13
No.47	$X_{21} \subset \mathbb{P}(1,5,7,8)$	A_4, A_7	11
No.48	$X_{21} \subset \mathbb{P}(2,3,7,9)$	$A_1, 2 \times A_2, A_8$	9
No.49	$X_{21} \subset \mathbb{P}(3,5,6,7)$	$3 \times A_2, A_4, A_5$	7
No.50	$X_{22} \subset \mathbb{P}(1,3,7,11)$	A_2, A_6	14
No.51	$X_{22} \subset \mathbb{P}(1,4,6,11)$	A_3, A_1, A_5	13
No.52	$X_{22} \subset \mathbb{P}(2,4,5,11)$	$5 \times A_1, A_3, A_4$	10
No.53	$X_{24} \subset \mathbb{P}(1,3,8,12)$	$2 \times A_2, A_3$	16
No.54	$X_{24} \subset \mathbb{P}(1,6,8,9)$	A_1, A_2, A_8	11
No.55	$X_{24} \subset \mathbb{P}(2,3,7,12)$	$2 \times A_1, 2 \times A_2, A_6$	10
No.56	$X_{24} \subset \mathbb{P}(2,3,8,11)$	$3 \times A_1, A_{10}$	9
No.57	$X_{24} \subset \mathbb{P}(3,4,5,12)$	$2 \times A_2, 2 \times A_3, A_4$	8
No.58	$X_{24} \subset \mathbb{P}(3,4,7,10)$	A_1, A_6, A_9	6

Table 1. Reid's List of 95 Codimension 1 Weighted K3 Surfaces.

	$X_{\mathbf{w}}$	singularities	$b_2(X_{\mathbf{w}})$
No.59	$X_{24} \subset \mathbb{P}(3,6,7,8)$	$4 \times A_2, A_1, A_6$	7
No.60	$X_{24} \subset \mathbb{P}(4,5,6,9)$	$2 \times A_1, A_4, A_2, A_8$	6
No.61	$X_{25} \subset \mathbb{P}(4,5,7,9)$	A_3, A_6, A_8	5
No.62	$X_{26} \subset \mathbb{P}(1,5,7,13)$	A_4, A_6	12
No.63	$X_{26} \subset \mathbb{P}(2,3,8,13)$	$3 \times A_1, A_2, A_7$	10
No.64	$X_{26} \subset \mathbb{P}(2,5,6,13)$	$4 \times A_1, A_4, A_5$	9
No.65	$X_{27} \subset \mathbb{P}(2,5,9,11)$	A_1, A_4, A_{10}	7
No.66	$X_{27} \subset \mathbb{P}(5,6,7,8)$	A_4, A_5, A_2, A_6	5
No.67	$X_{28} \subset \mathbb{P}(1,4,9,14)$	A_1, A_8	13
No.68	$X_{28} \subset \mathbb{P}(3,4,7,14)$	$A_2, A_1, 2 \times A_6$	7
No.69	$X_{28} \subset \mathbb{P}(4,6,7,11)$	$2 \times A_1, A_5, A_{10}$	5
No.70	$X_{30} \subset \mathbb{P}(1,4,10,15)$	A_3, A_4, A_1	14
No.71	$X_{30} \subset \mathbb{P}(1,6,8,15)$	A_1, A_2, A_7	12
No.72	$X_{30} \subset \mathbb{P}(2,3,10,15)$	$3 \times A_1, 2 \times A_2, A_4$	6
No.73	$X_{30} \subset \mathbb{P}(2,6,7,15)$	$5 \times A_1, A_2, A_6$	9
No.74	$X_{30} \subset \mathbb{P}(3,4,10,13)$	A_3, A_1, A_{12}	5
No.75	$X_{30} \subset \mathbb{P}(4,5,6,15)$	$A_3, 2 \times A_1, 2 \times A_4, A_2$	7
No.76	$X_{30} \subset \mathbb{P}(5,6,8,11)$	A_1, A_7, A_{10}	4
No.77	$X_{32} \subset \mathbb{P}(2,5,9,16)$	$2 \times A_1, A_4, A_8$	8
No.78	$X_{32} \subset \mathbb{P}(4,5,7,16)$	$2 \times A_3, A_4, A_6$	6
No.79	$X_{33} \subset \mathbb{P}(3,5,11,14)$	A_4, A_{13}	5
No.80	$X_{34} \subset \mathbb{P}(3,4,10,17)$	A_2, A_3, A_1, A_9	7
No.81	$X_{34} \subset \mathbb{P}(4,6,7,17)$	$A_3, 2 \times A_1, A_5, A_6$	6
No.82	$X_{36} \subset \mathbb{P}(1,5,12,18)$	A_4, A_5	13
No.83	$X_{36} \subset \mathbb{P}(3,4,11,18)$	$2 \times A_2, A_1, A_{10}$	7
No.84	$X_{36} \subset \mathbb{P}(7,8,9,12)$	A_6, A_7, A_3, A_2	4
No.85	$X_{38} \subset \mathbb{P}(3,5,11,19)$	A_2, A_4, A_{10}	6
No.86	$X_{38} \subset \mathbb{P}(5,6,8,19)$	A_4, A_5, A_1, A_7	5
No.87	$X_{40} \subset \mathbb{P}(5,7,8,20)$	$2 \times A_4, A_6, A_3$	5

Table 1. Reid's List of 95 Codimension 1 Weighted K3 Surfaces.

	$X_{\mathbf{w}}$	singularities	$b_2(X_{\mathbf{w}})$
No.86	$X_{38} \subset \mathbb{P}(5,6,8,19)$	A_4, A_5, A_1, A_7	5
No.87	$X_{40} \subset \mathbb{P}(5,7,8,20)$	$2 \times A_4, A_6, A_3$	5
No.88	$X_{42} \subset \mathbb{P}(1,6,14,21)$	A_1, A_2, A_6	13
No.89	$X_{42} \subset \mathbb{P}(2,5,14,21)$	$3 \times A_1, A_4, A_6$	9
No.90	$X_{42} \subset \mathbb{P}(3,4,14,21)$	$2 \times A_2, A_3, A_1, A_6$	8
No.91	$X_{44} \subset \mathbb{P}(4,5,13,22)$	A_1, A_4, A_{12}	5
No.92	$X_{48} \subset \mathbb{P}(3,5,16,24)$	$2 \times A_2, A_4, A_7$	7
No.93	$X_{50} \subset \mathbb{P}(7,8,10,25)$	A_6, A_7, A_1, A_4	4
No.94	$X_{54} \subset \mathbb{P}(4,5,18,27)$	A_3, A_1, A_4, A_8	6
No.95	$X_{66} \subset \mathbb{P}(5,6,22,33)$	A_4, A_1, A_2, A_{10}	5

Table 1. Reid's List of 95 Codimension 1 Weighted K3 Surfaces.

	$X_{\mathbf{w}}$	singularities	$b_2(X_{\mathbf{w}})$
No.1	$X_{2,3} \subset \mathbb{P}(1,1,1,1,1)$		22
No.2	$X_{3,3} \subset \mathbb{P}(1,1,1,1,2)$	A_1	21
No.3	$X_{3,4} \subset \mathbb{P}(1,1,1,2,2)$	$2 \times A_1$	20
No.4	$X_{4,4} \subset \mathbb{P}(1,1,1,2,3)$	A_2	20
No.5	$X_{4,4} \subset \mathbb{P}(1,1,2,2,2)$	$4 \times A_1$	18
No.6	$X_{4,5} \subset \mathbb{P}(1,1,2,2,3)$	$2 \times A_1, A_2$	18
No.7	$X_{4,6} \subset \mathbb{P}(1,1,2,3,3)$	$2 \times A_2$	18
No.8	$X_{4,6} \subset \mathbb{P}(1,2,2,2,3)$	$6 \times A_1$	16
No.9	$X_{5,6} \subset \mathbb{P}(1,1,2,3,4)$	A_1, A_3	18
No.10	$X_{5,6} \subset \mathbb{P}(1,2,2,3,3)$	$3 \times A_1, 2 \times A_2$	15
No.11	$X_{6,6} \subset \mathbb{P}(1,1,2,3,5)$	A_4	18
No.12	$X_{6,6} \subset \mathbb{P}(1,2,2,3,4)$	$4 \times A_1, A_3$	15
No.13	$X_{6,6} \subset \mathbb{P}(1,2,3,3,3)$	$4 \times A_2$	14
No.14	$X_{6,6} \subset \mathbb{P}(2,2,2,3,3)$	$9 \times A_1$	13

Table 2. Fletcher's List of 84 Codimension 2 Weighted K3 surfaces.

	$X_{\mathbf{w}}$	singularities	$b_2(X_{\mathbf{w}})$
No.15	$X_{6,7} \subset \mathbb{P}(1,2,2,3,5)$	$3 \times A_1, A_4$	15
No.16	$X_{6,7} \subset \mathbb{P}(1,2,3,3,4)$	$A_1, 2 \times A_2, A_3$	14
No.17	$X_{6,8} \subset \mathbb{P}(1,1,3,4,5)$	A_4	18
No.18	$X_{6,8} \subset \mathbb{P}(1,2,2,3,5)$	$2 \times A_2, A_4$	14
No.19	$X_{6,8} \subset \mathbb{P}(1,2,3,4,4)$	$2 \times A_1, 2 \times A_3$	14
No.20	$X_{6,8} \subset \mathbb{P}(2,2,3,3,4)$	$6 \times A_1, 2 \times A_2$	12
No.21	$X_{6,9} \subset \mathbb{P}(1,2,3,4,5)$	A_1, A_3, A_4	14
No.22	$X_{7,8} \subset \mathbb{P}(1,2,3,4,5)$	$2 \times A_1, A_2, A_4$	14
No.23	$X_{6,10} \subset \mathbb{P}(1,2,3,5,5)$	$2 \times A_4$	14
No.24	$X_{6,10} \subset \mathbb{P}(2,2,3,4,5)$	$7 \times A_1, A_3$	12
No.25	$X_{8,9} \subset \mathbb{P}(1,2,3,4,7)$	$2 \times A_1, A_6$	14
No.26	$X_{8,9} \subset \mathbb{P}(1,3,4,4,5)$	$2 \times A_3, A_4$	13
No.27	$X_{8,9} \subset \mathbb{P}(2,3,3,4,5)$	$2 \times A_1, 3 \times A_2, A_4$	10
No.28	$X_{8,10} \subset \mathbb{P}(1,2,3,5,7)$	A_2, A_6	14
No.29	$X_{8,10} \subset \mathbb{P}(1,2,4,5,6)$	$3 \times A_1, A_5$	14
No.30	$X_{8,10} \subset \mathbb{P}(1,3,4,5,5)$	$A_2, 2 \times A_4$	12
No.31	$X_{8,10} \subset \mathbb{P}(2,3,4,4,5)$	$4 \times A_1, A_2, 2 \times A_3$	10
No.32	$X_{9,10} \subset \mathbb{P}(1,2,3,5,8)$	A_1, A_7	14
No.33	$X_{9,10} \subset \mathbb{P}(1,3,4,5,6)$	A_2, A_3, A_5	12
No.34	$X_{9,10} \subset \mathbb{P}(2,2,3,5,7)$	$5 \times A_1, A_6$	11
No.35	$X_{9,10} \subset \mathbb{P}(2,3,4,5,5)$	$2 \times A_1, A_3, 2 \times A_4$	9
No.36	$X_{8,12} \subset \mathbb{P}(1,3,4,5,7)$	A_4, A_6	12
No.37	$X_{8,12} \subset \mathbb{P}(2,3,4,5,6)$	$4 \times A_1, 2 \times A_2, A_4$	10
No.38	$X_{9,12} \subset \mathbb{P}(2,3,4,5,7)$	$3 \times A_1, A_4, A_6$	9
No.39	$X_{10,11} \subset \mathbb{P}(2,3,4,5,7)$	$2 \times A_1, A_2, A_3, A_6$	9
No.40	$X_{10,12} \subset \mathbb{P}(1,3,4,5,9)$	A_2, A_8	12
No.41	$X_{10,12} \subset \mathbb{P}(1,3,5,6,7)$	$2 \times A_2, A_6$	12

Table 2. Fletcher's List of 84 Codimension 2 Weighted K3 surfaces.

	$X_{\mathbf{w}}$	singularities	$b_2(X_{\mathbf{w}})$
No.42	$X_{10,12} \subset \mathbb{P}(1,2,5,6,6)$	$A_1, 2 \times A_5$	11
No.43	$X_{10,12} \subset \mathbb{P}(2,3,4,5,8)$	$3 \times A_1, A_3, A_7$	9
No.44	$X_{10,12} \subset \mathbb{P}(2,3,5,5,7)$	$2 \times A_4, A_6$	8
No.45	$X_{10,12} \subset \mathbb{P}(2,4,5,5,6)$	$5 \times A_1, 2 \times A_4$	9
No.46	$X_{10,12} \subset \mathbb{P}(3,3,4,5,7)$	$4 \times A_2, A_6$	8
No.47	$X_{10,12} \subset \mathbb{P}(3,4,4,5,6)$	$2 \times A_2, 3 \times A_3, A_1$	8
No.48	$X_{11,12} \subset \mathbb{P}(1,4,5,6,7)$	A_1, A_4, A_6	11
No.49	$X_{10,14} \subset \mathbb{P}(1,2,5,7,9)$	A_8	14
No.50	$X_{10,14} \subset \mathbb{P}(2,3,5,7,7)$	$A_2, 2 \times A_6$	8
No.51	$X_{10,14} \subset \mathbb{P}(2,4,5,6,7)$	$5 \times A_1, A_3, A_5$	9
No.52	$X_{10,15} \subset \mathbb{P}(2,3,5,7,8)$	A_1, A_6, A_7	8
No.53	$X_{12,13} \subset \mathbb{P}(3,4,5,6,7)$	$2 \times A_2, A_1, A_4, A_6$	7
No.54	$X_{12,14} \subset \mathbb{P}(1,3,4,7,11)$	A_{10}	12
No.55	$X_{12,14} \subset \mathbb{P}(1,4,6,7,8)$	A_1, A_3, A_7	11
No.56	$X_{12,14} \subset \mathbb{P}(2,3,4,7,10)$	$4 \times A_1, A_9$	9
No.57	$X_{12,14} \subset \mathbb{P}(2,3,5,7,9)$	A_2, A_4, A_8	8
No.58	$X_{12,14} \subset \mathbb{P}(3,4,5,7,7)$	$A_4, 2 \times A_6$	6
No.59	$X_{12,14} \subset \mathbb{P}(4,4,5,6,7)$	$3 \times A_3, 2 \times A_1, A_4$	7
No.60	$X_{12,15} \subset \mathbb{P}(1,4,5,6,11)$	A_1, A_{10}	11
No.61	$X_{12,15} \subset \mathbb{P}(3,4,5,6,9)$	$3 \times A_2, A_1, A_8$	7
No.62	$X_{12,15} \subset \mathbb{P}(3,4,5,7,8)$	A_3, A_6, A_7	6
No.63	$X_{12,16} \subset \mathbb{P}(2,5,6,7,8)$	$4 \times A_1, A_4, A_6$	8
No.64	$X_{14,15} \subset \mathbb{P}(2,3,5,7,12)$	A_1, A_2, A_{11}	8
No.65	$X_{14,15} \subset \mathbb{P}(2,5,6,7,9)$	$2 \times A_1, A_5, A_8$	7
No.66	$X_{14,15} \subset \mathbb{P}(3,4,5,7,10)$	A_3, A_4, A_9	6
No.67	$X_{14,15} \subset \mathbb{P}(3,5,6,7,8)$	$2 \times A_2, A_5, A_7$	6
No.68	$X_{14,16} \subset \mathbb{P}(1,5,7,8,9)$	A_4, A_8	10

Table 2. Fletcher's List of 84 Codimension 2 Weighted K3 surfaces.

	$X_{\mathbf{w}}$	singularities	$b_2(X_{\mathbf{w}})$
No.69	$X_{14,16} \subset \mathbb{P}(3,4,5,7,11)$	A_2, A_4, A_{10}	6
No.70	$X_{14,16} \subset \mathbb{P}(4,5,6,7,8)$	$A_1, 2 \times A_3, A_4, A_5$	6
No.71	$X_{15,16} \subset \mathbb{P}(2,3,5,8,13)$	$2 \times A_1, A_{12}$	8
No.72	$X_{15,16} \subset \mathbb{P}(3,4,5,8,11)$	$2 \times A_3, A_{10}$	6
No.73	$X_{14,18} \subset \mathbb{P}(2,3,7,9,11)$	$2 \times A_2, A_{10}$	8
No.74	$X_{14,18} \subset \mathbb{P}(2,6,7,8,9)$	$5 \times A_1, A_2, A_7$	8
No.75	$X_{12,20} \subset \mathbb{P}(4,5,6,7,10)$	$2 \times A_1, 2 \times A_4, A_6$	6
No.76	$X_{16,18} \subset \mathbb{P}(1,6,8,9,10)$	A_1, A_2, A_9	10
No.77	$X_{16,18} \subset \mathbb{P}(4,6,7,8,9)$	$2 \times A_1, 2 \times A_3, A_2, A_6$	6
No.78	$X_{18,20} \subset \mathbb{P}(4,5,6,9,14)$	$2 \times A_1, A_2, A_{13}$	5
No.79	$X_{18,20} \subset \mathbb{P}(4,5,7,9,13)$	A_6, A_{12}	4
No.80	$X_{18,20} \subset \mathbb{P}(5,6,7,9,11)$	A_2, A_6, A_{10}	4
No.81	$X_{18,22} \subset \mathbb{P}(2,5,9,11,13)$	A_4, A_{12}	6
No.82	$X_{20,21} \subset \mathbb{P}(3,4,7,10,17)$	A_1, A_{16}	5
No.83	$X_{18,30} \subset \mathbb{P}(6,8,9,10,15)$	$2 \times A_1, 2 \times A_2, A_7, A_4$	5
No.84	$X_{24,30} \subset \mathbb{P}(8,9,10,12,15)$	A_1, A_3, A_8, A_2, A_4	6

Table 2. Fletcher's List of 84 Codimension 2 Weighted K3 surfaces.

References

[1] Baily, W.L.; *On the Embedding of V-Manifolds in Projective Space,* Amer. J. Math. 79 (1957), 403-430.

[2] Belcastro, S.M.; *Picard Lattices of families of K3 surfaces* Ph.D. thesis, University of Michigan: Ann Arbor, 1997.

[3] Barth, W., Peters, C., Van de Ven, A.; *Compact Complex Surfaces,* Erg. Math., 3. Folge, Band 4, Springer, Berlin, 1984.

[4] Boyer, C.P., Galicki, K.; *3-Sasakian Manifolds, Surveys in Differential Geometry VI: Essays on Einstein Manifolds; A supplement to the Journal of Differential Geometry* pp. 123-184, (eds. C. LeBrun, M. Wang) Internation Press, Cambridge (1999).

[5] Boyer, C.P., Galicki, K.; *On Sasakian-Einstein Geometry,* Internat. J. Math. 11, 2000, no. 7, 873-909.

[6] Boyer, C.P., Galicki, K., Nakamaye, M.; *On the Geometry of Sasakian-Einstein 5-Manifold,* Math. Ann. 325 (2003), 485-524.

[7] Boyer, C.P., Galicki, K., Kollár, J.; *Einstein Metrics on Spheres,* Annals of Mathematics, 162 (2005), 557-580.

[8] Boyer, C.P., Galicki, K., Matzeu, P.; *On eta-Einstein Sasakian Geometry,* Comm. Math. Phys., Volume 262, Issue 1 (2006), pp.177-208.

[9] Boyer, C.P., Galicki, K.; *Sasakian Geometry,* Oxford University Press, (2008).

[10] Cuadros Valle, J.; *Null Sasaki η-Einstein structure in 5-manifolds,* Geom. Dedicata, DOI 10.1007/s10711-013-9859-9, 2013.

[11] Durfee, Alan H.; *Fifteen characterization of rational double points and simple critical points,* Enseign. Math. II 25 (1979), 131-163.

[12] Hamm, H.; *Lokale topologische Eigenschaften komplexer Räume* Math. Ann. 191 (1971), 235-252.

[13] Iano-Fletcher, A.R.; *Working with Weighted Complete Intersections,* Explicit Birational Geometry of 3-folds, London Math. Soc. Lecture Notes Ser., vol. 281, Cambridge Univ. Press, Cambridge (2000), 101-173.

[14] Kawamata Y.; *Pluricanonical systems on minimal algebraic varieties,* Invent. math. 79, 567-588 (1985).

[15] Kobayashi, R.; *Einstein-Kähler V-metrics on open Satake V-surfaces with isolated quotient singularities,* Math. Ann., 272 (1985) no. 385-298.

[16] Kobayashi, R., Todorov, A.N.; *Polarized Period Map for Generalized K3 Surfaces and the Moduli of Einstein Metrics,* Tohoku Math. J. 39 (1987), no. 3, 341-363.

[17] Lazarsfeld R.; *Positivity in Algebraic Geometry I,* Ergebnisse der Mathematik, Springer-Verlag, Berlin, 2004.

[18] Kollár J.; *Einstein Metrics of Five dimensional Seifert Bundles,* Journal of Geometric Analysis, Vol 15 (2005), 445-476.

[19] Looijenga, E.J.N.; *Isolated Singular Points of Complete Intersections,* London, Mathematical Society Lecture Note Series, vol. 77, Cambridge University Press, Cambridge, 1984.

[20] O'Neill, B.; *The fundamental equation of a submersion,* Michigan Math. J. 13 (1966), 459-469. MR 0200865 (33 # 751).

[21] Orlik, P., Wagreich, P.; *Singularities of Algebriac Singularities with C^* action,* Math. Ann. 193 (1971), 121-135. MR 45 no. 1914.

[22] Milnor, J.; *Singular Points of Complex Hypersurfaces,* Annals of Mathematical Studies, Vol.61, Princeton University Press, Princeton, NJ, 1968.

[23] Molino, P.; *Riemannian foliations,* Progress in Mathematics, vol. 73, Birkhäuser, Boston Inc., Boston MA, 1988, MR 89b: 53054.

[24] Moerdijk, I., Mrcun J.; *Introduction to foliation and Lie groupies,* Cambridge Studies in Advanced Mathematics, vol. 91, Cambridge University Press, Cambridge, 2003, MR 2012261.

[25] Moerdijk, I., Pronk, D.A. J.; *Orbifolds, Sheaves and Gruopoids,* K-Theory 12 (1997), n0. 1, 3-21. MR 98i:22004.

[26] Orlik, P., Wagreich, P.; *Seifert n-manifolds,* Invent. math. 28 (1975), 137-159. MR 50 no. 13596.

[27] Reid, M.; *Young Person's Guide to Canonical Singularities,* Proc. of Symp. in Pure Math, vol. 46, Part I, A.M.S., 1987, 345-414.

[28] Ross, J. Thomas, R.; *Weighted Projective Embedding, stability of orbifolds, and constant scalar Kähler metrics,* J. Differential Geom. Vol. 88, Number 1 (2011), 109-159.

[29] Satake, I.; *On a generalization of the notion of manifold,* Proc. Natl. Acad. Sci. USA 42 (1956), 359-363.

[30] Séminaire Palaiseau; *Géométrie des Surfaces K3: Modules et Périodes,* Société Mathématique de France. Astérisque, 1985.

[31] Smale, S; *On the Structure of 5-Manifolds,* Ann. of Math. (2), 75 (1962), 38-46.

[32] Siu, Y.-T.; *Every K3 surface is Kähler,* Invent. Math. 73, (1983), 139-150. MR 0707352.

[33] Tosatti V.; *Limits of Calabi-Yau metrics when the Kähler Class degenerates,* J. Eur.Math. Soc. (JEMS) 11, 2009, no. 4, 755-776.

[34] Thurston, W.; *The Geometry and Topology of Three Manifolds,* Mimeographed Notes, Princeton University, 1979.

[35] Voisin, C.; *Hodge Theory and Algebraic Complex Geometry. I,* Cambridge Studies in Advanced Mathematics, vol. 76, Cambridge University Press, Cambridge 2002.

Resumen

Describimos una correspondencia entre dos enlaces asociados a un mismo espacio K3 que soporta a lo más, singularidades cíclicas de tipo *orbifold*. Esta dualidad se hace evidente cuando dos elementos, uno en el interior y el otro en la frontera del cono de Kähler, son identificados. Denominamos a esta correspondencia ∂-*dualidad.* También discutimos las consecuencias de ∂-dualidad al nivel de estructuras riemaniannas.

Palabras Clave: Geometría diferencial, geometría algebraica, espacio de órbitas, superficies K3, submersiones riemannianas.

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Invariant measures on polynomial quadratic Julia sets with no interior

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Abstract

We characterize invariant measures for quadratic polynomial Julia sets with no interior. We prove that besides the harmonic measure —the only one that is even and invariant—, all others are generated by a suitable odd measure.

MSC(2010): 30D05, 37F05; 37F50, 37L40.

Keywords: Holomorphic dynamics, iteration of polynomials, Mandelbrot set, invariant measures.

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1. Introduction

For a given a degree $d \geq 2$ polynomial, the **filled Julia set** is the set of points that have bounded orbit under iteration. We denote this set by K. It is well known that this set is a compact invariant subset of $\mathbb{C}$. For this and several other facts related to iteration of rational functions we refer the reader to [3].

In this paper we are concerned with the algebra of continuous functions defined on K. For them, our starting point is the following classical setting.

Let $C(K)$ be the algebra of continuous functions defined on a compact set $K \subset \mathbb{C}$ with values in $\mathbb{C}$. We denote by $Pol(K)$ the linear space of polynomial restrictions to K.

Theorem 1.1 (Lavrientiev, Mergelyan [1],[4])**.** *Let K be a compact set of the plane whose interior is empty. If the complement of K is connected, then $Pol(K)$ is dense in $C(K)$ in the uniform topology.* $\square$

Along this work, $P(z) = z^2 + c$ is a degree two polynomial whose filled Julia set K has no interior. Therefore K is compact with empty interior and connected complement, and Lavrientiev's theorem applies.

2. The harmonic decomposition

Let $P(z) = z^2 + c$ be a degree two polynomial whose filled Julia set K has no interior. As this set is symmetric by the involution $z \mapsto -z$, it is safe to define even and odd objects using a standard procedure.

Given $f \in C(K)$, continuous, its **even** and **odd** parts are defined by the averages

$$\mathcal{E}(f)(z) = \frac{f(z) + f(-z)}{2}, \qquad \mathcal{O}(f)(z) = \frac{f(z) - f(-z)}{2},$$

respectively.

Lemma 2.1. *The odd and even parts of f are continuous functions with norm not bigger than $||f||$. If f is a (restriction of a) polynomial, so are $\mathcal{E}(f)$ and $\mathcal{O}(f)$.*

Proof. Both claims are elementary. □

A continuous function f is **even** if $\mathcal{E}(f) = f$, and **odd** if $\mathcal{O}(f) = f$. Alternatively, as $f = \mathcal{O}(f) + \mathcal{E}(f)$ holds, we have that f to be even is equivalent to $\mathcal{O}(f) = 0$ (that is to $f(z) = f(-z)$), while f to be odd is equivalent to $\mathcal{E}(f) = 0$ (or to $f(z) = -f(-z)$).

Also note that $\mathcal{E}, \mathcal{O} : C(K) \to C(K)$, which recover in turn the symmetric and antisymmetric part, are norm 1 operators. Both $\mathcal{E}$ and $\mathcal{O}$ are projections.

Lemma 2.2 (Reduction lemma)**.** *A continuous function $f \in C(K)$ is even if and only if there exists $g \in C(K)$ such that $f(z) = g(P(z))$. In particular, we have $\mathcal{E}(f \circ P) = f \circ P$ for all $f \in C(K)$. On the other side, if $g(P(z))$ is continuous, then $g(z)$ is continuous. We always have $||f|| = ||g||$. Anyway, f is a polynomial if and only if g is a polynomial.*

Proof. If f is even, as K is closed and $P : K \to K$ is surjective and proper, we have that f factors through P. Conversely, $g(P(z))$ is always even and continuous.

It is clear that f peaks at z_0 if and only if g peaks at $P(z_0)$ and that f is continuous at z_0 if and only if g is continuous at $P(z_0)$.

That g is a polynomial (when f is) was already indicated in Lemma 2.1. □

This lemma gives rise to a unique **even—odd decomposition**

$$f(z) = f_0(z) + g_0(P(z)),$$

where f_0 is odd and g_0 continuous. If we again split g_0 into its odd and even parts as $g_0(z) = f_1(z) + g_1(P(z))$, we get

$$f(z) = f_0(z) + f_1(P(z)) + g_1(P^{\circ 2}(z)).$$

We can continue this process indefinitely.

Proposition 2.3. *Fix $n \geq 0$. For every $f \in C(K)$ there are unique odd continuous functions $f_0, f_1, \dots f_n$ and $g_n \in C(K)$ subject to*

$$f(z) = f_0(z) + f_1(P(z)) + \cdots + f_n(P^{\circ n}(z)) + g_n(P^{\circ n+1}(z)).$$

Here we have $||f_i|| \leq ||f||$ and $||g_n|| \leq ||f||$.

Proof. Apply induction to the odd—even decomposition of f. □

Corollary 2.4. *In the decomposition above we have*

$$|f_0(z) + f_1(P(z)) + \cdots + f_n(P^{\circ n}(z))| \leq 2||f||.$$

Proof. Indeed, the partial sum is bounded by $|f(z)| + |g_n(P^{\circ n+1}(z))|$. □

The decomposition displayed in Proposition 2.3 is much simpler for polynomials as the process eventually reaches a deadlock.

Proposition 2.5. *Given a polynomial $F \in Pol(K)$, there exist a constant $H(F)$ subject to $|H(F)| \leq ||F||$, and a finite number of odd polynomials, say $F_0, F_1, \dots F_n$, such that*

$$F(z) = H(F) + F_0(z) + F_1(P(z)) + \cdots + F_n(P^{\circ n}(z)).$$

Those elements are uniquely determined.

Proof. A trivial induction in the degree of F. □

The assignment $H : Pol(K) \to \mathbb{C}$ clearly is linear and annihilates all odd polynomials, hence the symmetry formula $H(\mathcal{E}(f)) = H(f)$. Also, for $F \in Pol(K)$ we have $H(F \circ P) = H(F)$ and $|H(F)| \leq ||F||$.

Theorem 2.6. *Suppose the filled Julia set $K = K(P)$ has no interior. Then there exists a unique norm 1 even invariant measure supported on the Julia set that agrees with H on polynomials. In other words, for all $f \in C(K)$ the measure H satisfies*

- $H(f) = \int f(z)\,dH(z) = \int f(P(z))\,dH(z)$ *(invariance),*
- $H(f) = \int f(z)\,dH(z) = \int \mathcal{E}(f)(z)\,dH(z) = H(\mathcal{E}(f))$ *(symmetry).*

Proof. In fact, when K has no interior, polynomial restrictions to K are dense in $C(K)$. As $H : Pol(K) \to \mathbb{C}$ is continuous, it can be extended uniquely to all of $C(K)$. Since all other properties are satisfied for polynomials, they are satisfied for continuous functions as well. □

As Lyubich proved (cf. [2]), the harmonic measure already satisfies the properties stated in the theorem, so H is actually the **harmonic measure of** K. This corollary is actually true for all polynomial Julia sets. Functions for which the harmonic integral vanish (i.e, f such that $H(f) = 0$) are **harmonic free functions**. For simplicity, we will write H_f for $H(f)$.

Next we retrace our steps with these results in mind. First we apply the odd—even decomposition to the function $f(z) - H_f$ in order to obtain

$$f(z) - H_f = f_0(z) + f_1(P(z)) + \ldots + f_n(P^{\circ n}(z)) + e_n(P^{\circ n+1}(z)).$$

where $f_0, \ldots, f_n$ are odd.

Notice that here we have $0 = H(f_0) = H(f_1) = \ldots = H(f_n)$ because H is even and invariant. We also get $H(e_n) = 0$ by linearity (together with invariance). From our previous work we get further estimates.

Lemma 2.7. *We have $||e_n|| \leq ||f - H_f|| \leq 2||f||$.* □

Corollary 2.8. *For $n < m$ we get*

$$||f_{n+1}(P^{\circ n+1}(z)) + \ldots + f_m(P^{\circ m}(z))|| \leq 2||e_n||.$$

Proof. From Proposition 2.3 with

$$e_n(P^{\circ n+1}(z)) = f_{n+1}(P^{\circ n+1}(z)) + \ldots + f_m(P^{\circ m}(z)) + e_m(P^{\circ m+1}(z))$$

in the role of f we get $||e_m|| \leq ||e_n||$. Then we apply several times Lemma 2.2 and reduce to

$$e_n(z) = f_{n+1}(z) + \ldots + f_m(P^{\circ m-n}(z)) + e_m(P^{\circ m-n+1}(z)).$$

From here we conclude

$$||f_{n+1}(z) + \ldots + f_m(P^{\circ m-n}(z))|| \leq ||e_n|| + ||e_m|| \leq 2||e_n||.$$

□

Lemma 2.9. *If K have no interior, then $||e_n|| \to 0$.*

Proof. Given $\epsilon > 0$, choose a polynomial Q so that $|f(z) - H_f - Q(z)| \leq \epsilon$ on K. Expand Q as $Q(z) = H_Q + \sum_{i=0}^{N} Q_i(P^{\circ i}(z))$. Then, by uniqueness, for $n > N$ we get

$$\begin{aligned} f(z) - H_f - Q(z) &= H_Q + \sum_{i=0}^{N} f_i(P^{\circ i}(z)) - Q_i(P^{\circ i}(z)) \\ &+ \sum_{i=N+1}^{n} f_i(P^{\circ i}(z)) + e_n(P^{\circ n+1}(z)). \end{aligned}$$

Finally, Lemma 2.7 yields $||e_n|| \leq 2||f - H_f - Q|| \leq 2\epsilon$ when applied to $f - H_f - Q$. □

The expansion in the next theorem is **the harmonic decomposition of f**.

Theorem 2.10. *Let K have no interior. Then for $f \in C(K)$ there are odd continuous functions $f_0, f_1, \ldots$ such that*

$$f(z) = H_f + f_0(z) + f_1(P(z)) + f_2(P^{\circ 2}(z)) + \ldots;$$

the convergence here is uniform.

Proof. In fact, for $m \geq n \geq N$ we have

$$||f_{n+1}(P^{\circ n+1}(z)) + \ldots + f_m(P^{\circ m}(z))|| \leq 2||e_n|| \to 0.$$

So, the partial sums form a Cauchy sequence since the discrepancy e_n tends to 0. □

The next result is trivial after inspecting grand orbits. Anyhow, we present an alternative proof.

Lemma 2.11 (Lyubich [2])**.** *If K has no interior, the only invariant continuous functions are the constants.*

Proof. In fact, if $f = f \circ P$, then by matching their harmonic decompositions we get $H_f = H_{f\circ P}$ together with $f_0 = 0, f_1 = f_0, f_2 = f_1, \ldots$. Hence $f(z) = H_f$ is a constant. □

3. The dual decomposition

For the study of measures supported in K we will take the functional analysis approach. Thus, a "measure" on K "is" a linear functional (with values in $\mathbb{C}$) defined on $C(K)$. We denote by $\mathcal{M}(K)$ the space of (complex valued) measures.

Given a measure ν, the **odd** and **even parts** are given by

$$\mathcal{O}(\nu)(f) = \nu(\mathcal{O}(f)) \qquad \mathcal{E}(\nu)(f) = \nu(\mathcal{E}(f)).$$

Evidently, we get $||\mathcal{O}(\nu)||, ||\mathcal{E}(\nu)|| \leq ||\nu||$ (because at the level of functions we have $||\mathcal{O}||, ||\mathcal{E}|| \leq 1$). Also, note the equality $\nu = \mathcal{O}(\nu) + \mathcal{E}(\nu)$. The measure $\mathcal{O}(\nu)$ is odd in the sense that it kills all even functions, while $\mathcal{E}(\nu)$ is even as it kills the odd functions.

Example 3.1. For the delta mass δ_{z_0} based at a point $z_0 \in K$, the even part $\mathcal{E}(\delta_{z_0})$ is given by $\frac{1}{2}\sum_{P(\hat{z})=P(z_0)} \delta_{\hat{z}} = \frac{\delta_{z_0}+\delta_{-z_0}}{2}$. In fact, we get

$$\mathcal{E}(\delta_{z_0})(f) = \mathcal{E}(f)(z_0) = \frac{1}{2}\sum_{P(\hat{z})=P(z_0)} f(\hat{z}) = \left(\frac{1}{2}\sum_{P(\hat{z})=P(z_0)} \delta_{\hat{z}}\right)(f).$$

As a by-product we obtain

$$\mathcal{O}(\delta_{z_0}) = \frac{\delta_{z_0}-\delta_{-z_0}}{2}.$$

It is important to set some notation straight. Instead of the customary $d\nu(z)$ we will use $\nu(z)$ most of the time. In this way, given $f \in C(K)$, we write

$$\nu(f) = \int f\,\nu = \int f(z)\,\nu(z)$$

when needed. We will even use $\nu(z)$, meaning ν, when the context calls for it.

The measure $\nu \circ P$ (or $\nu(P(z))$ in brief) is by convention the even measure that satisfies

$$\int g(P(z))\nu(P(z)) = \int g(z)\nu(z).$$

Example 3.2. The harmonic measure is even as the relation $H(f) = H(\mathcal{E}(f))$ is equivalent to $H(f) = \mathcal{E}(H)(f)$.

Also, for $f \in C(K)$ we set $h_f(z) = f(z)\,H(z)$, where $H(z)$ is the standard harmonic measure as defined in Section 2. Then we have $h_f(P(z)) = f(P(z))\,H(z)$. In fact, as both of the above measures are even, it is enough to check the equality

$$\int g(P(z))\,h_f(P(z)) = \int g(P(z))f(P(z))\,H(z).$$

For them, however, we readily get

$$\begin{aligned}\int g(P(z))\, h_f(P(z)) &= \int g(z)\, h_f(z)\\ &= \int g(z)f(z)\, H(z)\\ &= \int g(P(z))f(P(z))\, H(z),\end{aligned}$$

where the fist equality is given by convention, the second by definition of h_f, and the third by the invariance of the harmonic measure. This notable fact is what justifies our convention for the dynamical push-forward of the measure.

Example 3.3. We claim that

$$\mathcal{E}(\delta_{z_0}) = \frac{\delta_{z_0} + \delta_{-z_0}}{2} = \delta_{z_1} \circ P$$

holds (as usual, we have $P(z_0) = z_1$). In fact, let $f(z) = f_0(z) + g(P(z))$ with f_0 odd. Then we have

$$\begin{aligned}\int f_0(z) + g(P(z))\, \delta_{z_1}(P(z)) &= \int g(P(z))\, \delta_{z_1}(P(z))\\ &= \int g(z)\, \delta_{z_1}(z)\\ &= g(z_1).\end{aligned}$$

On the other side, oddness of f_0 implies $\frac{1}{2}\sum_{P(\hat{z})=z_1} f_0(\hat{z}) = 0$, so we get

$$\int f_0(z) + g(P(z))\, (\frac{1}{2}\sum_{P(\hat{z})=z_1} \delta_{\hat{z}}) = \frac{1}{2}\sum_{P(\hat{z})=z_1} (f_0(\hat{z}) + g(P(\hat{z}))) = g(z_1).$$

Thus, the two values coincide, and the measures agree.

An easy induction delivers $\delta_{z_n} \circ P^{\circ n} = \dfrac{1}{2^n} \displaystyle\sum_{P^{\circ n}(\hat{z})=z_n} \delta_{\hat{z}}$ as well, for $z_n = P^{\circ n}(z_0)$.

Lemma 3.4. *The measures τ and $\tau \circ P$ have the same norm.*

Proof. Notice that $\tau(f) = \tau \circ P(f \circ P)$ implies $||\tau|| \leq ||\tau \circ P||$.

Now take $f + g \circ P$ with f odd subject to $||f + g \circ P|| \leq 1$. Then $||g|| = ||g \circ P|| = ||\mathcal{E}(f + g \circ P)|| \leq 1$ forces

$$||\tau \circ P(f + g \circ P)|| = ||\tau \circ P(g \circ P)|| = ||\tau(g)|| \leq ||\tau||\,||g|| \leq ||\tau||.$$

□

Lemma 3.5. *All even measures have the form $\tau \circ P$ for some $\tau \in \mathcal{M}(K)$.*

Proof. Let ν be a measure that kills all odd functions. For the functional

$$\tau(f) = \int f(P(z))\,\nu(z),$$

the convention $\int f(z)\,\tau(z) = \int f(P(z))\,\tau(P(z))$ joining forces with the symbolism $\tau(f) = \int f(z)\,\tau(z)$ leads us to $\nu(z) = \tau(P(z))$. □

In view of Lemma 3.5, we have a natural splitting

$$\nu(z) = \nu_0(z) + \sigma(P(z)),$$

where ν_0 is odd. However, before iterating this odd–even decomposition, practice gained in the manipulation of continuous functions suggests we better subtract the "harmonic" part first. For that, we set

$$H_\nu = \nu(1) = \int \nu(z).$$

Whenever we have $H_\nu = 0$, we say that ν is **harmonic free**.

Proposition 3.6. *Fix $n \geq 0$. There are unique odd measures $\nu_0, \dots \nu_n$ and a measure τ_n such that*

$$\nu(z) = H_\nu\, dH(z) + \nu_0(z) + \cdots + \nu_n(P^{\circ n}(z)) + \tau_n(P^{\circ n+1}(z)).$$

This decomposition is unique provided $H_{\tau_n} = 0$. In this case we have $||\tau_n|| \leq 2||\nu||$ and $||\nu_i|| \leq ||\nu||$.

Proof. Trivial. □

What is not trivial is the following asymptotic decomposition.

Theorem 3.7. *If K has no interior, the partial sums*

$$H_\nu \, dH(z) + \nu_0(z) + \cdots + \nu_n(P^{\circ n}(z))$$

converge $*$-weak to ν.

Proof. Key here is to understand how

$$\nu(z) = H_\nu \, dH(z) + \nu_0(z) + \cdots + \nu_n(P^{\circ n}(z)) + \tau_n(P^{\circ n+1}(z))$$

acts on the function

$$f(z) = H_f + f_0(z) + f_1(P(z)) + \ldots + f_n(P^{\circ n}(z)) + e_n(P^{\circ n+1}(z)).$$

To begin with, by definition H_f is the way how $dH(z)$ acts on f. Therefore $H_\nu \, dH(z)$ paired against f gives $H_f H_\nu$.

Next, $\nu_i(P^{\circ i}(z))$ acts on $g(P^{\circ m}(z))$, with $m > i$, as $\nu_i(z)$ acts on $G(P^{\circ m-i}(z))$, hence kills them all since ν_i is odd and the said functions are even. This applies to H_f, $f_m(P^{\circ m}(z))$, for $m > i$, and to $e_n(P^{\circ n+1}(z))$. When $m < i$ then $\nu_i(P^{\circ i}(z))$ acts on $f_m(P^{\circ m}(z))$ in the same way as $\nu_i(P^{\circ i-m}(z))$ acts on $f_m(z)$, thus annihilating them. We also have $\displaystyle\int f_i(P^{\circ i}(z))\nu_i(P^{\circ i}(z)) = \int f_i(z)\nu_i(z)$ by reduction, the surviving term at this stage.

Finally, it should be clear by now that $\tau_n(P^{\circ n+1}(z))$ annihilates all the f_i. Also, evaluating at the constant function 1 we get

$$\nu(1) = H(\nu)H(1) + \nu_0(1) + \ldots + \nu_n(1) + \tau_n(1).$$

Since we have relations $\nu_i(1) = 0$ and $H_\nu H(1) = \nu(1)H(1) = \nu(1)$, we conclude the equality $\tau_n(1) = 0$. Therefore, $\tau_n(P^{\circ n+1}(z))$ acts merely on $e_n(P^{\circ n+1}(z))$.

Collecting our findings we obtain

$$\int f(z)\nu(z) = H_f H_\nu + \sum_{i=0}^{n} \int f_i(z)\nu_i(z) + \int e_n(z)\tau_n(z).$$

By the above formula, the action on f of $\nu(z) - \tau_n(P^{\circ n+1}(z))$ (i.e, of $H_\nu\, dH(z) + \nu_0(z) + \cdots + \nu_n(P^{\circ n}(z)))$ is $H_f H_\nu + \sum_{i=0}^{n} \int f_i(z)\nu_i(z)$; which in turn equals $\int f(z)\nu(z) - \int e_n(z)\tau_n(z)$. However,

$$\left| \int e_n(z)\tau_n(z) \right| \leq ||e_n||\, ||\tau_n|| \leq 2||\nu||\, ||e_n||$$

converges to 0, so we are done. □

Example 3.8. We try the decomposition of a delta mass. Let $z_0 \in K$. Then we have

$$\delta_{z_0}(z) = dH(z) + \Delta_0(z) + \ldots + \Delta_n(P^{\circ n}(z)) + \ldots,$$

since the harmonic part is $\delta_{z_0}(1) = 1$.

Now take $f \in C(K)$ odd (so that $f(z) + f(-z) = 0$ for all $z \in K$). Then the formula

$$\int f(z)\, \Delta_0(z) = \int f(z)\, \delta_{z_0}(z) = f(z_0) = \frac{f(z_0)}{2} - \frac{f(-z_0)}{2},$$

shows that the odd part of δ_{z_0} is $\dfrac{\delta_{z_0} - \delta_{-z_0}}{2}$ (compare also Example 3.3).

In general, (we use here the convention $z_n = P^{\circ n}(z_0)))$ for f odd we get

$$\int f(P^{\circ n}(z))\, \Delta_n(P^{\circ n}(z)) = \int f(P^{\circ n}(z))\, \delta_{z_0}(z) = f(z_n),$$

and we conclude that Δ_n is the odd part of δ_{z_n}, that is $\dfrac{\delta_{z_n} - \delta_{-z_n}}{2}$. In short, we have

$$\delta_{z_0} = 1 + \sum_{i=0}^{\infty} \mathcal{O}(\delta_{z_i}) \circ P^{\circ i}.$$

A measure ν is **invariant** when for all $f \in C(K)$ we have

$$\int f(z)\,\nu(z) = \int f(P(z))\,\nu(z).$$

The harmonic measure $dH(z)$ and the delta masses δ_{z_f} located at fixed points z_f are prototypical examples of invariant measures. This is in sharp contrast with the function case where we only have one invariant object. Other examples of invariant measures are averages along periodic orbits.

The following is a characterization of invariant measures using the canonical decomposition.

Theorem 3.9. *Suppose K has no interior. If $\nu(z) = \alpha + \nu_0(z) + \nu_1(P(z)) + \ldots$ is an invariant measure, then $\nu_0 = \nu_1 = \nu_2 = \ldots$.*

Proof. For any odd test function f we get thanks to invariance

$$\begin{aligned} \int f(z)\nu_n(z) &= \int f(P^{\circ n}(z))\nu_n(P^{\circ n}(z)) \\ &= \int f(P^{\circ n}(z))\nu(z) \\ &= \int f(z)\nu(z) \\ &= \int f(z)\nu_0(z). \end{aligned}$$

Therefore ν_n and ν_0 are the same functional. $\square$

As an extra remark, we should indicate that not all odd functions give rise to invariant measures. For instance, we will see briefly that the odd part of a delta mass seldom determines an invariant measure.

Corollary 3.10. *If K has no interior, then the space of even invariant measures supported in K is one-dimensional.* $\square$

Theorem 3.11. *Suppose K has no interior. Let ν_0 be an odd measure. Then the partial sums $\mu_n = \nu_0 + \nu_0 \circ P + \ldots + \nu_n \circ P^{\circ n}$ converge (∗-weak) to an invariant measure if and only if there is a constant M so that $||\mu_n|| \leq M$.*

Proof. If the sequence μ_n converges ∗-weak, then their norms certainly form a bounded sequence.

On the other side, if $||\mu_n||$ is bounded, it carries ∗-weakly convergent subsequences. Therefore it is enough to prove that for all $f \in C(K)$ the limit of $\mu_n(f)$ exists. Given $\epsilon > 0$, let N be such that for $n \geq N$ we have

$$f(z) = H_f + f_0(z) + \ldots + f_n(P^{\circ n}) + e_n(P^{\circ n+1}(z)),$$

with $||e_n|| \leq \epsilon$. When we take $m > n \geq N$, we get

$$|\mu_m(f) - \mu_n(f)| = |(\mu_n - \mu_m)(e_N \circ P^{\circ N+1})| \leq 2M||e_N|| \leq 2M\epsilon.$$

□

Example 3.12. Let $z_0 \in K$ be a non-periodic point outside the orbit of the critical point (any point with a countable number of exceptions would do). We use Theorem 3.11 to prove that the odd part of the delta mass δ_{z_0} does not generate an invariant measure.

If $\pm z_{-1}$ are the two preimages of z_0, the measures $\delta_{\pm z_{-1}}(P^{\circ i}(z))$ have total mass 1 and support $(P^{\circ i})^{-1}(\pm z_{-1})$, mutually disjoint sets. The bottom line is that

$$\sum_{i=0}^{n-1} (\frac{\delta_{z_{-1}} - \delta_{-z_{-1}}}{2}) \circ P^{\circ i}$$

has norm n.

4. Iteration and reduction

In this section we study the iteration process as an operator acting both on continuous functions and on measures. For better understanding, we introduce in parallel the process of reduction.

The **iteration operator it** is defined in continuous functions as $\mathbf{it}(f)(z) = f(P(z))$ and in measures as $\mathbf{it}(\nu)(z) = \nu(P(z))$. The **reduction operator red** is defined as follows. If $\varphi(z) = \mathcal{O}(\varphi)(z) + \psi(P(z))$, then we set $\mathbf{red}(\varphi)(z) = \psi(z)$, both for functions and measures.

When K has no interior and

$$\varphi(z) = H_\varphi + \varphi_0(z) + \sum_{i=1}^{\infty} \varphi_i(P^{\circ i}(z)),$$

with φ_i odd, holds, then we write

$$\mathbf{red}(\varphi)(z) = H_\varphi + \sum_{i=1}^{\infty} \varphi_i(P^{\circ i-1}(z)).$$

Proposition 4.1. *The adjoint operator of* $\mathbf{red} : C(K) \to C(K)$ *is given by* $\mathbf{it} : \mathcal{M}(K) \to \mathcal{M}(K)$*, while the adjoint of* $\mathbf{it} : C(K) \to C(K)$ *is* $\mathbf{red} : \mathcal{M}(K) \to \mathcal{M}(K)$*. Both are norm* 1 *operators.*

Proof. For $f \in C(K)$ let $f(z) = \mathcal{O}(f)(z) + g(P(z))$ and for $\nu \in \mathcal{M}(K)$ let $\nu(z) = \mathcal{O}(\nu)(z) + \tau(P(z))$. Then we have

$$\begin{aligned} \int f(z)\,\mathbf{red}^*(\nu)(z) &= \int \mathbf{red}(f)(z)\,\nu(z) = \int g(z)\,\nu(z) \\ &= \int g(P(z))\,\nu(P(z)) \\ &= \int \mathcal{O}(f)(z) + g(P(z))\,\nu(P(z)) \\ &= \int f(z)\,\mathbf{it}(\nu)(z). \end{aligned}$$

Therefore, we obtain $\mathbf{it}(\nu) = \mathbf{red}^*(\nu)$.

The other identity is tackled in a similar way.

About the norm, this should be obvious by now. □

Next we comment briefly about the operators $I-\lambda\,\mathbf{it}$ and $I-\lambda\,\mathbf{red}$, with $\lambda \in \mathbb{C}$.

Lemma 4.2. *Both in $C(K)$ and in $\mathcal{M}(K)$ the operators $I-\lambda\,\mathbf{it}$ and $I-\lambda\,\mathbf{red}$ are invertible for $|\lambda|<1$.*

Proof. In fact, both **it** and **red** have norm 1. □

Lemma 4.3. *Both in $C(K)$ and in $\mathcal{M}(K)$, for $|\lambda|>1$, the operators $I-\lambda\,\mathbf{it}$ are closed, injective but not surjective, while the $I-\lambda\,\mathbf{red}$ are closed, surjective but not injective.*

Proof. We first attack the surjectivity of $I-\lambda\,\mathbf{red}$. Given ψ in the appropriate space, we define $\varphi(z) = -\sum_{i=0}^{\infty} \psi(P^{\circ i+1}(z))/\lambda^{i+1}$. From

$$\begin{aligned}
\lambda\,\mathbf{red}(\varphi)(z) &= -\lambda \sum_{i=0}^{\infty} \psi(P^{\circ i}(z))/\lambda^{i+1} \\
&= -\sum_{i=0}^{\infty} \psi(P^{\circ i}(z))/\lambda^{i} \\
&= -\psi(z) - \sum_{i=1}^{\infty} \psi(P^{\circ i}(z))/\lambda^{i} \\
&= -\psi(z) + \varphi(z),
\end{aligned}$$

we get $\{I-\lambda\,\mathbf{red}\}(\varphi)=\psi$, and the operator is surjective. Evidently, a surjective operator has closed range. Also, for any odd ψ, the element $\sum_{i=0}^{\infty} \psi(P^{\circ i}(z))/\lambda^{i}$ is well defined (since $|\lambda|>1$) and belongs to the kernel of $I-\lambda\,\mathbf{red}$.

The properties for the operator $I-\lambda\,\mathbf{it}$ follow by duality. □

When $|\lambda|=1$, the study of those operators is not simple. We will be concerned specially with the case $\lambda=1$, since they help characterize invariant measures.

Proposition 4.4. *For $\nu \in \mathcal{M}(K)$ the following properties are equivalent.*

- *The measure ν is invariant;*
- *the condition $\{I - \mathbf{it}\}(\nu) = \mathcal{O}(\nu)$ holds;*
- *the measure $\{I - \mathbf{it}\}(\nu)$ is odd;*
- *the measure ν belongs to the kernel of $I - \mathbf{red}$.*

Proof. Everything is trivial. □

Proposition 4.5. *If K has empty interior, the kernel of $I - \mathbf{it}$ is one dimensional: it consists of the constants or of the multiples of the harmonic measure, depending in the case. These operators are not closed.*

Proof. It is clear that the constants (or constant multiples of H) are the only members of the kernel of $I - \mathbf{it}$.

To prove that this operator acting on continuous functions is not closed, we note that the space of all functions annihilated by the harmonic measure is a codimension one space in where $I - \mathbf{it}$ acts injectively. Therefore it is enough to construct a sequence of harmonic free functions φ_n of norm greater or equal to 1 such that $||\{I - \mathbf{it}\}(\varphi_n)||$ converges to 0. With that in mind, let z_f be a non-critical fixed point of P. Let $F : K \to [-1, 1]$ be any continuous function such that $F(z_f) = 1$ and $F(-z_f) = -1$. Write $F_0 = \mathcal{O}(F)$. Notice that $F(z_f) = -F(-z_f)$ implies $F_0(z_f) = F(z_f) = 1$. Therefore we get $1 \leq ||F_0|| \leq ||F|| = 1$. Now for $\varphi_n(z) = \dfrac{1}{n}\displaystyle\sum_{i=0}^{n-1} F_0(P^{\circ i}(z))$ we have $\varphi_n(z_f) = 1$, and so $||\varphi_n|| \geq 1$. However by construction the function $\{I - \mathbf{it}\}(\varphi_n)(z) = \dfrac{F_0(z) - F_0(P^{\circ n}(z))}{n}$ has norm at most $2/n$.

For measures we proceed similarly: for z_0 a point that is not eventually periodic (compare Example 3.12), we take the odd measure $\nu_0 = \mathcal{O}(\delta_{z_0})$ and define $\varphi_n(z) = (1/n)\sum_{i=0}^{n-1} \nu_0(P^{\circ i}(z))$. A trivial calculation gives then $||\varphi_n|| = 1$ and $||\{I - \mathbf{it}\}(\varphi_n)|| = 2/n$. □

Corollary 4.6. *If K has empty interior, the image of $Id - \mathbf{red}$ is dense in the space of harmonic free objects. This operator is not closed.*

Proof. This follows from Proposition 4.5 by duality. □

References

[1] T. Gamelin; *Uniform Algebras*, American Mathematical Society, 2005.

[2] M. Lyubich; *The measure of maximal entropy of a rational endomorphism of a Riemann sphere,* Funktsional. Anal. i Prilozhen. 16 (1982), no. 4, 78-79. (Russian).

[3] J. Milnor; *Dynamics in One Complex Variable, Introductory Lectures,* Vieweg, 1999.

[4] W. Rudin; *Real and Complex Analysis*, McGraw Hill, 1974.

Resumen

En este artículo caracterizamos medidas invariantes sobre conjuntos de Julia sin interior asociados con polinomios cuadráticos. Probamos que más allá de la medida armónica —la única par e invariante—, el resto son generadas por su parte impar.

Palabras clave: Dinámica holomorfa, iteración de polinomios, conjunto de Maldelbrot, medidas invariantes.

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