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CONTENIDO

Hexagonalidad y estructura transversalmente afín

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Resumen

El presente trabajo prueba que la hexagonalidad del web $\mathcal{G}_{\mathcal{F}}$, imagen directa de la foliación $\mathcal{F}$ por la aplicación de Gauss $\mathcal{G}_{\mathcal{F}} : \mathbb{P}^2_{\mathbb{C}} \dashrightarrow \check{\mathbb{P}}^2_{\mathbb{C}}$, implica que la foliación $\mathcal{F}$ es transversalmente afín.

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1. Introducción

Existe una noción geométrica de hexagonalidad de un $3-$web $\mathcal{W}$ en $\mathbb{C}^3$ introducida por Blaschke y Dubourdieu [2] y caracterizada por la equivalencia de $\mathcal{W}$ con el $3-$web trivial dado por $\mathrm{d}x \cdot \mathrm{d}y \cdot \mathrm{d}z = 0$. No obstante, otra forma de caracterizar dicha noción es a través del anulamiento de su curvatura, una $2-$forma meromorfa con polos contenidos en el discriminante del web $\mathcal{W}$. La hexagonalidad para un $d-$web $\mathcal{W}$, con $d > 3$, se analiza a partir de la hexagonalidad de todos sus $3-$subwebs [16].

La aplicación de Gauss $\mathcal{G}_{\mathcal{F}}$ asociada a una foliación $\mathcal{F}$ sobre $\mathbb{P}^2_{\mathbb{C}}$ es una aplicación racional $\mathcal{G}_{\mathcal{F}} : \mathbb{P}^2_{\mathbb{C}} \dashrightarrow \check{\mathbb{P}}^2_{\mathbb{C}}$ definida por $\mathcal{G}_{\mathcal{F}}(p) = T_p\mathcal{F}$, donde $T_p\mathcal{F}$ es la recta tangente de la foliación $\mathcal{F}$ en un punto regular p, cuyo conjunto de indeterminación coincide con el conjunto singular de $\mathcal{F}$. Nuestro punto de partida son las siguientes construcciones.

i) Si la foliación $\mathcal{F}$ es de grado d, en uso de la dualidad proyectiva, obtenemos en $\check{\mathbb{P}}^2_{\mathbb{C}}$ un $d-$web denotado por $\mathrm{Leg}\mathcal{F}$.

ii) La imagen directa $\mathcal{G}_*\mathcal{F}$ de la foliación $\mathcal{F}$ es el $d-$web sobre $\check{\mathbb{P}}^2_{\mathbb{C}}$ que se obtiene por la superposición de las $d-$foliaciones definidas por la aplicación de recubrimiento $\mathcal{G}_{\mathcal{F}}$.

En [7] los autores asocian a foliaciones de grado 3 sobre $\mathbb{P}^2_{\mathbb{C}}$ una trivolución birracional: toda recta genérica ℓ es tangente a una foliación $\mathcal{F}$ en tres puntos; la aplicación buscada que intercambia estos puntos en general es multivaluada. Ellos obtienen un criterio para que dicha aplicación sea birracional y construyen ejemplos de $3-$webs hexagonales a partir de foliaciones que admiten trivoluciones no triviales asociadas a una foliación $\mathcal{F}$ de grado 3.

En este artículo demostramos que la hexagonalidad del web $\mathcal{G}_*\mathcal{F}$ implica que la foliación $\mathcal{F}$ es transversalmente afín, donde $\mathcal{G}_{\mathcal{F}} : \mathbb{P}^2 \dashrightarrow \check{\mathbb{P}}^2_{\mathbb{C}}$ es la aplicación de Gauss asociada a la foliación $\mathcal{F}$.

Definición 1.1. Una **estructura transversalmente afín singular** para una foliación $\mathcal{F}$ sobre el plano proyectivo complejo $\mathbb{P}^2_{\mathbb{C}}$ consiste de un par (ω, η) de $1-$formas meromorfas que verifican las siguientes condiciones:

i) ω define la foliación $\mathcal{F}$,

ii) $\mathrm{d}\omega = \omega \wedge \eta$,

iii) $\mathrm{d}\eta = 0$.

Un criterio para determinar cuando una foliación admite una estructura transversalmente afín es el criterio de Singer (ver [15]).

Teorema 1.2 (**Criterio de Singer**). *Sea ω una forma racional en $\mathbb{C}^2$. Entonces ω tiene una integral primera liouvilliana si y solamente si la foliación definida por $\mathcal{F}$ admite una estructura transversalmente afín.*

Webs

Un **germen de $k-$web singular** de codimensión uno sobre $(\mathbb{C}^n, 0)$ es una clase de equivalencia $[\omega]$ de gérmenes de $k-$formas simétricas, esto es, secciones de $\mathrm{Sym}^k\Omega^1(\mathbb{C}^n, 0)$ módulo multiplicación por $\mathcal{O}^*(\mathbb{C}^n, 0)$, tales que cualquier representante ω definido en una vecindad conexa U del origen verifica las siguientes propiedades:

1. el conjunto de ceros de ω tiene codimensión mayor o igual a 2;
2. ω puede ser vista como un polinomio homogéneo de grado k libre de factores cuadrados en el anillo $\mathcal{O}(\mathbb{C}^n, 0)[\mathrm{d}x_1, \ldots, \mathrm{d}x_k]$;
3. para un punto genérico $p \in U$, $\omega(p)$ es un producto de k $1-$formas;
4. para un punto genérico $p \in U$, el germen de ω en p es el producto de $k-$gérmenes de $1-$formas integrables.

Un $k-$**web global $\mathcal{W}$ de codimensión uno** sobre una superficie compleja $\mathbf{S}$ está dado por una cobertura por abiertos $\mathcal{U} = \{U_i\}$ de $\mathbf{S}$ y $k-$formas simétricas $\omega_i \in \mathrm{Sym}^k\Omega^1_{\mathbf{S}}(U_i)$ que verifican las siguientes condiciones.

a) Para cada intersección no vacía $U_i \cap U_j$ de elementos de $\mathcal{U}$ existe una función no nula $g_{ij} \in \mathcal{O}^*(U_i \cap U_j)$ tal que $\omega_i = g_{ij}\omega_j$.

b) El conjunto singular $\mathrm{Sing}(\omega_i)$ de ω_i tiene codimensión al menos dos.

c) Para cada $U_i \in \mathcal{U}$ y $p \in U_i$ genérico, el germen de ω_i en p es un producto de k gérmenes de $1-$formas integrables que no son colineales dos a dos.

Cuando $k = 1$, recuperamos la definición de foliación singular de codimensión uno.

El subconjunto de $\mathbf{S}$ donde la condición (c) falla es denominado **discriminante** de $\mathcal{W}$ y se denota por $\Delta(\mathcal{W})$. Asimismo, el **conjunto singular** $\Sigma_{\mathcal{W}}$ de $\mathcal{W}$ es definido por $\Sigma_{\mathcal{W}} \cap U_i = \mathrm{Sing}(\omega_i)$ y está contenido en $\Delta(\mathcal{W})$.

Si la superficie $\mathbf{S}$ es compacta, entonces un $k-$**web global** es un elemento del espacio $H^0(\mathbf{S}, \mathrm{Sym}^k\Omega^1_{\mathbf{S}} \otimes \mathcal{N})$, donde $\mathcal{N} \in \mathrm{Pic}(\mathbf{S})$ es un fibrado lineal cuyo germen de cualquier representante en cualquier punto de $\mathbf{S}$ verifica las condiciones (1-4).

Web asociado a una foliación

De la secuencia de Euler

$$0 \to \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}} \to \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(1)^{\oplus 3} \to T\mathbb{P}^2_{\mathbb{C}} \to 0$$

podemos deducir la siguiente secuencia exacta

$$0 \to \mathrm{Sym}^{k-1}(\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(1)^{\oplus 3}) \otimes \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}} \to \mathrm{Sym}^k(\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(1)^{\oplus 3}) \to \mathrm{Sym}^k T\mathbb{P}^2_{\mathbb{C}} \to 0.$$

De ello se sigue que un $k-$web de grado d sobre $\mathbb{P}^2_{\mathbb{C}}$ queda determinado por un polinomio bihomogéneo $P(x, y, z; a, b, c)$ de grado d en las coordenadas (x, y, z) y grado k en las coordenadas (a, b, c), respectivamente. Las coordenadas $(a : b : c)$ son las coordenadas homogéneas naturales en el plano dual proyectivo:

$$\begin{aligned} T^*_{(x:y:z)}\mathbb{P}^2_{\mathbb{C}} &= \{\omega = a\mathrm{d}x + b\mathrm{d}y + c\mathrm{d}z \in T^*\mathbb{C}^3 : \omega(R) = 0\} \\ &= \{a\mathrm{d}x + b\mathrm{d}y + c\mathrm{d}z : ax + by + cz = 0\}, \end{aligned}$$

donde R es el campo radial. Existe una identificación natural de $\mathbb{P}T^*\mathbb{P}^2_{\mathbb{C}}$ con la variedad de incidencia

$$\mathcal{I} = \{((x : y : z), (a : b : c)) : \ ax + by + cz = 0\} \subset \mathbb{P}^2_{\mathbb{C}} \times \check{\mathbb{P}}^2_{\mathbb{C}}.$$

Sean $\mathcal{W}$ un $k-$web de grado d sobre $\mathbb{P}^2_{\mathbb{C}}$ y $P(x, y, z; a, , b, c)$ un polinomio bihomogéneo que define $\mathcal{W}$. Entonces $\mathbf{S}_{\mathcal{W}} \subset \mathbb{P}T^*\mathbb{P}^2_{\mathbb{C}}$, en adelante el **gráfico de $\mathcal{W}$ sobre** $\mathbb{P}T^*\mathbb{P}^2_{\mathbb{C}}$, está dado por

$$\begin{aligned} \mathbf{S}_{\mathcal{W}} = \{((x : y : z); (a : b : c)) \in \mathbb{P}^2_{\mathbb{C}} \times \check{\mathbb{P}}^2_{\mathbb{C}} : \\ ax + by + cz = 0, \ P(x, y, z; a, b, c) = 0\} \end{aligned}$$

bajo la identificación entre $\mathcal{I}$ y $\mathbb{P}T^*\mathbb{P}^2_{\mathbb{C}}$.

Supongamos que $\mathcal{W}$ es un web irreducible de grado $d > 0$ y consideremos las restricciones π y $\check{\pi}$ a $\mathbf{S}_{\mathcal{W}}$ de las proyecciones naturales de $\mathbb{P}^2_{\mathbb{C}} \times \check{\mathbb{P}}^2_{\mathbb{C}}$ sobre $\mathbb{P}^2_{\mathbb{C}}$ y $\check{\mathbb{P}}^2_{\mathbb{C}}$ respectivamente. La **distribución de contacto** $\mathcal{D}$ sobre $\mathbb{P}T^*\mathbb{P}^2_{\mathbb{C}}$ es dada por

$$\mathcal{D} = \ker(a\mathrm{d}x + b\mathrm{d}y + c\mathrm{d}z) = \ker(x\mathrm{d}a + y\mathrm{d}b + z\mathrm{d}c).$$

La foliación $\mathcal{F}_{\mathcal{W}}$ inducida por $\mathcal{D}$ sobre $\mathbf{S}_{\mathcal{W}}$ se proyecta a través de π sobre un $k-$web $\mathcal{W}$ y, a través de $\check{\pi}$, sobre un $d-$web $\check{\mathcal{W}}$ en $\check{\mathbb{P}}^2_{\mathbb{C}}$. El $d-$web es llamado **transformado de Legendre** de $\mathcal{W}$ y se suele denotar por $\mathrm{Leg}\mathcal{W}$.

En los libros clásicos de ecuaciones diferenciales ordinarias uno encuentra que el transformado de Legendre es una transformación involutiva que envía una ecuación diferencial polinomial de la forma $F(x, y, p) = 0$ a otra de la forma $F(P, XP - Y, X) = 0$, donde $p = \dfrac{\mathrm{d}y}{\mathrm{d}x}$ y $P = \dfrac{\mathrm{d}Y}{\mathrm{d}X}$; ver por ejemplo [9, pág. 40].

Consideremos una carta afín (x, y) de $\mathbb{P}^2_{\mathbb{C}}$ y una carta afín (p, q) de $\check{\mathbb{P}}^2_{\mathbb{C}}$ cuyas coordenadas corresponden a la recta $\{y = px + q\}$. Si un web $\mathcal{W}$ es definido por una ecuación diferencial implícita $F(x, y, p) = 0$, con $p = \dfrac{\mathrm{d}y}{\mathrm{d}x}$, entonces el web Leg$\mathcal{W}$ es definido por la ecuación implícita afín

$$\check{F}(p, q; x) = F(x, px + q, p) = 0, \quad \text{donde } x = -\frac{\mathrm{d}q}{\mathrm{d}p}; \tag{1.1}$$

en este caso el discriminante es el divisor dado por $\check{\Delta}_{\mathcal{F}} = \{\mathrm{disc}(\check{F}) = 0\}$, donde $\mathrm{disc}(\check{F})$ es el $x-$discriminante.

Observación 1.3. Si $\mathcal{F} : X = A(x, y)\dfrac{\partial}{\partial x} + B(x, y)\dfrac{\partial}{\partial y}$ es una foliación no saturada pero libre de cuadrados, entonces su transformado de Legendre está bien definido.

Curvatura de webs

Sea $\mathcal{W}$ un $k-$web completamente descomponible sobre una superficie $\mathbf{S}$, esto es $\mathcal{W} = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k$ donde cada foliación $\mathcal{F}_i$ es definida por la $1-$forma ω_i con singularidades aisladas. De acuerdo con [16], para cada terna (r, s, t) con $1 \leq r < s < t \leq k$ definimos $\eta_{rst} = \eta(\mathcal{F}_r \boxtimes \mathcal{F}_s \boxtimes \mathcal{F}_t)$ como la única $1-$forma meromorfa que verifica las relaciones $\mathrm{d}(\delta_{st}\omega_r) = \eta_{rst} \wedge \delta_{st}\omega_r$, $\mathrm{d}(\delta_{tr}\omega_s) = \eta_{rst} \wedge \delta_{tr}\omega_s$ y $\mathrm{d}(\delta_{rs}\omega_t) = \eta_{rst} \wedge \delta_{rs}\omega_t$; donde la función δ_{ij} satisface la relación $\omega_i \wedge \omega_j = \delta_{ij}\, \mathrm{d}x \wedge \mathrm{d}y$. Las $1-$formas η_{rst} están bien definidas módulo la suma de una $1-$forma holomorfa cerrada. La curvatura del web $\mathcal{W}$ es $K(\mathcal{W}) = \mathrm{d}\eta(\mathcal{W})$, donde $\eta(\mathcal{W}) = \displaystyle\sum_{1 \leq r < s < t \leq k} \eta_{rst}$; es decir, equivale a la suma de las curvaturas de todos los $3-$subwebs de $\mathcal{W}$. Se verifica que $K(\mathcal{W})$ es una $2-$forma meromorfa intrínsecamente asociada a $\mathcal{W}$; esto es, para cualquier aplicación holomorfa dominante φ se tiene $K(\varphi^*\mathcal{W}) = \varphi^*(K(\mathcal{W}))$.

2. Resultado principal

Uno de los asuntos discutidos en [7] es la posible relación entre la existencia de trivoluciones no triviales asociadas a una foliación $\mathcal{F}$ de grado 3 y la hexagonalidad del web Leg$\mathcal{F}$. En esta sección brindamos una respuesta parcial a esta pregunta.

Teorema 2.1. *Sea $\mathcal{F}$ una foliación de grado d sobre $\mathbb{P}^2_{\mathbb{C}}$. Si el web Leg$\mathcal{F}$ es hexagonal, entonces la foliación $\mathcal{F}$ admite una estructura transversalmente afín singular.*

Prueba. El hecho de que Leg$\mathcal{F}$ sea hexagonal es equivalente a que $\mathcal{G}^*_{\mathcal{F}}\mathrm{Leg}\mathcal{F} = \mathcal{F} \boxtimes \mathcal{W}^{\perp}$ sea hexagonal, donde $\boxtimes$ denota la superposición de una foliación con un $(d-1)-$web y $\mathcal{G}_{\mathcal{F}}$ es la aplicación de Gauss asociada a la foliación $\mathcal{F}$. Al aplicar [1, teorema 18.2] al recubrimiento topológico asociado al núcleo de la monodromía del web $\mathcal{W}^{\perp}$ deducimos que existe una superficie algebraica normal $S^{\perp}$ irreducible y un recubrimiento ramificado $\rho : S^{\perp} \longrightarrow \mathbb{P}^2_{\mathbb{C}}$ tal que $\rho^*\mathcal{W}^{\perp}$ es totalmente descomponible. De esta manera se tendrá

$$\mathcal{W} = \rho^*(\mathcal{G}^*_{\mathcal{F}}\mathrm{Leg}\mathcal{F}) = \rho^*(\mathcal{F}) \boxtimes \rho^*(\mathcal{W}^{\perp}) = \mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \cdots \boxtimes \mathcal{F}_d,$$

donde $\mathcal{F}_1 = \rho^*\mathcal{F}$. Asimismo, la curvatura de $\mathcal{W}$ queda definida por $K(\mathcal{W}) = \mathrm{d}\eta(\mathcal{W})$, donde

$$\eta(\mathcal{W}) = \eta(\mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \cdots \boxtimes \mathcal{F}_d) = \sum_{1 \leq r < s < t \leq d} \eta_{rst}.$$

Supongamos ahora que ω_i sea una forma racional que define $\mathcal{F}_i$. Puesto que $\mathcal{W}$ es hexagonal, existe $f_i \in \mathbb{C}(S^{\perp})$ y una $1-$forma racional η sobre $S^{\perp}$ sujetas a

$$\sum_{i=1}^{2} f_i\omega_i = 0, \quad \mathrm{d}(f_i\omega_i) = f_i\omega_i \wedge \eta, \quad \mathrm{d}\eta = 0.$$

De esta manera la foliación $\mathcal{F}_1 = \rho^*\mathcal{F}$ admite una estructura transversalmente afín. De [6, Teorema 1.4] o [7, teorema 2.21] se concluye que la foliación $\mathcal{F}$ también admite una estructura transversalmente afín. □

El recíproco de la proposición no es cierto, aún en el caso Galois (ver [5]), como se ilustra en el siguiente ejemplo.

Ejemplo 2.2. La foliación dada por el campo vectorial

$$X(x,y) = x^3\frac{\partial}{\partial x} + (1 + x + \frac{x^2}{3})\frac{\partial}{\partial y}$$

admite una estructura transversalmente afín, puesto que posee una integral primera liouvilliana dada por $y - \frac{1}{3}\ln x + \frac{1}{x} + \frac{1}{2x^2}$.

Por otro lado, de la relación (1.1) deducimos

$$\check{F}(p,q,x) = pA(a, px + q) - B(x, px + q),$$

donde $x = -\frac{\mathrm{d}q}{\mathrm{d}p}$, $A(x,y) = x^3$ y $B(x,y) = 1 + x + \frac{x^3}{3}$. De esta manera, el web Leg$\mathcal{F}$ está dado por

$$\omega = p\mathrm{d}q^3 + \frac{1}{3}\mathrm{d}q^2\mathrm{d}p - \mathrm{d}q\mathrm{d}p^2 + \mathrm{d}p^3 = 0;$$

ver apéndice 3. Obsérvese que este web no es hexagonal, ya que su curvatura $K(\mathrm{Leg}\mathcal{F}) = \frac{81}{1+27p}\mathrm{d}p \wedge \mathrm{d}q$ es no nula. Además, se tiene que el grupo de transformaciones birracionales $D_{\mathcal{F}} = \{\tau \in \mathrm{Bir}(\mathbb{P}^2) : \mathcal{G}_{\mathcal{F}} = \mathcal{G}_{\mathcal{F}} \circ \tau\} = \langle\tau\rangle$ es isomorfo a $\mathbb{Z}_3$, donde $\mathcal{G}_{\mathcal{F}}$ es la aplicación de Gauss asociada a $\mathcal{F}$ y se cumple

$$\tau(x,y) = \Bigg(x + \frac{x(x+3)(-2x-3+\sqrt{3}i)}{2(x^2+3x+3)}, \\ y + \frac{(x+3)(1+x+\frac{x^2}{3})(-2x-3+\sqrt{3}i)}{x^2(x^2+3x+3)}\Bigg);$$

es decir $\mathcal{F}$ es Galois.

El resultado principal de Cousin y Pereira [8, Theorem A] conduce a una dicotomía para cada foliación transversalmente afín sobre una variedad compleja proyectiva cuyo primer número de Betti es cero.

Teorema 2.3 (Cousin-Pereira)**.** *Sea X una variedad compleja proyectiva tal que $H^1(X, \mathbb{C}) = 0$, y $\mathcal{F}$ una foliación singular transversalmente afín sobre X. Entonces al menos una de las siguientes afirmaciones se verifica:*

1. *existe un morfismo Galois finito $\rho : Y \longrightarrow X$ tal que $\rho^*\mathcal{F}$ queda definida por una $1-$forma racional cerrada; o*

2. *existe una foliación Ricatti $\mathcal{R}$ transversalmente afín sobre una superficie reglada S y una aplicación racional $q : X \dashrightarrow S$ tal que $\mathcal{F} = q^*\mathcal{R}$.*

Si particularizamos este resultado al caso $X = \mathbb{P}^2_{\mathbb{C}}$ obtenemos una descripción bastante precisa de las foliaciones $\mathcal{F}$ transversalmente afines en el plano proyectivo complejo que, por la proposición 2.1, son las únicas candidatas a verificar la condición que Leg$\mathcal{F}$ sea hexagonal.

Corolario 2.4. *Sea $\mathcal{F}$ una foliación sobre $\mathbb{P}^2_{\mathbb{C}}$ cuyo web dual Leg$\mathcal{F}$ es hexagonal. Entonces o existe un morfismo Galois genéricamente finito $p : S \longrightarrow \mathbb{P}^2_{\mathbb{C}}$ tal que $p^*\mathcal{F}$ es definida por una $1-$forma racional cerrada o bien existe una foliación Ricatti $\mathcal{R} : \mathrm{d}y + (a(x) + yb(x))\mathrm{d}x = 0$ transversalmente afín y una aplicación racional $q : \mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ tal que $\mathcal{F} = q^*\mathcal{R}$, donde $a(x)$, $b(x) \in \mathbb{C}(x)$.* □

3. Apéndice

Presentamos un algoritmo realizado en Maple por O. Ripoll (17) para calcular la curvatura de un $3-$web $\mathcal{W}$ conociendo los coeficientes de la función que lo define.

```
restart:with(LinearAlgebra):
kw:=proc(F) local a0,a1,a2,a3,R,alpha0,alpha1,alpha2,k:
a3:=coeff(F,p,0):a2:=coeff(F,p,1):a1:=coeff(F,p,2):a0:=coeff(F,p,3):
R:=Determinant(Matrix([[a0,a1,a2,a3,0],[0,a0,a1,a2,a3],
    [3*a0,2*a1,a2,0,0],[0,3*a0,2*a1,a2,0],[0,0,3*a0,2*a1,a2]]));
alpha0:=[diff(a0,y),diff(a0,x)+diff(a1,y),diff(a1,x)+diff(a2,y),
```

```
    diff(a2,x)+diff(a3,y),diff(a3,x)];
alpha1:=Determinant(Matrix([alpha0,[a0,a1,a2,a3,0],[-a0,0,a2,2*a3,0],
    [0,-2*a0,-a1,0,a3],[0,0,-3*a0,-2*a1,-a2]]));
alpha2:=Determinant(Matrix([alpha0,[0,a0,a1,a2,a3],[-a0,0,a2,2*a3,0],
        [0,-2*a0,-a1,0,a3],[0,0,-3*a0,-2*a1,-a2]]));
k:=simplify(diff(alpha2/R,x)+diff(alpha1/R,y)):
end proc:
```

Para ser precisos, si el $3-$web está dado por la condición

$$\mathcal{W}: \ F(x,y,p) = a_0(x,y)p^3 + a_1(x,y)p^2 + a_2(x,y)p + a_3(x,y) = 0,$$

donde $p = \dfrac{\mathrm{d}y}{\mathrm{d}x}$, en [17] se prueba que la curvatura de tal web está dada por la $2-$forma

$$K(\mathcal{W}) = \left(\frac{\partial}{\partial x}(A_2) - \frac{\partial}{\partial y}(A_1)\right) \mathrm{d}x \wedge \mathrm{d}y,$$

donde las expresions A_1 y A_2 están definidas por

$$A_1 = \frac{\alpha_1}{R}, \qquad A_2 = \frac{\alpha_2}{R},$$

donde $R = \mathrm{Res}(F, \partial_p(F))$ es la $p-$resultante de F y las funciones α_1 y α_2 son determinantes en términos de los coeficientes a_i, una vez omitida la dependencia de x, y:

$$\alpha_1 = \begin{vmatrix} \partial_y(a_0) & \partial_x(a_0)+\partial_y(a_1) & \partial_x(a_1)+\partial_y(a_2) & \partial_x(a_2)+\partial_y(a_3) & \partial_y(a_3) \\ a_0 & a_1 & a_2 & a_3 & 0 \\ -a_0 & 0 & a_2 & 2a_3 & 0 \\ 0 & -2a_0 & -a_1 & 0 & a_3 \\ 0 & 0 & -3a_0 & -2a_1 & -a_2 \end{vmatrix}$$

$$\alpha_2 = \begin{vmatrix} \partial_y(a_0) & \partial_x(a_0)+\partial_y(a_1) & \partial_x(a_1)+\partial_y(a_2) & \partial_x(a_2)+\partial_y(a_3) & \partial_y(a_3) \\ 0 & a_0 & a_1 & a_2 & a_3 \\ -a_0 & 0 & a_2 & 2a_3 & 0 \\ 0 & -2a_0 & -a_1 & 0 & a_3 \\ 0 & 0 & -3a_0 & -2a_1 & -a_2 \end{vmatrix}.$$

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Abstract

In this work we prove that the hexagonality of web $\mathcal{G}_{\mathcal{F}}$, the direct image of a foliation $\mathcal{F}$ by Gauss map $\mathcal{G}_{\mathcal{F}} : \mathbb{P}^2_{\mathbb{C}} \dashrightarrow \check{\mathbb{P}}^2_{\mathbb{C}}$, implies that $\mathcal{F}$ is transversally affine.

Keywords: Foliations, webs

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A large deviation principle for a natural sequence of point processes on a Riemannian two-dimensional manifold

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Abstract

We follow the techniques of Paul Dupuis, Vaios Laschos, and Kavita Ramanan in [8] to prove a large deviation principle for a sequence of point processes defined by Gibbs measures on a compact orientable two-dimensional Riemannian manifold. We see that the corresponding sequence of empirical measures converges to the solution of a partial differential equation and, in some cases, to the volume form of a constant curvature metric.

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1 Model and results

Let (M, g) be a compact oriented two-dimensional Riemannian manifold of genus different from one. Denote by *vol* the normalized volume form associated to g and the orientation. Define the 2-form

$$\Lambda = \frac{\operatorname{Ric} g}{2\pi\chi(M)}, \tag{1.1}$$

where $\operatorname{Ric} g$ is the Ricci curvature seen as a 2-form and $\chi(M)$ denotes the Euler characteristic. More precisely, write $\operatorname{Ric} g = K_g vol$, where K_g is the Gaussian curvature of g, while the usual symmetric Ricci curvature would be $K_g\, g$. We shall think of Λ as a signed measure.

It is known that there exists a continuous symmetric function

$$G : M \times M \to \mathbb{R} \cup \{\infty\}$$

such that the function $G_x : M \to \mathbb{R} \cup \{\infty\}$ defined by $G_x(y) = G(x, y)$ is integrable and satisfies

$$\Delta G_x = -\delta_x + \Lambda \tag{1.2}$$

for every $x \in M$. More precisely, for every $f \in C^\infty(M)$ and $x \in M$, we have

$$\int_M G(x,y)\Delta f(y) = -f(x) + \int_M f(y) d\Lambda(y), \tag{1.3}$$

here $\Delta : C^\infty(M) \to \Omega^2(M)$ is the usual Laplacian, i.e. $\Delta = d * d$, where $*$ is the Hodge star operator and d is the exterior derivative. Moreover, such G is unique up to an additive constant and we can choose G such that

$$\int_M G(x,y) d\Lambda(y) = 0 \tag{1.4}$$

for every $x \in M$. See [6] for a proof and more information.

Take an integer $n \geq 2$. We consider a system of n indistinguishable particles with total charge 1 interacting via the electrostatic force. In other words, each particle has charge $1/n$ and the two-particle interaction potential is G. This means that the total energy will be $H_n : M^n \to \mathbb{R} \cup \{\infty\}$, defined by

$$H_n(x_1, ..., x_n) = \frac{1}{n^2} \sum_{i<j} G(x_i, x_j).$$

Choose a sequence of positive numbers $\{\beta_n\}_{n\geq 2}$ and a positive number $\beta > 0$ such that $\beta_n \to \beta$. We define the **Gibbs non-normalized measure associated to H_n and β_n** as the finite measure γ_n on M^n given by

$$d\gamma_n = \exp\left(-n\beta_n H_n\right)\, dvol^{\otimes n}.$$

The **Gibbs probability measure** will be the probability measure

$$\mathbb{P}_n = \frac{\gamma_n}{Z_n},$$

where $Z_n = \gamma_n(M^n)$ is called the **partition function**. The probability measure $\mathbb{P}_n$ describes a system of n particles with Hamiltonian H_n and inverse temperature $n\beta_n$.

For any metrizable compact space E we endow $\mathcal{P}(E)$, the space of probability measures on E, with the smallest topology such that for every continuous function $f : E \to \mathbb{R}$ the application $\mu \to \int_E f d\mu$ is continuous. We can see that this is again a metrizable compact space. See Appendix for a short proof, and [5] for extra information. Furthermore, a sequence of probability measures on E, say $\{\mu_n\}_{n\in\mathbb{N}}$, converges to $\mu \in \mathcal{P}(E)$ if and only if $\int_E f d\mu_n$ converges to $\int_E f d\mu$ for every continuous function $f : E \to \mathbb{R}$. In fact, it is enough to verify that $\int_E f d\mu_n$ converges to $\int_E f d\mu$ for f belonging to a countable dense family of the space of continuous functions with the uniform topology.

The space M^n is to be 'injected' in $\mathcal{P}(M)$ by means of the continuous application

$$i_n : M^n \to \mathcal{P}(M)$$

$$(x_1, ..., x_n) \mapsto \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i},$$

and we will study the limit of the sequence of probabilites $i_n(\mathbb{P}_n)$, the **pushforward laws of** $\mathbb{P}_n$.

Define the **macroscopic energy** as

$$\begin{aligned} W : \mathcal{P}(M) &\to \mathbb{R} \cup \{\infty\} \\ \mu &\mapsto \int_{M \times M} G(x, y)\, d\mu(x)\, d\mu(y), \end{aligned}$$

and the **free energy** as

$$\begin{aligned} F : \mathcal{P}(M) &\to \mathbb{R} \cup \{\infty\} \\ \mu &\mapsto \frac{\beta}{2} W(\mu) + D\left(\mu \,\|\, vol\right), \end{aligned}$$

where $D\left(\mu \,\|\, vol\right)$ denotes the relative entropy of μ with respect to vol, also known as the **Kullback - Leibler divergence**, defined by

$$D(\mu \| vol) = \int_M \log\left(\frac{d\mu}{dvol}\right) d\mu$$

if μ is absolutely continuous with respect to vol, and $D(\mu \| vol) = \infty$ otherwise. Now we can state our main result.

Theorem 1.1 (Laplace principle)**.** *For every continuous $f : \mathcal{P}(M) \to \mathbb{R}$ we have the convergence*

$$\frac{1}{n} \log \int_{M^n} e^{-nf \circ i_n} d\gamma_n \xrightarrow[n \to \infty]{} - \inf_{\mu \in \mathcal{P}(M)} \{f(\mu) + F(\mu)\}.$$

To state a large deviation principle as an easy corollary we need to define first the function

$$\begin{aligned} I : \mathcal{P}(M) &\to \mathbb{R} \cup \{\infty\} \\ \mu &\mapsto F(\mu) - \inf F. \end{aligned}$$

Corollary 1.2 (A large deviation principle)**.** *For every closed set $C \subset \mathcal{P}(M)$ we have*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_n(i_n^{-1}(C)) \leq -\inf_{x\in C} I(x),$$

and for every open set $O \subset \mathcal{P}(M)$ we have

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}_n(i_n^{-1}(O)) \geq -\inf_{x\in O} I(x).$$

This tells us that to understand the limiting behavior of $i_n(\mathbb{P}_n)$ we must study the free energy F. The first two main properties will be studied in Section 2.

Proposition 1.3 (Convexity and lower semicontinuity of F)**.** *The function F is strictly convex and lower semicontinuous.*

Thus, F achieves its minimum at only one point. The following theorem characterizes this minimum.

Theorem 1.4 (Minimum of F)**.** *The function F achieves its minimum at a probability measure μ_{eq} that is absolutely continuous with respect to vol and such that $\rho = \dfrac{d\mu_{eq}}{dvol}$ is a C^∞ strictly positive everywhere function that satisfies the differential equation*

$$\Delta \log \rho = \beta\, \mu_{eq} - \beta\Lambda. \tag{1.5}$$

Remark 1.5 (Equivalent formulation: equation on the Ricci curvature)**.** If we define the metric $\bar{\omega} = \rho\, g$, we have that μ_{eq} is the volume form associated to $\bar{\omega}$, and Equation 1.5 can be written as

$$Ric\,\bar{\omega} = \left(2\pi\chi(M) + \frac{\beta}{2}\right) Ric\, g - \frac{\beta}{2}\mu_{eq}$$

(because of the identity $\Delta \log \rho = 2Ric\, g - 2Ric\,\bar{\omega}$).

Finally, by Corollary 1.2 and an application of the Borel-Cantelli lemma we get the following.

Corollary 1.6 (Convergence of the empirical measures)**.** *If $\{X_n\}_{n\geq 2}$ is a sequence of random elements in $\mathcal{P}(M)$ such that $X_n \sim i_n(\mathbb{P}_n)$, then*

$$X_n \xrightarrow{\text{a.s.}} \mu_{eq},$$

where μ_{eq} is the unique minimizer of F.

2 Lower semicontinuity and convexity of I

In this section we prove Proposition 1.3. It is well known that $D(\cdot\|vol)$ is lower semicontinuous and strictly convex (see [9, Lemma 1.4.3]). What we need to establish is lower semicontinuity and convexity for W.

Proof of the lower semicontinuity of W. For positive m set $G_m(x,y) = G(x,y) \wedge m = \min\{G(x,y), m\}$. Then

$$\mu \mapsto \int_{M\times M} G_m(x,y)\, d\mu(x)\, d\mu(y)$$

is a continuous function of μ. As W is the increasing limit of functions as m tends to infinity, we get that W is lower semicontinuous. □

Proof of the convexity of W. To prove convexity it is enough to show that for every $\mu, \nu \in \mathcal{P}(M)$ we have

$$W\left(\frac{1}{2}\mu + \frac{1}{2}\nu\right) \leq \frac{1}{2}W(\mu) + \frac{1}{2}W(\nu) \tag{2.1}$$

due to the lower semicontinuity of W. Inequality 2.1 is equivalent to

$$\frac{1}{2}W(\mu) + \frac{1}{2}W(\nu) \geq \int_{M\times M} G(x,y)\, d\mu(x)\, d\nu(y). \tag{2.2}$$

This inequality is easy to verify if μ and ν are differentiable, *i.e.*, given by differentiable forms. Indeed, in that case, the functions

$$f(x) = \int_M G(x,y)\, d\mu(y) \quad \text{and} \quad g(x) = \int_M G(x,y)\, d\nu(y)$$

satisfy

$$\Delta f = -\mu + \Lambda \quad \text{and} \quad \Delta g = -\nu + \Lambda$$

because of 1.3, are differentiable due to the ellipticity of the Laplacian, and have zero integral with respect to Λ because of 1.4. So, in terms of f and g Inequality 2.2 reads

$$-\frac{1}{2}\int_M f\,\Delta f \; - \frac{1}{2}\int_M g\,\Delta g \geq -\int_M f\,\Delta g,$$

which is equivalent to

$$\int_M \|\nabla(f-g)\|^2\, dvol \geq 0.$$

For the general case we need two lemmas.

Lemma 2.1. *Let μ be a continuous probability measure, i.e., given by a continuous 2-form. Then there exists a sequence μ_n of differentiable probability measures such that*

$$\mu_n \to \mu \qquad \textit{and} \qquad W(\mu_n) \to W(\mu).$$

Proof. As μ is continuous, we can write $d\mu = \rho\, dvol$ with ρ continuous. Take a sequence of differentiable functions $\{\rho_n\}_{n\in\mathbb{N}}$ such that $\rho_n \to \rho$ uniformly. We can assume $\rho_n \geq 0$ (redefine $\rho_n = \rho_n + \|\rho - \rho_n\|_\infty$) and $\int_M \rho_n\, dvol = 1$ (because $\int_M \rho_n dvol \to \int_M \rho\, dvol$). Define μ_n by means of $d\mu_n = \rho_n dvol$. We notice that

$$\mu_n \to \mu$$

holds due to the uniform convergence and, as $\rho_n \otimes \rho_n \to \rho \otimes \rho$ uniformly, we obtain

$$\int_{M\times M} G(x,y)\rho_n(x)\rho_n(y)\, dvol(x)\, dvol(y)$$
$$\to \int_{M\times M} G(x,y)\rho(x)\rho(y)\, dvol(x)\, dvol(y)$$

by the dominated convergence theorem outside the diagonal (because G is $vol \otimes vol$ - integrable and the sequence ρ_n is uniformly bounded). $\square$

To approximate arbitrary probability measures with continuous ones we refer to [12, Lemma 6.3.1]. It states the following.

Lemma 2.2. *Let μ be any probability measure such that $W(\mu) < \infty$. Then there exists a sequence μ_n of continuous probability measures such that*

$$\mu_n \to \mu \qquad \textit{and} \qquad W(\mu_n) \to W(\mu).$$

□

To complete the proof of the convexity, let $\mu, \nu \in \mathcal{P}(M)$. Take two sequences $\{\mu_n\}_{n\in\mathbb{N}}$ and $\{\nu_n\}_{n\in\mathbb{N}}$ of differentiable measures such that $\mu_n \to \mu$, $W(\mu_n) \to W(\mu)$, and $\nu_n \to \nu$, $W(\nu_n) \to W(\nu)$. We want to take the lower limit to the sequence of inequalities

$$\frac{1}{2}W(\mu_n) + \frac{1}{2}W(\nu_n) \geq \int_{M\times M} G(x,y)\, d\mu_n(x) d\nu_n(y).$$

For this, notice that $(\mu,\nu) \mapsto \int_{M\times M} G(x,y)\, d\mu(x)d\nu(y)$ is lower semicontinuous. This can be seen as a consequence of the fact that it can be reexpresed as the increasing limit as m goes to infinity of the continuous functions $(\mu,\nu) \mapsto \int_{M\times M} G_m(x,y)\, d\mu(x)d\nu(y)$ where $G_m(x,y) = G(x,y)\wedge m$. Then, we get

$$\begin{aligned} \frac{1}{2}W(\mu) + \frac{1}{2}W(\nu) &\geq \liminf_{n\to\infty} \int_{M\times M} G(x,y)\, d\mu_n(x)d\nu_n(y) \\ &\geq \int_{M\times M} G(x,y)\, d\mu(x)d\nu(y), \end{aligned}$$

and the proof is complete. □

3 The minimum of F

Now we prove Theorem 1.4. Let $\rho \in C^\infty(M)$ be a differentiable positive solution of the equation (see [7])

$$\Delta \log \rho = \beta\, \mu_{eq} - \beta\Lambda,$$

where $d\mu_{eq} = \rho\, dvol$. We will prove that the functional F achieves its minimum at μ_{eq}. For this we shall calculate the derivative of F at μ_{eq} and prove that it is zero. We start with the following result.

Lemma 3.1 (Derivative of the energy)**.** *Let μ_0 and μ_1 be two probability measures. Define $\mu_t = t\mu_1 + (1-t)\mu_0$, for $t \in [0,1]$. If $W(\mu_0) < \infty$ and $W(\mu_1) < \infty$ then $W(\mu_t)$ is differentiable at $t = 0$, and satisfies*

$$\frac{d}{dt} W(\mu_t)\,|_{t=0} \; = 2 \int_{M\times M} G(x,y)\, d\mu_0(x)\; (d\mu_1(y) - d\mu_0(y))\,.$$

Proof. As $W(\mu_0)$ and $W(\mu_1)$ are finite, due to the convexity of W we have that

$$\begin{aligned} W(\mu_t) = t^2 &\int_{M\times M} G(x,y) d\mu_1(x)\, d\mu_1(y) + \\ &+ 2t(1-t) \int_{M\times M} G(x,y) d\mu_0(x)\, d\mu_1(y) + \\ &+ (1-t)^2 \int_{M\times M} G(x,y) d\mu_0(x)\, d\mu_0(y) \end{aligned}$$

is finite. The linear term (in the variable t) is given by

$$2 \int_{M\times M} G(x,y)\, d\mu_0(x)\; (d\mu_1(y) - d\mu_0(y))\,,$$

which is the sought derivative. □

Lemma 3.2 (Derivative of the entropy)**.** *Let μ_0 and μ_1 be two probability measures. Define $\mu_t = t\mu_1 + (1-t)\mu_0$, for $t \in [0,1]$. If $D(\mu_0 \| vol) < \infty$, $D(\mu_1 \| vol) < \infty$ and $\int_M \left|\log\left(\frac{d\mu_0}{d\, vol}\right)\right| d\mu_1 < \infty$, then $D(\mu_t \| vol)$ is differentiable at $t = 0$, and satisfies*

$$\frac{d}{dt} D(\mu_t \| vol)\,|_{t=0} \; = \int_M \log\left(\frac{d\mu_0}{d\, vol}(y)\right) (d\mu_1(y) - d\mu_0(y))\,.$$

Proof. We use the notation

$$\rho_0 = \frac{d\mu_0}{d\, vol} \quad \text{ and } \quad \rho_1 = \frac{d\mu_1}{d\, vol}.$$

As $D(\mu_0\|vol)$ and $D(\mu_1\|vol)$ are finite, by the convexity of the entropy we get that $D(\mu_t\|vol)$ is also finite. In particular, we have

$$\int_M |\log\left(t\rho_1(x)+(1-t)\rho_0(x)\right)|d\mu_t<\infty$$

and, as $\mu_t = t\mu_1 + (1-t)\mu_0$, if $0<t<1$, we get

$$\int_M |\log\left(t\rho_1(x)+(1-t)\rho_0(x)\right)|d\mu_0<\infty$$

and

$$\int_M |\log\left(t\rho_1(x)+(1-t)\rho_0(x)\right)|d\mu_1<\infty.$$

Keeping this in mind it makes sense to write

$$\begin{aligned}
D(\mu_t\|vol) &= \int_M \log\left(t\rho_1(x)+(1-t)\rho_0(x)\right) t\rho_1(x)dvol(x)+\\
&\quad+\int_M \log\left(t\rho_1(x)+(1-t)\rho_0(x)\right)(1-t)\rho_0(x)dvol(x)\\
&= t\int_M \log\left(t\rho_1(x)+(1-t)\rho_0(x)\right)\left(\rho_1(x)-\rho_0(x)\right)\,dvol(x)+\\
&\quad+\int_M \log\left(t\rho_1(x)+(1-t)\rho_0(x)\right)\rho_0(x)\,dvol(x),
\end{aligned}$$

which together with

$$D(\mu_0\|vol) = \int_M \log(\rho_0(x))\,\rho_0(x)\,dvol(x)$$

yields

$$\begin{aligned}
\frac{1}{t}\left(D(\mu_t\|vol)-D(\mu_0\|vol)\right) &= \int_M \log\left(t\rho_1(x)+(1-t)\rho_0(x)\right)d\mu_1(x)+\\
&\quad-\int_M \log\left(t\rho_1(x)+(1-t)\rho_0(x)\right)d\mu_0(x)\\
&\quad+\int_M \frac{1}{t}\left[\log\left(t\rho_1(x)+(1-t)\rho_0(x)\right)\right]\rho_0(x)dvol(x)\\
&\quad-\int_M \frac{1}{t}\log\rho_0(x)\rho_0(x)\,dvol(x).
\end{aligned}$$

For the first two terms we notice that, as $t \to 0$, we get

$$\log(\,t\rho_1(x) + (1-t)\rho_0(x)\,) \to \log\rho_0(x),$$

for every $x \in M$. We know that $|\log(t\rho_1(x) + (1-t)\rho_0(x)\,)|$ is bounded by $\left|\log\left(\frac{1}{2}\rho_1 + \frac{1}{2}\rho_0\,\right)\right| + |\log\rho_0(x)|$, for $0 < t \leq \frac{1}{2}$, and we can use the dominated convergence theorem to reach

$$\int_M \log(t\rho_1(x) + (1-t)\rho_0(x)\,)\,d\mu_1(x) \to \int_M \log\rho_0(x)\,d\mu_1(x)$$

and

$$\int_M \log(t\rho_1(x) + (1-t)\rho_0(x)\,)\,d\mu_0(x) \to \int_M \log\rho_0(x)\,d\mu_0(x).$$

Finally, we notice the convergence

$$\frac{1}{t}\left[\log(t\rho_1(x) + (1-t)\rho_0(x)\,) - \log\rho_0(x)\right]\rho_0(x) \uparrow [\rho_1(x) - \rho_0(x)]$$

as $t \downarrow 0$ for every $x \in M$, and since each term is integrable, we can use the monotone convergence theorem to obtain

$$\int_M \frac{1}{t}\left[\log(t\rho_1(x) + (1-t)\rho_1(x)\,) - \log\rho_0(x)\right]\rho_0(x)\,dvol(x) \to 0.$$

$\square$

Now we are in position to prove Theorem 1.4.

Proof of Theorem 1.4. Let μ be any probability measure different from μ_{eq} and such that $F(\mu) < \infty$. Define $\mu_t = t\mu + (1-t)\mu_{eq}$, for $t \in [0,1]$. Multiply the equality

$$\Delta\log\rho = \beta\,\rho\,vol - \beta\Lambda,$$

by $G(x,y)$ and integrate in one variable to get

$$-\log\rho(y) + \int_M \log\rho(x)\,d\Lambda(x) = \beta\int_M G(x,y)\rho(x)\,dvol(x)$$

for every $y \in M$. Then, by Lemma 3.1 and 3.2, we have

$$\begin{aligned}
\frac{d}{dt}F(\mu_t)|_{t=0} &= \beta \int_{M\times M} G(x,y)d\mu_{eq}(x)\ (d\mu(y) - d\mu_{eq}(y)) + \\
&\quad + \int_{M\times M} \log\rho(y)\,(d\mu(y) - d\mu_{eq}(y)) \\
&= \int_M \left(\beta \int_M G(x,y)\rho(x)\,dvol(x) + \log\rho(y)\right)(d\mu(y) - d\mu_{eq}(y)) \\
&= \int_M \left(\int_M \log\rho(x)\,d\Lambda(x)\right)(d\mu(y) - d\mu_{eq}(y)) \\
&= \left(\int_M \log\rho(x)\,d\Lambda(x)\right)\int_M (d\mu(y) - d\mu_{eq}(y)) = 0.
\end{aligned}$$

This implies, due to the strict convexity of $F(\mu_t)$, the inequality

$$F(\mu_{eq}) > F(\mu).$$

□

4 Laplace principle

Theorem 1.1 will be proved in this section. For this, we first understand some limiting properties of the energy. Write

$$W_n(x_1, ..., x_n) = 2H_n(x_1, ..., x_n) = \frac{1}{n^2}\sum_{i\neq j} G(x_i, x_j).$$

The easiest property we need to establish is the following.

Proposition 4.1. *For $\mu \in \mathcal{P}(M)$, we have*

$$\int_{M^n} W_n d\mu^{\otimes n} \to W(\mu).$$

Proof. We integrate to get

$$\int_{M^n} W_n d\mu^{\otimes n} = \frac{n(n-1)}{n^2}\int_{M\times M} G(x,y)\,d\mu(x)\,d\mu(y).$$

Then we take limits to complete the proof. □

For more general $\tau_n \in \mathcal{P}(M^n)$ (not necessarily of the form $\mu^{\otimes n}$) we can obtain a bound for below of the lim inf.

Proposition 4.2. *For each n choose $\tau_n \in \mathcal{P}(M^n)$. Suppose there exists a probability distribution on $\mathcal{P}(M)$, say ζ (that is $\zeta \in \mathcal{P}(\mathcal{P}(M))$), such that $i_n(\tau_n) \to \zeta$. Then we have*

$$\int_{\mathcal{P}(M)} W d\zeta \leq \liminf_{n\to\infty} \int_{M^n} W_n d\tau_n.$$

Proof. As usual, for each $m \geq 0$ define $G_m(x,y) = G(x,y) \wedge m$. For each n take a random element $(X_1^n, ..., X_n^n) \in M^n$ with law τ_n and $\mu \in \mathcal{P}(M)$ with law ζ. Define $\mu_n = i_n(X_1^n, ..., X_n^n)$. We have then

$$\begin{aligned}\int_{M\times M} G_m(x,y) d\mu_n(x)\, d\mu_n(y) &= \frac{1}{n^2}\sum_{i\neq j} G_m(X_i^n, X_j^n) + \frac{m}{n} \\ &\leq \frac{1}{n^2}\sum_{i\neq j} G(X_i^n, X_j^n) + \frac{m}{n},\end{aligned}$$

from which, taking expected values, we obtain

$$\mathbb{E}\left[\int_{M\times M} G_m(x,y) d\mu_n(x)\, d\mu_n(y)\right] \leq \mathbb{E}\left[W_n(X_1^n, ..., X_n^n)\right] + \frac{m}{n}. \quad (4.1)$$

We have thus

$$\mathbb{E}\left[\int_{M\times M} G_m(x,y) d\mu_n(x)\, d\mu_n(y)\right] \to \mathbb{E}\left[\int_{M\times M} G_m(x,y) d\mu(x)\, d\mu(y)\right]$$

by the continuity of G_m. So, by letting $n \to \infty$ in Inequality 4.1, we reach

$$\mathbb{E}\left[\int_{M\times M} G_m(x,y) d\mu(x)\, d\mu(y)\right] \leq \liminf_{n\to\infty} \mathbb{E}\left[W_n(X_1^n, ..., X_n^n)\right].$$

By letting $m \to \infty$, we finally conclude

$$\mathbb{E}\left[\int_{M\times M} G(x,y) d\mu(x)\, d\mu(y)\right] \leq \liminf_{n\to\infty} \mathbb{E}\left[W_n(X_1^n, ..., X_n^n)\right]$$

by the monotone convergence theorem. □

Remark 4.3. In the previous proposition we may choose a sequence of increasing integers n_k and for each k a measure $\tau_k \in \mathcal{P}(M^{n_k})$ such that $i_{n_k}(\tau_k) \to \zeta$, and get the same result:

$$\int_{\mathcal{P}(M)} W d\zeta \leq \liminf_{k\to\infty} \int_{M^{n_k}} W_{n_k} d\tau_k.$$

Now we can start proving Theorem 1.1.

Proof of Theorem 1.1. Take $f : \mathcal{P}(M) \to \mathbb{R}$ continuous. Because of the identity

$$\frac{1}{n} \log \int_{M^n} e^{-nf\circ i_n} d\gamma_n = \frac{1}{n} \log \int_{M^n} e^{-n\left(f\circ i_n + \frac{\beta_n}{2} W_n\right)} dvol^{\otimes n},$$

we only need to prove

$$\frac{1}{n} \log \int_{M^n} e^{-n\left(f\circ i_n + \frac{\beta_n}{2} W_n\right)} dvol^{\otimes n} \to - \inf_{\mu\in\mathcal{P}(M)} \{f(\mu) + F(\mu)\}.$$

For that we use the following result (see [9, Proposition 4.5.1]).

Lemma 4.4 (Variational formulation)**.** *Let E be a Polish space, μ a probability measure on E and $g : E \to \mathbb{R} \cup \{\infty\}$ a measurable function bounded from below. Under those hypothesis, the relation*

$$\log \int_E e^{-g} d\mu = - \inf_{\tau\in\mathcal{P}(E)} \left\{ \int_E g\, d\tau + D(\tau\|\mu) \right\}.$$

holds □

In our case, we have

$$\frac{1}{n} \log \int_{M^n} e^{-n\left(f\circ i_n + \frac{\beta_n}{2} W_n\right)} dvol^{\otimes n} =$$
$$= - \inf_{\tau\in\mathcal{P}(M^n)} \left\{ \int_{M^n} f\circ i_n\, d\tau + \frac{\beta_n}{2} \int_{M^n} W_n\, d\tau + \frac{1}{n} D(\tau\|vol^{\otimes n}) \right\}.$$

Let us start with an upper limit inequality. More precisely, we prove the relation

$$\limsup_{n\to\infty} \inf_{\tau\in\mathcal{P}(M^n)} \left\{ \int_{M^n} f\circ i_n \, d\tau + \frac{\beta_n}{2}\int_{M^n} W_n \, d\tau + \frac{1}{n} D(\tau \| vol^{\otimes n}) \right\} \\ \leq \inf_{\mu\in\mathcal{P}(M)} \{ f(\mu) + F(\mu) \}. \tag{4.2}$$

For this, we need to see that for every probability measure $\mu \in \mathcal{P}(M)$ we get

$$\limsup_{n\to\infty} \inf_{\tau\in\mathcal{P}(M^n)} \left\{ \int_{M^n} f\circ i_n \, d\tau + \frac{\beta_n}{2}\int_{M^n} W_n \, d\tau + \frac{1}{n} D(\tau \| vol^{\otimes n}) \right\} \\ \leq f(\mu) + F(\mu). \tag{4.3}$$

It will be enough to find, for every $n \geq 2$, a probability measure $\tau_n \in \mathcal{P}(M^n)$ such that

$$\limsup_{n\to\infty} \left\{ \int_{M^n} f\circ i_n \, d\tau_n + \frac{\beta_n}{2}\int_{M^n} W_n \, d\tau_n + \frac{1}{n} D(\tau_n \| vol^{\otimes n}) \right\} \\ \leq f(\mu) + F(\mu).$$

We choose the simplest one: $\tau_n = \mu^{\otimes n}$. If so, by the law of large numbers in the compact space M, we have

$$i_n(\tau_n) \to \delta_\mu.$$

Indeed, take a sequence $\{X_k\}_{k\in\mathbb{N}}$ of independent and identically distributed random elements of M with law μ and take any continuous function $g : M \to \mathbb{R}$. Then, $\{g(X_k)\}_{k\in\mathbb{N}}$ is a sequence of independent and identically distributed bounded random variables. By the strong law of large numbers we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} g(X_k) = \mathbb{E}[g(X_1)]$$

almost surely. This can be written as

$$\lim_{n\to\infty} \int_M g \; d[i_n(X_1, ..., X_n)] = \int_M g \, d\mu,$$

and taking a countable dense family of functions we get

$$\lim_{n\to\infty} i_n(X_1, ..., X_n) = \mu$$

almost surely. By the dominated convergence theorem, the almost sure convergence implies the convergence of their laws, and so, as the law of $i_n(X_1, ..., X_n)$ is $i_n(\tau_n)$ and μ is deterministic (of law δ_μ), we obtain

$$i_n(\tau_n) \to \delta_\mu.$$

Hence, we get

$$\lim_{n\to\infty} \int_{M^n} f \circ i_n \, d\tau_n = f(\mu).$$

The second term has already been studied in Proposition 4.1: we have

$$\lim_{n\to\infty} \int_{M^n} W_n d\tau_n = W(\mu).$$

Finally, we use

$$D(\tau_n \| vol^{\otimes n}) = nD(\mu \| vol)$$

to get

$$\begin{aligned} \lim_{n\to\infty} \left\{ \int_{M^n} f \circ i_n \, d\tau_n + \frac{\beta_n}{2} \int_{M^n} W_n \, d\tau_n + \frac{1}{n} D(\tau_n \| vol^{\otimes n}) \right\} \\ = f(\mu) + \frac{\beta}{2} W(\mu) + D(\mu \| vol). \end{aligned}$$

The second and final step is to prove the **lower bound**

$$\begin{aligned} \liminf_{n\to\infty} \inf_{\tau\in\mathcal{P}(M^n)} \left\{ \int_{M^n} f \circ i_n \, d\tau + \frac{\beta_n}{2} \int_{M^n} W_n \, d\tau + \frac{1}{n} D(\tau \| vol^{\otimes n}) \right\} \\ \geq \inf_{\mu\in\mathcal{P}(M)} \{ f(\mu) + F(\mu) \}. \end{aligned} \tag{4.4}$$

We proceed by contradiction. Suppose this is not true, *i.e.* we have

$$\liminf_{n\to\infty} \inf_{\tau\in\mathcal{P}(M^n)} \left\{ \int_{M^n} f\circ i_n \, d\tau + \frac{\beta_n}{2}\int_{M^n} W_n \, d\tau + \frac{1}{n} D(\tau\|vol^{\otimes n}) \right\} < \inf_{\mu\in\mathcal{P}(M)} \{ f(\mu) + F(\mu)\}.$$

Then we can find $C \in \mathbb{R}$ subject to

$$\inf_{\tau\in\mathcal{P}(M^n)} \left\{ \int_{M^n} f\circ i_n \, d\tau + \frac{\beta_n}{2}\int_{M^n} W_n \, d\tau + \frac{1}{n} D(\tau\|vol^{\otimes n}) \right\} < C < \inf_{\mu\in\mathcal{P}(M)} \{ f(\mu) + F(\mu)\}$$

for every n along a subsequence. For each of those n we pick $\tau_n \in \mathcal{P}(M^n)$ such that

$$\int_{M^n} f\circ i_n \, d\tau_n + \frac{\beta_n}{2}\int_{M^n} W_n \, d\tau_n + \frac{1}{n} D(\tau_n\|vol^{\otimes n}) < C$$

The idea now is to take the limit (or just the limit of a subsequence) and derive a contradiction. To achieve that we use the following lemma.

Lemma 4.5. *There exists a subsequence of* $\{\tau_n\}$*, that we will still call* $\{\tau_n\}$ *for ease of notation, and a probability distribution* ζ *(i.e.* $\zeta \in \mathcal{P}(\mathcal{P}(M))$*) on* $\mathcal{P}(M)$*, such that* $i_n(\tau_n) \to \zeta$ *and*

$$\int_{\mathcal{P}(M)} D(\cdot\|vol)\, d\zeta \;\leq\; \liminf_{n\to\infty} \frac{1}{n} D(\tau_n\|vol^{\otimes n}).$$

Proof. Given a probability measure $\tau_n \in \mathcal{P}(M^n)$ we can construct a n-tuple of random probabilities in M by means of marginals. More precisely, there exists a random variable $(\mathcal{T}_n^1, \mathcal{T}_n^2, ..., \mathcal{T}_n^n)$ on $\mathcal{P}(M)^n$ and a random variable $(X_1, ..., X_n) \in M^n$ with law τ_n, such that

$$\int_M g \, d\mathcal{T}_n^i = \mathbb{E}[g(X_i)|X_1, ..., X_{i-1}],$$

for every continuous function $g : M \to \mathbb{R}$.

We can prove (see Proposition 7.2 in the Appendix for an idea of the proof, or see [9, Theorem C.3.1] for a complete proof) that

$$D(\tau_n \| vol^{\otimes n}) = \mathbb{E}\left[\sum_{i=1}^{n} D(\mathcal{T}_n^i \| vol)\right]$$

holds. So, by the convexity of $D(\cdot\|vol)$ we get

$$\mathbb{E}\left[D\left(\frac{1}{n}\sum_{i=1}^{n}\mathcal{T}_n^i \middle\| vol\right)\right] \leq \frac{1}{n}\mathbb{E}\left[\sum_{i=1}^{n} D(\mathcal{T}_n^i \| vol)\right] = \frac{1}{n} D(\tau_n \| vol^{\otimes n}).$$

The compactness of $\mathcal{P}(\mathcal{P}(M)\times\mathcal{P}(M))$ allows us to extract a subsequence of $\left(\frac{1}{n}\sum_{i=1}^{n}\mathcal{T}_n^i, \frac{1}{n}\sum_{i=1}^{n}\delta_{X_i}\right) \in \mathcal{P}(M)\times\mathcal{P}(M)$ such that $\left(\frac{1}{n}\sum_{i=1}^{n}\mathcal{T}_n^i,\right.$ $\left.\frac{1}{n}\sum_{i=1}^{n}\delta_{X_i}\right)$ converges in law to, say, $(\chi, \tilde{\chi})$. Then, we get $\chi = \tilde{\chi}$ almost surely (see Proposition 7.4 in the Appendix or [8, Lemma 3.5]). Denote by ζ the common law of χ and $\tilde{\chi}$. The fact that $D(\cdot\|vol)$ is lower semicontinuous and bounded from below implies that it can be written as an increasing pointwise limit of bounded continuous functions, and then the function $\alpha \mapsto \int_{\mathcal{P}(M)} D(\cdot\|vol)d\alpha$ is also lower semicontinuous. In particular, we get

$$\int_{\mathcal{P}(M)} D(\cdot\|vol)\, d\alpha \leq \liminf \mathbb{E}\left[D\left(\frac{1}{n}\sum_{i=1}^{n}\mathcal{T}_n^i \middle\| vol\right)\right].$$

□

We can now complete the proof by noticing that Lemma 4.5 and Proposition 4.2 imply

$$\int_{\mathcal{P}(M)} \left(f + \frac{\beta}{2}W + D(\cdot\|vol)\right) d\zeta \leq C < \inf_{\mu\in\mathcal{P}(M)}\{f(\mu) + F(\mu)\},$$

or, equivalently,

$$\int_{\mathcal{P}(M)} (f + F)\, d\zeta \leq C < \inf_{\mu\in\mathcal{P}(M)}\{f(\mu) + F(\mu)\},$$

which is impossible. □

5 Convergence of $i_n(\mathbb{P}_n)$

We prove the corollaries in this section: Corollary 1.2, about the large deviation principle, and Corollary 1.6, about the convergence of the empirical measures.

Proof of Corollary 1.2. By [9, Theorem 1.2.3] and the fact that I is lower semicontinuous the following Laplace principle implies the large deviation principle: for every continuous function $f : \mathcal{P}(M) \to \mathbb{R}$ we have

$$\frac{1}{n} \log \int_{M^n} e^{-nf\circ i_n} d\mathbb{P}_n \overset{n\to\infty}{\longrightarrow} - \inf_{\mu\in\mathcal{P}(M)} \{f(\mu) + I(\mu)\}.$$

Using the measures γ_n and the definition of I it is enough to prove that for every continuous function $f : \mathcal{P}(M) \to \mathbb{R}$ we have

$$\frac{1}{n} \log \int_{M^n} e^{-nf\circ i_n} \frac{d\gamma_n}{Z_n} \overset{n\to\infty}{\longrightarrow} - \inf_{\mu\in\mathcal{P}(M)} \{f(\mu) + F(\mu) - \inf F\}.$$

However, by Theorem 1.1 applied to the function $f = 0$, we get

$$\frac{1}{n} \log Z_n \underset{n\to\infty}{\longrightarrow} - \inf F,$$

and combining this with the same theorem for general f, we get

$$\frac{1}{n} \log \int_{M^n} e^{-nf\circ i_n} d\gamma_n \overset{n\to\infty}{\longrightarrow} - \inf_{\mu\in\mathcal{P}(M)} \{f(\mu) + F(\mu)\},$$

and the proof is finished. □

Proof of Corollary 1.6 . Take random probabilities $\{X_n\}_{n\geq 2}$ coupled in any way but such that $X_n \sim i_n(\mathbb{P}_n)$. For any closed set C that does not contain μ_{eq}, we have $\inf_{x\in C} I(x) > 0$ due to the semicontinuity of I. The property

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_n(i_n^{-1}(C)) \leq - \inf_{x\in C} I(x)$$

implies that there exists $A > 0$ and $N \in \mathbb{N}$ such that

$$\frac{1}{n} \log \mathbb{P}_n(i_n^{-1}(C)) \leq -A$$

for every $n > N$. Hence we have

$$\mathbb{P}_n(i_n^{-1}(C)) \leq e^{-nA}$$

for every $n > N$, which yields

$$\sum_{n=1}^{\infty} \mathbb{P}_n(i_n^{-1}(C)) < \infty.$$

By the Borel-Cantelli lemma we get then

$$\mathbb{P}(\text{there exists } M \in \mathbb{N} \text{ such that } i > M \text{ implies } X_i \notin C) = 1.$$

Take a countable local base $\{O_i\}_{i\in\mathbb{N}}$ around μ_{eq} and apply the previous argument for every $C = O_i^c$ to obtain almost sure convergence. $\square$

6 Final comments

This work has been inspired on the article by Robert Berman [4] where a slightly different model is treated. Our proof of the large deviation principle is an adaptation of the article by Paul Dupuis, Vaios Laschos, and Kavita Ramanan [8] to the case of compact manifolds.

Here we have studied just one kind of limiting behavior for a sequence of point processes on a surface. There are two main issues that, to our knowledge, are still open: the **fluctuations** and the **local behaviour**.

By **fluctuations** we mean the following. Take $f \in C^{\infty}(M)$ and μ_n a sequence with law $i_n(\mathbb{P}_n)$. We have proved, in Corollary 1.6, the convergence

$$\int f d\mu_n \to \int f \, d\mu_{eq},$$

what we could rewrite as

$$\int f d\mu_n = \int f\, d\mu_{eq} + o(1).$$

The idea is to find the next order terms (to prove a central limit type theorem). More precisely, to find a sequence $\alpha_n \to \infty$ such that

$$\alpha_n \left(\int f d\mu_n - \int f\, d\mu_{eq} \right)$$

converges weakly, and describe such limit.

When we talk about **local behavior** we take $x \in M$ and a chart

$$\phi : U \to T_x M$$

such that $\phi(x) = 0$ and $d\phi_x = id|_{T_x M}$. We fix n points $(X_1, ..., X_n)$ distributed according to $\mathbb{P}_n$. We get a point process in $T_x M$ with points $\phi(X_1), ..., \phi(X_n)$ (when $X_i \in U$). We then scale this point process by $\sqrt{n}$ and find the limit (in some sense) point process. We ask how this point process depends on $x \in M$.

These questions are already answered in the case of some determinantal point processes (see [1] and [3]) and in the one dimensional case (see [10]). Very recent results about fluctuations on $\mathbb{R}^2$ can be found in [2] and [11].

7 Appendix

Here we deal with several tools used along this paper.

Proposition 7.1. *Let E be a compact metrizable space. Then $\mathcal{P}(E)$, the space of probability measures on E, is a compact metrizable space.*

Proof. By the Stone-Weierstrass theorem we know that the space of continuous functions on E is separable in the topology of uniform convergence. Choose a dense countable set $\{f_m\}_{m \in \mathbb{N}}$.

Let d be a metric in E that induces its topology. Define $\bar{d} : \mathcal{P}(E) \times \mathcal{P}(E) \to \mathbb{R}$ by

$$\bar{d}(\mu, \nu) = \sum_{m \in \mathbb{N}} \frac{1}{2^m} \wedge \left| \int_E f_m d\mu - \int_E f_m d\nu \right|.$$

We can see that the topology induced by $\bar{d}$ is the smallest topology such that $\mu \mapsto \int_E f_m d\mu$ is continuous for every $m \in \mathbb{N}$. But by density and uniform convergence the functional $\mu \mapsto \int_E f_m d\mu$ is continuous for every $m \in \mathbb{N}$ if and only if $\mu \mapsto \int_E f d\mu$ is continuous for any continuous function $f : E \to \mathbb{R}$. So, the topology induced by $\bar{d}$ is the weak topology of $\mathcal{P}(E)$.

To see that $\mathcal{P}(E)$ is compact it is enough to show that it is sequentially compact. Take a sequence $\{\mu_n\}_{n \in \mathbb{N}}$ of probability measures on E. By a diagonal procedure we can choose a subsequence $\{\mu_{n_i}\}_{i \in \mathbb{N}}$ such that $\int_E f_m d\mu_{n_i}$ converges as i goes to infinity for every $m \in \mathbb{N}$. This implies that $\int_E f d\mu_{n_i}$ converges as i goes to infinity for every continuous function $f : E \to \mathbb{R}$. Indeed, we can prove that $\{\int_E f d\mu_{n_i}\}_{i \in \mathbb{N}}$ is Cauchy. For this, take $\epsilon > 0$ and choose $m \in \mathbb{N}$ such that $\|f_m - f\| < \epsilon/3$. Take a number M such that if $i, j > M$ then $|\int_E f_m d\mu_{n_i} - \int_E f_m d\mu_{n_j}| < \epsilon/3$. Then, whenever $i, j > M$, we have

$$\begin{aligned}
\left| \int_E f d\mu_{n_i} - \int_E f d\mu_{n_j} \right| \leq & \left| \int_E f d\mu_{n_i} - \int_E f_m d\mu_{n_i} \right| + \\
& + \left| \int_E f_m d\mu_{n_i} - \int_E f_m d\mu_{n_j} \right| + \\
& + \left| \int_E f_m d\mu_{n_j} - \int_E f_m d\mu_{n_j} \right| \\
& < \epsilon
\end{aligned}$$

Define $\Lambda : C(E) \to \mathbb{R}$ as $\Lambda(f) = \lim_{i \to \infty} \int_E f d\mu_{n_i}$. Then Λ is a positive linear functional and so, there exists a positive measure μ on E such that $\Lambda(f) = \int_E f d\mu$ for every $f \in C(E)$. As $\Lambda(1) = \lim_{i \to \infty} \int_E 1 d\mu_{n_i} = 1$, we obtain $\mu \in \mathcal{P}(E)$. In this way, we have extracted a subsequence of $\{\mu_n\}_{n \in \mathbb{N}}$ that converges. □

In what follows, instead of writing $d\mu(x)$ we write $\mu(dx)$.

As in the proof of Lemma 4.5, given a probability measure $\mu \in \mathcal{P}(M^n)$ we can construct a n-tuple of random probabilities $(\mu_1, \mu_2, ..., \mu_n)$ in $\mathcal{P}(M)^n$ and a random element $(X_1, ..., X_n) \in M^n$ with law μ such that

$$\int_M f d\mu_i = \mathbb{E}[f(X_i)|X_1, ..., X_{i-1}]$$

holds.

Proposition 7.2 (Chain rule). *We have*

$$D(\mu \| vol^{\otimes_n}) = \mathbb{E}\left[\sum_{i=1}^n D(\mu_i \| vol)\right].$$

Sketch of the proof. We will give an idea of the proof ignoring issues of measurability and finiteness of the entropy. For extra details we refer to [9, Theorem C.3.1].

We consider M^n with a probability measure μ. In this case the random element with law μ is $(X_1, ..., X_n)$ where $X_i : M^n \to M$ is the projection onto the i-th coordinate. Suppose that

$$\tilde{\mu}_k : M^{k-1} \to \mathcal{P}(M)$$

is a transition kernel from $(X_1, ..., X_{k-1})$ to X_k.

If we define

$$\mu_k = \tilde{\mu}_k \circ \pi_{k-1},$$

where $\pi_{k-1} : M^n \to M^{k-1}$ is the projection onto the first $k-1$ coordinates, we see that $(\mu_1, ..., \mu_n)$ satisfies the properties of the definition. If we assume all entropies are finite, we get

$$\begin{aligned}
\mathbb{E}\left[\sum_{k=1}^{n} D(\mu_k \| vol)\right] &= \sum_{k=1}^{n} \mathbb{E}\left[D(\mu_k \| vol)\right] \\
&= \sum_{k=1}^{n} \int_{M^{k-1}} D(\tilde{\mu}_k(x) \| vol)\, [\pi_{k-1}(\mu)](dx) \\
&= \sum_{k=1}^{n} \int_{M^{k-1}} \left(\int_M \log\left(\frac{\tilde{\mu}_k(x,dy)}{vol(dy)}\right) \mu_k(x,dy) \right) [\pi_{k-1}(\mu)](dx) \\
&= \sum_{k=1}^{n} \int_{M^{k-1}\times M} \log\left(\frac{\tilde{\mu}_k(x,dy)}{vol(dy)}\right) [\pi_k(\mu)]\,(dx,dy) \\
&= \sum_{k=1}^{n} \int_{M^n} \log\left(\rho_k(x)\right) \mu(dx),
\end{aligned}$$

where $\rho_k : M^n \to [0,\infty]$ is equal to $\rho_k = \frac{\tilde{\mu}_k(x,dy)}{vol(dy)} \circ \pi_k$.

Then we just have to notice the equality

$$\prod_{i=1}^{n} \rho_i(x) = \frac{\mu(dx)}{vol^{\otimes_n}(dx)},$$

that follows from the definition.

□

Lemma 7.3. *Let $(X_1, ..., X_n) \in M^n$ and $(\mu_1, ..., \mu_n) \in \mathcal{P}(M)^n$ be random elements as before. Consider the random measures*

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} \mu_i \quad , \quad \hat{\nu} = \frac{1}{n}\sum_{i=1}^{n} \delta_{X_i}.$$

Then we have

$$\mathbb{P}\left(\left|\int_M f(x)\hat{\mu}(dx) - \int_M f(y)\hat{\nu}(dy)\right| > \epsilon\right) \leq 4\frac{\|f\|_\infty^2}{n\epsilon^2}.$$

Proof. By Chebyshev's inequality, we need to understand the quantity

$$\mathrm{Var}\ \left(\int_M f(x)\hat{\mu}(dx) - \int_M f(y)\hat{\nu}(dy)\right).$$

The first term is

$$\int_M f(y)\hat{\mu}(dy) = \frac{1}{n}\sum_{k=1}^{n}\int_M f(y)\,\mu_k(dy) = \frac{1}{n}\sum_{k=1}^{n}\mathbb{E}[f(X_k)|X_1,...,X_{k-1}]$$

while the second is

$$\int_M f(x)\hat{\nu}(dx) = \frac{1}{n}\sum_{i=1}^{n} f(X_i).$$

We can see that both have the same expected value, and if $i < j$, we have

$$\mathbb{E}\Big[\Big(f(X_i) - \mathbb{E}[f(X_i)|X_1,...,X_{i-1}]\Big)\mathbb{E}[f(X_j)|X_1,...,X_{j-1}]\Big] =$$
$$= \mathbb{E}\Big[\Big(f(X_i) - \mathbb{E}[f(X_i)|X_1,...,X_{i-1}]\Big)f(X_j)\Big]$$

because $\Big(f(X_i) - \mathbb{E}[f(X_i)|X_1,...,X_{i-1}]\Big)$ is $(X_1,...,X_{j-1})$ measurable. Then we get

$$\mathbb{E}\Big[\Big(f(X_i){-}\mathbb{E}[f(X_i)|X_1,...,X_{i-1}]\Big)\Big(f(X_j){-}\mathbb{E}[f(X_j)|X_1,...,X_{j-1}]\Big)\Big] = 0.$$

So we have

$$\begin{aligned}\mathrm{Var}\ &\left(\int_M f(x)\hat{\mu}(dx) - \int_M f(y)\hat{\nu}(dy)\right) = \\ &= \frac{1}{n^2}\sum_{i=1}^{n}\mathbb{E}\left[\Big(f(X_i) - \mathbb{E}[f(X_i)|X_1,...,X_{i-1}]\Big)^2\right] \\ &\leq \frac{1}{n^2}\sum_{i=1}^{n}4\|f\|_\infty^2 = \frac{1}{n}4\|f\|_\infty^2,\end{aligned}$$

and by Chebyshev's inequality we conclude our claim. □

Proposition 7.4. *Using the notation of the proof in Lemma 4.5, if we have*

$$(\hat{\mu}_n, \hat{\nu}_n) = \left(\frac{1}{n} \sum_{i=1}^{n} \tau_n^i, \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}, \right) \to (\chi, \tilde{\chi})$$

in law, then we have $\chi = \tilde{\chi}$ *almost surely.*

Proof. For any continuous $f : M \to \mathbb{R}$, the function

$$\begin{aligned} T_f : \mathcal{P}(M) \times \mathcal{P}(M) &\to \mathbb{R} \\ (\mu, \nu) &\mapsto \int_M f(x)\mu(dx) - \int_M f(y)\nu(dy) \end{aligned}$$

is continuous. By Lemma 7.3, for every continuous f, we get

$$\mathbb{P}\left(|T_f(\hat{\mu}_n, \hat{\nu}_n)| > \epsilon\right) \leq 4\frac{\|f\|_\infty^2}{n\epsilon^2},$$

and, by the Portmanteau theorem (taking the lower limit on both sides), we reach

$$\mathbb{P}\left(|T_f(\chi, \tilde{\chi})| > \epsilon\right) = 0$$

for every $\epsilon > 0$. Thus we have

$$\mathbb{P}\left(|T_f(\chi, \tilde{\chi})| = 0\right) = 1.$$

Next, choose a dense sequence $\{f_m\}_{m \in \mathbb{N}}$ in the space of continuous functions on M endowed with the topology of uniform convergence in order to obtain

$$\mathbb{P}\left(|T_{f_m}(\chi, \tilde{\chi})| = 0 \text{ for all } m\right) = 1.$$

But, by density, we have

$$\{|T_{f_m}(\chi, \tilde{\chi})| = 0 \text{ for all } m\} = \{|T_f(\chi, \tilde{\chi})| = 0 \text{ for all continuous } f\},$$

which means $\chi = \tilde{\chi}$ almost surely. □

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Resumen

Siguiendo las técnicas desarrolladas por Paul Dupuis, Vaios Laschos y Kavita Ramanan en [8], se establecerá un principio de grandes desviaciones para una secuencia de procesos puntuales definidos por medidas de Gibbs en una variedad riemanniana bidimensional compacta y orientable. Veremos que la correspondiente secuencia de medidas empíricas converge a la solución de una ecuación diferencial parcial y, en ciertos casos, a la forma de volumen de una métrica de curvatura constante.

Palabras clave: Medidas de Gibbs; gas de Coulomb; medida empírica; principio de grandes desvíos; sistemas de partículas interactuantes; potencial singular; variedad de Einstein 2-dimensional; entropía relativa.

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Singularidades presimples y simples de foliaciones de codimensión uno

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Resumen

Tanto en dimensión dos como en dimensión tres la reducción de singularidades de foliaciones holomorfas de codimensión uno pasa por dos etapas: la primera es pasar de la singularidad dada a la sigularidad presimple, la segunda de la singularidad presimple a la simple. En este artículo establecemos las formas normales de la singularidades presimples siguiendo los artículos [2], [4] y [3]. También caracterizamos las singularidades presimples con hojas cerradas.

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Palabras clave: Foliaciones holomorfas, singularidades, resolución de singularidades.

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1. Introducción

Uno de los aspectos de la teoría de campos vectoriales holomorfos es entender su comportamiento en torno de sus *puntos singulares*, es decir, puntos donde el campo se anula. En un espacio ambiente de dimensión dos, el tipo más sencillo de singularidades queda definido por campos lineales.

Sea X es un campo vectorial sobre $(\mathbb{C}^2, \mathbf{0})$ con parte lineal, D_0X, no nula. Si D_0X satisface ciertas propiedades (por ejemplo, cuando los autovalores se encuentran en el dominio de Poincaré o Siegel [8]), entonces X es linealizable. Cuando el campo no tiene parte lineal, es posible aplicar un proceso de reducción de singularidades. Este proceso transforma sus singularidades en un número finito de ellas, cuyo campo, cerca de estas singularidades, admite parte lineal y su comportamiento es relativamente sencillo.

En dimensión dos una singularidad es llamada **simple** si la parte lineal del campo es no nula y tiene autovalores λ_1, λ_2 sujetos a la condición

- $\lambda_1.\lambda_2 \neq 0$ y $\dfrac{\lambda_1}{\lambda_2} \notin \mathbb{Q}_+$ ó
- $\lambda_1 = 0$ y $\lambda_2 \neq 0$ ó $\lambda_1 \neq 0$ y $\lambda_2 = 0$ (referido en adelante como **silla nodo**).

Por el teorema de Seidenberg [9], toda singularidad se puede reducir a singularidades simples mediante un número finito de explosiones. La singularidades simples fueron estudiadas muchos años antes de la publicación del teorema de Seidenberg, lográndose con ello formas normales (las llamadas formas normales de Poincaré, Dulac y Siegel).

Cuando el espacio ambiente es de dimensión tres, la existencia de una reducción de singularidades para foliaciones de codimensión uno fueron estudiadas por F. Cano y D. Cerveau [4] en el caso no-dicrítico y por F.Cano [3] en el caso dicrítico.

Cuando la dimensión es superior a tres, aún no se cuenta con un resultado análogo. En este artículo, describimos brevemente las singularidades presimples y simples de una foliación de codimensión uno en un espacio ambiente de dimensión finita, objetos que generalizan las singularidades presimples y simples de dimensión dos. Propondremos los modelos formales para singularidades presimples y, a partir de estos, definiremos las singularidades simples. Los modelos formales son deducidos directamente de la *"jordanización"* formal [7] de álgebras conmutativas de campos de vectores [3], y de una forma normal formal de una 1-forma integrable. En la última sección caracterizamos las singularidades presimples con la propiedad de que todas sus hojas sean cerradas.

2. Foliaciones singulares adaptadas

Sea X una variedad analítica compleja de dimensión n. Fijemos la siguiente notación.

- El **haz de gérmenes de funciones holomorfas sobre** X será denotado $\mathcal{O}_X$. La fibra en cada punto $P \in X$ será denotado por $\mathcal{O}_{X,P}$.
- El haz **ideal maximal** del haz $\mathcal{O}_X$ será denotado $\mathfrak{m}_X$. La **fibra** en cada punto $P \in X$ es el ideal maximal del anillo local $\mathcal{O}_{X,P}$ denotado por $\mathfrak{m}_{X,P}$.

Un subconjunto analítico de codimensión uno, $E \subset X$, es un **divisor con cruzamientos normales** si para todo punto $P \in X$ existe una vecindad coordenada $(U, (x_1, x_2, \ldots, x_n))$, donde $P = (0, \ldots, 0)$, tal que

$$E|_U : \prod_{i \in A} x_i = 0; \quad A \subset \{1, 2, \ldots, n\}.$$

Si $E \subset X$ es un divisor con cruzamientos normales denotamos por $e = e(E, P)$ el número de componentes irreducibles de E que pasan por el punto P. Obsérvese que se satisface

$$e(E, P) \leq n, \quad \text{para } P \in X.$$

Denotemos por $\Omega_X[-E]$ el haz de gérmenes de 1-formas meromorfas con polos simples a lo largo del divisor E. Una sección ω alrededor de un punto $P \in X$ de este haz en las coordenadas $(U, (x_1, x_2, \ldots, x_n))$, se expresa como

$$\omega = \sum_{i \in A} a_i \frac{dx_i}{x_i} + \sum_{i \notin A} a_i dx_i,$$

donde a_i son funciones holomorfas en U.

Una **foliación** $\mathcal{F}$ en X de codimensión uno **adaptada** al divisor E es un par $(\mathcal{F}, E)$, donde $\mathcal{F}$ es un $\mathcal{O}_X$-submódulo de $\Omega_X[-E]$ sujeto a las siguientes condiciones:

a) $\mathcal{F}$ es localmente libre de rango uno,

b) $\mathcal{F}$ es **integrable**, es decir satisface $\mathcal{F} \wedge d\mathcal{F} = 0$, con d el diferencial exterior y

c) $\mathcal{F}$ es **saturado**, es decir, el cociente $\dfrac{\Omega_X[-E]}{\mathcal{F}}$ es libre de torsión.

Observación 2.1. El **$\mathcal{O}_X$-módulo de torsión del cociente** $\dfrac{\Omega_X[-E]}{\mathcal{F}}$ es definido como

$$\operatorname{Tor}\left(\frac{\Omega_X[-E]}{\mathcal{F}}\right) = \Big\{\overline{\eta} \ : \ \text{ existe } g \in \mathcal{O}_X^* - \{0\} \text{ tal que } g.\overline{\eta} = 0\Big\}. \tag{2.1}$$

Dado que $\dfrac{\Omega_X[-E]}{\mathcal{F}}$ es libre de torsión, se tiene

$$\operatorname{Tor}\left(\frac{\Omega_X[-E]}{\mathcal{F}}\right) = \{0\},$$

es decir, dado $\eta \in \Omega_X[-E]$, existe $g \in \mathcal{O}_X^*$ tal que $g.\eta \in \mathcal{F}$, en donde se tiene necesariamente $\eta \in \mathcal{F}$. Como $\mathcal{F}$ tiene rango uno, existe una 1-forma $\omega \in \Omega_X[-E]$ que genera $\mathcal{F}$, esto es, se cumple $\mathcal{F} = \omega.\mathcal{O}_X$. Localmente, en las coordenadas $(x_1, \ldots, x_n)$, esta 1-forma se expresa cual

$$\omega = \sum_{i \in A} a_i \frac{dx_i}{x_i} + \sum_{i \notin A} a_i dx_i. \tag{2.2}$$

Observación 2.2. Si el cociente $\dfrac{\Omega_X[-E]}{\mathcal{F}}$ es libre de torsión entonces el máximo común divisor $\mathrm{mcd}(a_1, \ldots a_n)$, vale 1. En efecto, supongamos que existen gérmenes $g, b_1, \cdots, b_n$, no nulos ni unidades del anillo $\mathcal{O}_{X_P}$, tales que $gb_i = a_i$. De ser así, la 1-forma meromorfa

$$\eta = \sum_{i \in A} b_i \frac{dx_i}{x_i} + \sum_{i \notin A} b_i dx_i$$

satisface $g.\eta = w \in \mathcal{F}$. Se tendrá así $\eta \notin \mathcal{F}$, pues caso contrario b sería una unidad. Luego la clase $\overline{\eta} \neq 0$ representaría un elemento del módulo cociente.

Sea $(\mathcal{F}, E)$ una foliación adaptada. Definimos el **orden adaptado** de $(\mathcal{F}, E)$ en $P \in X$ como el orden $\mathfrak{m}_{X,P}-$ádico del submódulo $\mathcal{F}_P$ de $\Omega_X[-E]$, esto es,

$$\nu(\mathcal{F}, E, P) = \text{máx}\left\{k : \ \mathcal{F}_P \subset \mathfrak{m}^k_{X,P}\Omega_{X,P}[-E]\right\}.$$

Por definición el **lugar singular** $\mathrm{Sing}(\mathcal{F}, E)$ de la foliación adaptada al divisor E es el conjunto de puntos $P \in X$ que cumplen $\nu(\mathcal{F}, E, P) \geq 1$.

Para ω como en (2.2), el generador de $\mathcal{F}$, se tiene

$$\nu(\mathcal{F}, E, P) = \text{mín}\left\{\nu_P(a_i) : i = 1, \ldots, n\right\}.$$

Si consideramos $A^* = \{i : x_i \text{ no divide a } a_i\}$, entonces la foliación

$$\mathcal{G} : \Omega = \left(\prod_{i \in A^*} x_i\right).w,$$

admite un conjunto singular de codimensión mayor o igual que 2.

Observación 2.3. De hecho A^* es el conjunto de índices correspondiente a las componentes irreducibles F de E en torno a P que obedecen $\Omega \mid_F = 0$, es decir, F resulta ser separatriz de Ω.

Si i está en A, mas no en A^*, diremos que $x_i = 0$ es una **componente dicrítica** de E para $(\mathcal{F}, E)$.

Un subespacio analítico Y de X se dice que tiene **cruzamientos normales** con E si en las coordenadas (2.2) existe $B \subset \{1, \cdots, n\}$ tal que

$$I_{Y,P} = \sum_{i \in B} x_i \mathcal{O}_{X,P}.$$

Definimos los **órdenes adaptados** a $(\mathcal{F}, E)$ en Y como

$$\mu(\mathcal{F}, E, Y) = \text{mín}(\{\nu_Y(a_i) : i \notin B - A\} \cup \{\nu_Y(a_i) + 1 : i \in B - A\}),$$

$$\rho(\mathcal{F}, E, Y, P) = \text{mín}(\{\nu_P(a_i) : i \notin B - A\} \cup \{\nu_P(a_i) + 1 : i \in B - A\}),$$

donde $\nu_Y(a_i) = \text{máx}\{n : a_i \in I^n_{Y,p}\}$ y $\nu_P(a_i) = \text{máx}\{n : a_i \in \mathfrak{m}^n_{X,P}\}$.

En el caso $Y = \{P\}$, diremos que $\mu(\mathcal{F}, E, \{P\})$ es la **multiplicidad adaptada** de $(\mathcal{F}, E)$ en P.

Observación 2.4. Si $\mu(\mathcal{F}, E, Y) = r$, entonces se tiene

$$\rho(\mathcal{F}, E, Y, P) = r \qquad \text{ó} \qquad \rho(\mathcal{F}, E, Y, P) = r + 1,$$

es decir, siempre se cumple

$$\mu(\mathcal{F}, E, Y) \leq \rho(\mathcal{F}, E, Y, P).$$

Sea $\mathcal{F}$ una foliación en X de codimensión uno, E un divisor con cruzamientos normales en X e $Y \subset X$ una subvariedad de X. Decimos que Y es **centro permitido** en un punto $P \in Y$ si se satisfacen las condiciones

a) $Y \subset \text{Sing}(\mathcal{F})$,

b) $Y \subset X$ tiene cruzamientos normales con E en P y

c) $\mu(\mathcal{F}, E, Y) = \rho(\mathcal{F}, E, Y, P)$.

Decimos que Y es centro permitido para $(\mathcal{F}, E)$ si todo punto de Y es un **centro permitido**.

Ejemplo 2.5. Consideremos $E : x_2x_3 = 0$, $Y : x_1 = x_3 = 0$ y

$$\omega = x_1x_2dx_1 + x_3\frac{dx_2}{x_2} + x_1^2x_2\frac{dx_3}{x_3};$$

acá se tendrá $A = \{2,3\}$ y $B = \{1,3\}$. Un cálculo simple conduce a

$$\begin{aligned}\mu(\mathcal{F},E,Y) &= \text{mín}\left(\{\nu_Y(a_i) : i \notin B - A\} \cup \{\nu_Y(a_i)+1 : i \in B - A\}\right)\\ &= \text{mín}(\{\nu_Y(a_2), \nu_Y(a_3)\} \cup \{\nu_Y(a_1)+1\})\\ &= \text{mín}(\{1,2\} \cup \{2\}) = 1.\end{aligned}$$

Si $P = (0,0,0)$ tenemos

$$\begin{aligned}\rho(\mathcal{F},E,Y,P) &= \text{mín}(\nu_P(a_i) : i \notin B - A\} \cup \{\nu_P(a_i)+1 : i \in B - A\})\\ &= \text{mín}(\{\nu_P(a_2), \nu_P(a_3)\} \cup \{\nu_P(a_1)+1\})\\ &= \text{mín}(\{1,3\} \cup \{3\}) = 1.\end{aligned}$$

Si $Q = (0,y,0)$, donde $y \neq 0$, tenemos

$$\begin{aligned}\rho(\mathcal{F},E,Y,Q) &= \text{mín}(\{\nu_Q(a_i) : i \notin B - A\} \cup \{\nu_Q(a_i)+1 : i \in B - A\})\\ &= \text{mín}(\{\nu_Q(a_2), \nu_Q(a_3)\} \cup \{\nu_Q(a_1)+1\})\\ &= \text{mín}(\{1,2\} \cup \{2\}) = 1.\end{aligned}$$

De este modo la curva Y es un centro permitido para la foliación adaptada $(\mathcal{F},E)$.

Ejemplo 2.6. Para $E : x_2x_3 = 0$, $Y : x_1 = x_3 = 0$ y

$$\omega = x_1x_2dx_1 + x_3^3\frac{dx_2}{x_2} + x_1^2x_2\frac{dx_3}{x_3},$$

se tiene

$$\begin{aligned}\mu(\mathcal{F},E,Y) &= \text{mín}(\{\nu_Y(a_2), \nu_Y(a_3)\} \cup \{\nu_Y(a_1)+1\})\\ &= \text{mín}(\{3,2\} \cup \{2\}) = 2.\end{aligned}$$

Si $P = (0,0,0)$, tenemos

$$\begin{aligned}\rho(\mathcal{F},E,Y,P) &= \text{mín}(\{\nu_P(a_2), \nu_P(a_3)\} \cup \{\nu_P(a_1)+1\})\\ &= \text{mín}(\{3,3\} \cup \{3\}) = 3.\end{aligned}$$

Si $Q = (0, y, 0)$, donde $y \neq 0$, tenemos

$$\begin{aligned} \rho(\mathcal{F}, E, Y, Q) &= \text{mín}(\{\nu_Q(a_2), \nu_Q(a_3)\} \cup \{\nu_Q(a_1) + 1\}) \\ &= \text{mín}(\{3, 2\} \cup \{2\}) = 2. \end{aligned}$$

Como se tiene $\mu(\mathcal{F}, E, Y) = \rho(\mathcal{F}, E, Y, Q)$ y $\mu(\mathcal{F}, E, Y) < \rho(\mathcal{F}, E, Y, P)$, resulta que Y no es centro permitido en P.

3. Explosión con centro permitido

Consideremos el divisor $E : \prod_{i \in A} x_i = 0$, con $A \subset \{1, \cdots, n\}$, e $Y : x_i = 0$, donde $i \in B \subset \{1, \cdots, n\}$, un conjunto analítico que tiene cruzamientos normales con E.

Una **explosión con centro en** Y está formada por una terna (X', E', π), donde

$$X' = \left\{ \Big(x, [l]\Big) \in X \times \mathbb{P}^{k-1} \colon x_i l_j = x_j l_i,\ i, j \in B \right\},$$

con $x = (x_1, \cdots, x_n)$, $l = (l_i : i \in B)$, k es el número de elementos de B y π es la restricción a la subvariedad X' de la proyección

$$pr_1 : X \times \mathbb{P}^{k-1} \to X,$$

llamada la **aplicación de explosión**. La **transformada** de E por π es un divisor con cruzamientos normales $E' = \pi^{-1}(E \cup Y)$. Obsérvese que X' es una variedad de la misma dimensión que X y π, localmente (en las coordenadas $(x'_1, \dots x'_n)$ de X'), queda descrita como sigue: para $t \in B$, con t fijo, tenemos

$$\pi = \begin{cases} x_i = x'_i, & \text{para todo } i \notin B \text{ o } i = t, \\ x_i = (x'_i + \zeta_i) x'_t, & \text{para todo } i \text{ distinto de } t, \end{cases} \tag{3.1}$$

donde $\zeta_i \in \mathbb{C}$ es la constante de traslación.

Consideremos $\omega \in \Omega_{X,p}[-E]$ como en (2.2). Los términos de $\pi^*\omega$ quedan descritos como sigue.

Si $i \in B-(A\cup\{t\})$, tenemos

$$\sum_{i\in B\setminus(A\cup\{t\})}(a_i\circ\pi)\ d((x_i'+\zeta_i)x_t')=\sum_{i\in B\setminus(A\cup\{t\})}(a_i\circ\pi)\ (x_t'dx_i'+(x_i'+\zeta_i)dx_t').$$

Si $i \in B\cap(A-\{t\})$, se tiene

$$\sum_{i\in B\cap(A\setminus\{t\})}(a_i\circ\pi)\ \frac{d((x_i'+\zeta_i)x_t')}{(x_i'+\zeta_i)x_t'}=\sum_{i\in B\cap(A\setminus\{t\})}(a_i\circ\pi)\ \left(\frac{dx_i'}{x_i'+\zeta_i}+\frac{dx_t'}{x_t'}\right).$$

Si $i\notin B$, tenemos

$$\sum_{i\in A\setminus B}(a_i\circ\pi)\ \frac{dx_i'}{x_i'}+\sum_{i\notin(A\cup B)}(a_i\circ\pi)\ dx_i'.$$

Por otro lado, si $i=t\in B-A$, en uso de (2.2) aparecen términos de la forma $(a_t\circ\pi)dx_t'$. De manera similar, si $i=t\in B\cap A$, mediante (2.2) se obtienen términos de la forma $(a_t\circ\pi)\dfrac{dx_t'}{x_t'}$.

Al agrupar lo anterior, llegamos a

$$\begin{aligned}\pi^*\omega=&\left(\sum_{i\in B\cap A}(a_i\circ\pi)+\sum_{i\in B-A}(\overbrace{x_t'(x_i'+\zeta_i)}^{x_i}(a_i\circ\pi))\right)\ \frac{dx_t'}{x_t'}\\ &+\sum_{\{i\in B\cap(A-\{t\}):\zeta_i=0\}}(a_i\circ\pi)\ \left(\frac{dx_i'}{x_i'}\right)\\ &+\sum_{\{i\in B\cap(A-\{t\}):\zeta_i\neq0\}}(a_i\circ\pi)\ \left(\frac{dx_i'}{x_i'+\zeta_i}\right)\\ &+\sum_{i\in B-(A\cup\{t\})}(x_t'a_i\circ\pi)\ dx_i'+\sum_{i\notin(A\cup B)}(a_i\circ\pi)\ dx_i'\\ &+\sum_{i\in(A-B)}(a_i\circ\pi)\ \frac{dx_i'}{x_i'}.\end{aligned}$$

Observación 3.1. Sea $\mu(\mathcal{F}, E, Y)$ la máxima potencia de x'_t que divide a $\pi^*\omega$ en $\Omega[-E']$, donde E' es el transformado de E, es decir

$$\mu(\mathcal{F}, E, Y) = \text{máx}\left\{k \ : \ \frac{\pi^*\omega}{(x'_t)^k} \in \Omega[-E']\right\}.$$

Con esta notación, la forma que define la transforma estricta $(\mathcal{F}', E')$ de $(\mathcal{F}, E)$ queda definida por la $1-$forma

$$(\mathcal{F}', E') = \frac{\pi^*\omega}{(x'_t)^{\mu(\mathcal{F},E,Y)}}.$$

El siguiente teorema se usa en la primera etapa del proceso de reducción de singularidades, etapa que no discutiremos en este trabajo y que consiste en pasar de una singularidad arbitraria a una singularidad presimple.

Teorema 3.2. *[3, Theorem 2.7] Sea Y un centro permitido para $(\mathcal{F}, E)$, $\pi : X' \to X$ una explosión con centro en Y con $P' \in X'$ y $P = \pi(P')$. Entonces se cumple*

$$\nu(\mathcal{F}', E', P') \leq \nu(\mathcal{F}, E, P).$$

□

4. Singularidades dicriticas y no-dicríticas

En esta sección se enunciarán resultados sobre la primera etapa de la reducción de singularidades, la cual consiste en pasar de una singularidad arbitraria una singularidad presimple.

Sea $(\mathcal{F}, E)$ una foliación adaptada sobre X. Decimos que una componente irreducible F de E es una **componente dicrítica** de E para $(\mathcal{F}, E)$ si F no es una separatriz del saturado de $(\mathcal{F}, E)$.

Diremos que la foliación adaptada $(\mathcal{F}, E)$ es **dicrítica** si existe una sucesión finita

$$\Big\{X(i), \mathcal{F}(i), E(i), Y(i), \pi(i+1)\Big\}_{i=0}^{N-1}, \quad \text{con} \quad N \geq 1,$$

que obedece

a) $X(0) = X, (\mathcal{F}(0), E(0)) = (\mathcal{F}, E)$,

b) $\pi_{(i+1)} : X(i+1) \to X(i)$ es la aplicación de explosión con centro $Y(i) \subset (\mathcal{F}(i), E(i))$, para $i = 0, \cdots, N-1$,

d) $(\mathcal{F}(i+1), E(i+1))$ es la transformada estricta de $(\mathcal{F}(i), E(i))$ por $\pi(i+1)$ y

e) existe al menos una componente dicrítica de $E(N)$ para la foliación adaptada $(\mathcal{F}(N), E(N))$.

Caso contrario diremos que $(\mathcal{F}, E)$ es **no dicrítica**.

Observación 4.1. Si E no tiene componente dicrítica para $(\mathcal{F}, E)$, entonces $\mathcal{F}$ está generada por

$$\omega = \sum_{i \in A} a_i \frac{dx_i}{x_i} + \sum_{i \notin A} a_i dx_i, \quad a_i \in \mathcal{O}_{(\mathbb{C}^n,0)},$$

donde se tiene $A = A^* = \left\{ i : x_i \;\text{ no divide }\; a_i \right\}$. Así, el saturado resulta ser

$$\begin{aligned}
\mathcal{F} = \left(\prod_{j \in A} x_j\right) \omega &= \sum_{i \in A} \left(\prod_{j \in A} x_j\right) a_i \frac{dx_i}{x_i} + \sum_{i \notin A} \left(\prod_{j \in A} x_j\right) a_i dx_i \\
&= \sum_{\substack{i \in A \\ j \neq i}} a_i dx_i + \sum_{i \notin A} \left(\prod_{j \in A} x_j\right) a_i dx_i \\
&= \sum_{i \in A} b_i dx_i + \sum_{i \notin A} b_i dx_i \\
&= \sum_{i=1}^{n} b_i dx_i .
\end{aligned}$$

Obsérvese que con ello obtenemos

$$\begin{aligned}
\nu(\mathcal{F}, E, P) &= \min\{\nu_P(b_i) : i = 1, \ldots, n\} \\
&= \min(\{\nu_P(a_i) + e - 1 : i \in A\} \cup \{\nu_P(a_i) + e : i \notin A\}) \\
&= \mu(\mathcal{F}, E, P) + e(E, P) - 1.
\end{aligned}$$

En general escribiremos $b_i = b^i_\nu + b^i_{\nu+1} + \cdots$, donde b^i_k es polinomio homogéneo de grado k. También, para simplificar la notación usaremos $B_i = b^i_\nu$.

Lema 4.2. *Si* $\sum_{i=1}^{n} x_i B_i = 0$, *entonces* $(\mathcal{F}, E)$ *es dicrítico.*

Demostración. Sea π una explosión con centro en $P = (0, \ldots, 0)$ y fijemos $t = 1 \in B = \{1, \ldots, n\}$, para tener

$$\pi = \begin{cases} x_1 &= x'_1 \\ x_2 &= x'_1 x'_2 \\ x_3 &= x'_1 x'_3 \\ \\ \vdots \\ x_n &= x'_1 x'_n. \end{cases}$$

Si ponemos

$$\Omega = \sum_{i \in A} b_i dx_i + \sum_{i \notin A} b_i dx_i = \sum_{i=1}^{n} b_i dx_i,$$

tendremos

$$\begin{aligned}
\pi^*\Omega \quad &= \sum_{i=1}^{n} (b_i \circ \pi) dx_i \\
&= (b_1 \circ \pi) dx'_1 + \sum_{i=2}^{n} (b_i \circ \pi)(x'_i dx'_1 + x'_1 dx'_i) \\
&= \left((b_1 \circ \pi) + \sum_{i=2}^{n} (b_i \circ \pi) x'_i \right) dx'_1 + x'_1 \sum_{i=2}^{n} (b_i \circ \pi) dx'_i \\
&= \Bigg((x'_1)^\nu [(B_1(1, x'_2, \ldots, x'_n) + x'_1 b^1_{\nu+1}(1, x'_2, \ldots, x'_n) + \cdots) \\
&\quad + \sum_{i=2}^{n} (B_i(1, x'_2, \ldots, x'_n) + x'_1 b^i_{\nu+1}(1, x'_2, \ldots, x'_n) + \cdots) x'_i] \Bigg) dx'_1 \\
&\quad + (x'_1)^{\nu+1} \sum_{i=2}^{n} \left(B_i(1, x'_2, \ldots, x'_n) + x'_1 b^i_{\nu+1}(1, x'_2, \ldots, x'_n) + \cdots \right) dx'_i.
\end{aligned}$$

Lo anterior nos lleva a

$$\begin{aligned}\frac{\pi^*\Omega}{(x_1')^{\nu}} &= \Bigg([(B_1(1,x_2',\ldots,x_n') + x_1' b_{\nu+1}^1(1,x_2',\ldots,x_n') + \cdots) \\ &\quad + \sum_{i=2}^{n} (B_i(1,x_2',\ldots,x_n') + x_1' b_{\nu+1}^i(1,x_2',\ldots,x_n') + \cdots) x_i'] \Bigg) dx_1' \\ &\quad + x_1' \sum_{i=2}^{n} \left(B_i(1,x_2',\ldots,x_n') + x_1' b_{\nu+1}^i(1,x_2',\ldots,x_n') + \cdots \right) dx_i'.\end{aligned} \tag{4.1}$$

Por hipótesis se tiene $\displaystyle\sum_{i=1}^{n} x_i B_i = 0$, lo que conduce a

$$B_1(1,x_2',\ldots,x_n') + \sum_{i=2}^{n} B_i(1,x_2',\ldots,x_n') x_i' = 0.$$

Al reemplazar (4.1) resulta

$$\begin{aligned}\frac{\pi^*\Omega}{(x_1')^{\nu}} &= \Bigg((x_1' b_{\nu+1}^1(1,x_2',\ldots,x_n') + \cdots) \\ &+ \sum_{i=2}^{n} (x_1' b_{\nu+1}^i(1,x_2',\ldots,x_n') + \cdots) x_i' \Bigg) dx_1' \\ &+ x_1' \sum_{i=2}^{n} \left(B_i(1,x_2',\ldots,x_n') + x_1' b_{\nu+1}^i(1,x_2',\ldots,x_n') + \cdots \right) dx_i'.\end{aligned}$$

$$\begin{aligned}\frac{\pi^*\Omega}{(x_1')^{\nu+1}} &= \Bigg((b_{\nu+1}^1(1,x_2',\ldots,x_n') + \cdots) \\ &+ \sum_{i=2}^{n} (b_{\nu+1}^i(1,x_2',\ldots,x_n') + \cdots) x_i' \Bigg) dx_1' \\ &+ \sum_{i=2}^{n} \left(B_i(1,x_2',\ldots,x_n') + b_{\nu+1}^i(1,x_2',\ldots,x_n') + \cdots \right) dx_i'.\end{aligned}$$

Al hacer $x_1' = 0$ se tiene $\pi^*\Omega \neq 0$, de donde se deduce que $x_1' = 0$ no es invariante por $\mathcal{F}'$ (es decir, no es separatriz de $\mathcal{F}'$), acá $\mathcal{F}'$ es el transformado estricto del saturado $(\mathcal{F}, E)$. Concluimos que $(\mathcal{F}, E)$ es dicrítico. $\square$

Ejemplo 4.3. Consideremos $E : x_1x_3 = 0$, $Y : x_1 = x_3 = 0$ y

$$\mathcal{F} : \omega = (x_2x_3^2 + x_1^3)\frac{dx_1}{x_1} - x_3^2dx_2 - x_1^3\frac{dx_3}{x_3}.$$

Observamos que la foliación $(\mathcal{F}, E)$ es dicrítica en uso del lema 1, pues el saturado es

$$\Omega = (x_2x_3^3 + x_1^3x_3)dx_1 - x_1x_3^3dx_2 - x_1^4dx_3.$$

De esto deducimos la igualdad

$$x_1(x_2x_3^3 + x_1^3x_3) + x_2(-x_1x_3^3) + x_3(-x_1^4) = 0.$$

El siguiente teorema es otro resultado estructural que se usa en la primera etapa del proceso de reducción de singularidades.

Teorema 4.4. *[3, Theorem 3.3] Sea $(\mathcal{F}, E)$ una foliación adaptada en X no dicrítica, $Y \subset X$ una subvariedad de X, y $\pi : X' \to X$ una explosión con centro permitido en Y. Si $(\mathcal{F}', E')$ es la transformada estricta de $(\mathcal{F}, E)$ por π, entonces se tiene*

$$\mu(\mathcal{F}', E', P') \leq \mu(\mathcal{F}, E, P)$$

para todo $P' \in X'$, donde $P = \pi(P')$. $\square$

5. Singularidades presimples y simples

En esta sección se describen las formas normales de aquellas singularidades que aparecen tras de un número finito de explosiones. Estas singularidades son llamadas **singularidades simples** y generalizan las singularidades simples de dimensión dos.

Sea X una variedad compleja. Denotamos por $\Theta_{X,P}$ (respectivamente $\widehat{\Theta}_{X,P}$) el anillo de gérmenes de campos holomorfos (respectivamente de campos formales) sobre la variedad X en torno del punto P. Dado un campo vectorial formal $D \in \widehat{\mathfrak{m}}\widehat{\Theta}_{X,P}$ y un entero $k \geq 1$, queda inducida una derivación vía

$$\begin{array}{rcl} D^k : \mathfrak{m}/\mathfrak{m}^{k+1} & \to & \mathfrak{m}/\mathfrak{m}^{k+1} \\ f + \mathfrak{m}^{k+1} & \mapsto & D(f) + \mathfrak{m}^{k+1}. \end{array}$$

Debido a la identificación $\widehat{\mathfrak{m}}/\widehat{\mathfrak{m}}^{k+1} = \mathfrak{m}/\mathfrak{m}^{k+1}$, no hay inconveniente en sustituir $\mathfrak{m}$ por $\widehat{\mathfrak{m}}$ en la definición de D^k. Diremos que D es **semisimple** si D^k es semisimple para todo $k \geq 1$, además será **nilpotente** si D^k es nilpotente para todo $k \geq 1$.

Sea $D \in \widehat{\mathfrak{m}}\widehat{\Theta}_{X,P}$ un campo formal que se anula en el punto P y $(x_1, \dots, x_n)$ un sistema formal linealizable para D_S con autovalores $\lambda = (\lambda_1, \dots, \lambda_n)$. Adoptaremos la siguiente notación:

$$I = (i_1, \dots, i_n) \in \mathbb{N}^n, \quad x^I = x_1^{i_1} \dots x_n^{i_n}$$

junto con

$$\langle \lambda, I \rangle = \sum_{j=1}^{n} \lambda_j i_j, \quad |I| = \sum_{j=1}^{n} i_j.$$

Proposición 5.1. *Un campo vectorial $D \in \widehat{\mathfrak{m}}\widehat{\Theta}_{X,p}$ es semisimple si y solo si existe un sistema de coordenadas formales $(x_1, \dots, x_n)$ y números complejos $\lambda_1, \dots, \lambda_n$ con los cuales se tiene*

$$D = \sum_{j=1}^{n} \lambda_i x_i \frac{\partial}{\partial x_j}.$$

Demostración. Si existen coordenadas formales $(x_1, \dots, x_n)$ y escalares $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ sujetos a

$$D = \sum_{j=1}^{n} \lambda_i x_i \frac{\partial}{\partial x_j},$$

entonces una base de $\mathfrak{m}/\mathfrak{m}^{k+1}$ está dada por los monomios x^I, con $|I| \leq n$. En tal caso tendremos

$$D(x^I) =< \lambda, I > x^I,$$

esto es, una base de autovectores para D.

Recíprocamente, construiremos por inducción un sistema de coordenadas formales $(x_1, \cdots, x_n)$, sujeto a $D(x_i) = \lambda_i x_i$. Comencemos considerando una base $\left\{v_1^k, \ldots, v_{r_k}^k\right\}$ de $\mathfrak{m}/\mathfrak{m}^{k+1}$, para cada k, cuyos elementos son autovectores de D^k. Esta base satisface

$$\pi_k(v_i^{k+1}) = v_i^k, \text{ para todo } i = 1, 2, \ldots, r_k,$$

donde π_k es el homomorfismo natural $\mathfrak{m}/\mathfrak{m}^{k+2} \to \mathfrak{m}/\mathfrak{m}^{k+1}$. Estas bases pueden ser escogidas de la siguiente manera. Fijemos $\left\{v_1^k, \ldots, v_{r_k}^k\right\}$ una base cuyos elementos son autovectores de D^k. Buscaremos $v_1^{k+1}, \ldots, v_{r_{k+1}}^{k+1}$, autovectores de D^{k+1}. Para ello escojamos una base $\left\{u_1^{k+1}, \ldots, u_{r_{k+1}}^{k+1}\right\}$ cuyos elementos son autovectores de D^{k+1}. Si reordenamos los índices podemos suponer que $\left\{\pi_k(u_1^{k+1}), \ldots, \pi_k(u_{r_k}^{k+1})\right\}$ es una base de $\mathfrak{m}/\mathfrak{m}^{k+1}$ de vectores propios D^k. En la práctica tenemos una matriz cambio de base A_k para la que se tiene

$$\left(v_1^k, \ldots, v_{r_k}^k\right) = \left(\pi_k(u_1^{k+1}), \ldots, \pi_k(u_{r_k}^{k+1})\right) \cdot A_k.$$

Sea $v_i^{k+1} = u_i^{k+1}$ para $i > r_k$ y

$$\left(v_1^{k+1}, \ldots, v_{r_k}^{k+1}\right) = \left(u_1^{k+1}, \ldots, u_{r_k}^{k+1}\right) \cdot A_k.$$

Al cumplirse $x_i = \lim_k v_i^k$ en la topología de Krull para $i = 1, \ldots, n$, tenemos $D^k(x_i) = \lambda_i x_i$, $i = 1, \ldots, n$. □

Aprovechamos la construcción efectuada en la proposición 5.1 para destacar el siguiente corolario.

Corolario 5.2. *Sea $D \in \widehat{\mathfrak{m}}\widehat{\Theta}_{X,P}$ un campo semisimple y $(x'_1, \ldots x'_s)$ coordenadas formales con $1 \leq s \leq n$ tales que*

$$D(x'_i) = \lambda'_i x'_i, \quad \lambda'_i \in \mathbb{C}, \quad i = 1, \ldots, s.$$

Entonces existe un sistema de coordenadas formales $(x_1, \ldots, x_n)$ de $\hat{\mathcal{O}}_{X,P}$ en donde se tiene

$$D = \sum_{i=1}^n \lambda_i x_i \frac{\partial}{\partial x_i},$$

con $x_i = x'_i$ *y* $\lambda_i = \lambda'_i$ *para* $i = 1, \ldots, s$.

Demostración. Las coordenadas $x'_i, \ldots, x'_s$ proporcionan una base parcial cuyos elementos son autovectores de los D^k. Basta entonces completar esta base y hacer tender k a infinito. □

Proposición 5.3. *Un campo formal* $D \in \widehat{\mathfrak{m}}\widehat{\Theta}_{X,P}$ *es nilpotente si y solo si su parte lineal* D^1 *es nilpotente.*

Demostración. Por definición, si D es nilpotente resulta que D^1 es nilpotente.

Recíprocamente, si D^1 es nilpotente existe un entero $n \geq 1$ tal que el n-ésimo iterado $(D^1)^n$ de D^1 es nulo, es decir, cumple

$$D^n(f) \in \mathfrak{m}^2, \text{ para todo } f \in \mathfrak{m}.$$

Para mostrar que D^k es nilpotente hay que determinar $n_k \geq 1$ tal que el n_k-ésimo iterado $(D^k)^{n_k}$ es nulo, lo cual equivale a mostrar

$$D^{n_k}(f) \in \mathfrak{m}^{k+1}, \text{ para todo } f \in \mathfrak{m}.$$

En efecto, por inducción, para $k \geq 2$ asumamos que existe n_{k-1} tal que

$$D^{n_{k-1}}(g) \in \mathfrak{m}^k, \text{ para todo } g \in \mathfrak{m}. \tag{5.1}$$

Si ponemos $n_k = 3n_{k-1}$, entonces para todo $f \in \mathfrak{m}$ tendremos

$$D^{n_k}(f) = D^{2n_{k-1}}(D^{n_{k-1}}(f)) = D^{2n_{k-1}}(g),$$

donde $g = D^{n_{k-1}}(f)$. Al satisfacerse $g \in \mathfrak{m}^k \subset \mathfrak{m}^2$ podemos escribir

$$g = \sum_{i=1}^{r} a_i b_i, \quad \text{con } a_i, b_i \in \mathfrak{m},$$

obteniéndose así

$$D^{2n_{k-1}}(g) = \sum_{i=1}^{r} \sum_{j=0}^{2n_{k-1}} \binom{2n_{k-1}}{j} D^{2n_{k-1}-j}(a_i)\, D^j(b_i). \tag{5.2}$$

Como se cumple $D \in \widehat{\mathfrak{m}}\widehat{\Theta}_{X,P}$, tenemos $D^s(f) \in \mathfrak{m}^{k+s}$ para todo $f \in \mathfrak{m}^k$ y con ello

$$D^{2n_{k-1}-j}(a_i)\, D^j(b_i) \in \mathfrak{m}^{k+1}.$$

Puesto que se satisface $2n_{k-1} - j \geq n_{k-1}$, es decir $j \geq n_{k-1}$, para todo $0 \leq j \leq 2n_{k-1}$, cada término de (5.2) pertenece a $\mathfrak{m}^{k+1}$ por (5.1). Por lo expuesto D^k es nilpotente. □

Teorema 5.4 (Martinet)**.** *Para todo campo formal $D \in \widehat{\mathfrak{m}}\widehat{\Theta}_{X,P}$ existe un campo semisimple D_S y un campo nilpotente D_N tal que se tiene*

a) $D = D_S + D_N$,

b) $[D_S, D_N] = 0$. Además, D_S y D_N son los únicos con dichas propiedades.

Demostración. La unicidad es consecuencia de la unicidad de una descomposición de Jordan, $D^k = D^k_S + D^k_N$, con $[D^k_S, D^k_N] = 0$ para todo $k \in \mathbb{N}$.

Para la existencia, exhibiremos coordenadas $x = (x_1, \ldots, x_n)$ en donde $D_S = L_{\lambda,x} = \displaystyle\sum_{i=1}^{n} \lambda_i x_i \frac{\partial}{\partial x_i}$, así $\lambda = (\lambda_1, \ldots, \lambda_n)$.

Comencemos considerando una base $x^1 = (x^1_1, \ldots, x^1_n)$ tal que D^1 se exprese en forma canónica de Jordan cual

$$D^1 = L_{\lambda,x^1} + N^1 = \left(\sum_{i=1}^{n} \lambda_i x^1_i \frac{\partial}{\partial x_i} \right) + \left(\sum_{j=1}^{n} \sum_{|I|\geq 1} b^1_{j,I} [x^1]^I \frac{\partial}{\partial x^1_j} \right).$$

Como L_{λ,x^1} y N^1 conmutan hasta orden uno, se obtiene siempre $|I| \leq 1$; así $b^1_{j,I} \neq 0$ implica $\langle \lambda, I \rangle = \lambda_j$.

Supongamos por inducción que ya tenemos elegido un sistema de coordenadas a doble índice $x^{k,s} = (x^{k,s}_1, \ldots, x^{k,s}_n)$, $k \geq 1$, $s \in \{1, \ldots, n\}$, que satisface las siguientes propiedades:

• $x^{k,s} - x^{k-1,n} \in \mathfrak{m}^k \times \cdots \times \mathfrak{m}^k$,

• Para $|I| < k$ ó para $|I| = k$ con $j \leq s$ se tiene que $b^{k,s}_{j,I} \neq 0$ implica $\langle \lambda, I \rangle = \lambda_j$.

Luego, en estas coordenadas, el campo se expresa cual $D = L_{\lambda,x^{k,s}} + N^{k,s}$, donde

$$N^{k,s} = \sum_{j=2}^{n} \epsilon_j x_{j-1}^{k,s} \frac{\partial}{\partial x_j^{k,s}} + \sum_{j=1}^{n} \sum_{|I|\geq 1} b_{j,I}^{k,s} [x^{k,s}]^I \frac{\partial}{\partial x_j^{k,s}}.$$

Para hallar las coordenadas, consideramos dos casos $s < n$ y $s = n$.

Si $s < n$, definimos el cambio de coordenandas

$$x_j^{k,s} = \begin{cases} x_j^{k,s}, & j \neq s+1. \\ x_{k,s+1}^{k,s} + Q_{s+1}^{k,s}(x^{k,s}), & j = s+1, \end{cases}$$

donde $Q_{s+1}^{k,s}$ es un polinomio homogeneo de grado k dado por

$$Q_{s+1}^{k,s}(x^{k,s}) = \sum_{|I|=k} q_I (x^{k,s})^I.$$

Expresamos D en estas coordenadas para obtener

$$D(x_j^{k,s+1}) = \sum_{|I|\geq 1} b_{j,I}^{k,s+1} (x^{k,s+1})^I. \tag{5.3}$$

Por otro lado, el cambio de coordenadas

$$D(x_j^{k,s+1}) = \begin{cases} D(x_j^{k,s}) & j \neq s+1, \\ D(x_{s+1}^{k,s}) + Q(x_{s+1}^{k,s}) & j = s+1, \end{cases}$$

conduce a

$$D(x_j^{k,s+1}) = \sum_{|I|<k} b_{j,I}^{k,s} (x^{k,s+1})^I + \sum_{|I|\geq k} b_{j,I}^{k,s+1} (x^{k,s+1})^I. \tag{5.4}$$

Comparamos las identidades (5.3) y (5.4) y obtenemos

$$b_{j,I}^{k,s+1} = b_{j,I}^{k,s}, \text{ para } |I| < k, \text{ y } j \in \{1, \ldots, n\}.$$

Si $j \leq s$, se tiene $D(x_j^{k,s+1}) = D(x_j^{k,s})$, es decir, si

$$\begin{aligned}\lambda_j x_j^{k,s} + \epsilon_j z_{j-1}^{k,s} + \sum_{|I|\leq 2} b_{j,I}^{k,s}(x^{k,s})^I &= \lambda_j x_j^{k,s+1} + \epsilon_j z_{j-1}^{k,s+1} \\ &+ \sum_{|I|\leq k} b_{j,I}^{k,s}(x^{k,s+1})^I + \sum_{|I|>k} b_{j,I}^{k,s+1}(x^{k,s+1})^I\end{aligned}$$

se tiene $b_{j,I}^{k,s+1} = b_{j,I}^{k,s}$, cuando $|I| = k$ y $j \leq s$. Finalmente, para $j = s+1$ se obtendrá

$$\begin{aligned}D(x_{s+1}^{k,s+1}) =& D(x_{s+1}^{k,s} + Q) = D(x_{s+1}^{k,s}) + D(Q) \\ =& \lambda_{s+1}(x_{s+1}^{k,s+1} - Q) + \epsilon_{s+1} z_s^{k,s+1} + D(Q_{s+1}) \\ &+ \sum_{|I|\geq 2} b_{s+1,I}^{k,s}(x^{k,s})^I.\end{aligned}$$

Si escribimos $D(Q) = \sum_{|J|\geq 1} c_J (x^{k,s+1})^J$, la expresión anterior implica, cuando $|I| = k$, la igualdad

$$b_{s+1,I}^{k,s+1} = -\lambda_{s+1} q_I + c_I.$$

Luego tenemos

$$c_I = \langle \lambda, I\rangle q_I + \sum_{j=2}^{n} q_{I'_j} \epsilon_j (i_j + 1),$$

donde $I = (i_1, \ldots, i_n)$ y $I'_j = I - e_j + e_{j-1}$. En otras palabras se cumple

$$b_{s+1,I}^{k,s+1} = (\langle \lambda, I\rangle - \lambda_{s+1}) q_I + \sum_{j=2}^{n} q_{I'_j} \epsilon_j (i_j + 1).$$

Por inducción sobre el multiíndice I, con $|I| = k$, ordenados lexicográficamente, podemos fijar los q_I de modo que se tenga $b_{s+1,J}^{k,s+1} = 0$ cuando $\langle \lambda, I\rangle - \lambda_{s+1} \neq 0$. Para $s = n$ el tratamiento es análogo y se deja como ejercicio para el lector.

Por la inducción establecida, consideramos el sistema de coordenadas límite $x = \lim_k x^{k,n}$. De aquí se obtiene $D = L_{\lambda,x} + N$, con $N =$

$\sum_{j,I} b_{j,I} x^I \dfrac{\partial}{\partial x_j}$ nilpotente. Por construcción de la coordenada x, la relación $\langle \lambda, I \rangle - \lambda_j \neq 0$ implica $b_{jI} = 0$, lo cual equivale a $[L_{\lambda,x}, N] = 0$. $\square$

Dado un campo $D \in \widehat{\mathfrak{m}}\widehat{\Theta}_{X,P}$, el teorema de Martinet (ver Teorema 5.4) asegura una descomposición $D = D_S + D_N$, con D_S semisimple y D_N nilpotente, términos que además conmutan (esto es, $[D_S, D_N] = 0$). Por la proposición 5.1, existen coordenadas $(x_1, \ldots, x_n)$ en donde $D_S = \sum_{j=1}^{n} \lambda_j x_j \dfrac{\partial}{\partial x_j}$. De este modo se tiene

$$\left[D_S, x^I \frac{\partial}{\partial x_i} \right](f) = D_S \left(x^I \frac{\partial f}{\partial x_i} \right) - x^I \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{n} \lambda_j x_j \frac{\partial f}{\partial x_j} \right). \tag{5.5}$$

Al reemplazar las identidades

$$D_S \left(x^I \frac{\partial f}{\partial x_i} \right) = x^I \langle \lambda, I \rangle \frac{\partial f}{\partial x_i} + x^I \sum_{j=1}^{n} \lambda_j x_j \frac{\partial^2 f}{\partial x_j \partial x_i}$$

y

$$x^I \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{n} \lambda_j x_j \frac{\partial f}{\partial x_j} \right) = x^I \lambda_i \frac{\partial f}{\partial x_i} + x^I \sum_{j=1}^{n} \lambda_j x_j \frac{\partial^2 f}{\partial x_i \partial x_j},$$

en (5.5), obtenemos

$$\left[D_S, x^I \frac{\partial}{\partial x_i} \right](f) = (\langle \lambda, I \rangle - \lambda_i) x^I \frac{\partial f}{\partial x_i}. \tag{5.6}$$

De ello se desprende que los $x^I \frac{\partial}{\partial x_i}$ representan los autovectores de D_S asociados con los autovalores $\langle \lambda, I \rangle - \lambda_i$. Al escribir D_N como una serie formal

$$D_N = \sum_{j=1}^{n} \sum_{|I| \geq 1} a_{I,j} x^I \frac{\partial}{\partial x_j},$$

de la condición $[D_S, D_N] = 0$ y de (5.6) se recupera

$$0 = [D_S, D_N] = \sum_{j=1}^{n} \sum_{|I|\geq 1} a_{I,j} \left[D_S, x^I \frac{\partial}{x_j} \right] = \sum_{j=1}^{n} \sum_{|I|\geq 1} a_{I,j} \Big(\langle \lambda, I \rangle - \lambda_j \Big) x^I \frac{\partial}{\partial x_j}.$$

Así, obtenemos $a_{I,j}\Big(\langle \lambda, I\rangle - \lambda_j\Big) = 0$ para $j = 1, \dots, n$ y $|I| \geq 1$. De este modo la condición $(\langle \lambda, I\rangle - \lambda_j) \neq 0$ implica $a_{I,j} = 0$. En resumen, se tiene

$$D = D_S + D_N = \sum_{j=1}^{n} \lambda_j x_j \frac{\partial}{\partial x_j} + \sum_{j=1}^{n} \sum_{\substack{|I|\geq 1 \\ \langle \lambda, I\rangle = \lambda_j}} a_{I,j} x^I \frac{\partial}{\partial x_j}, \tag{5.7}$$

donde D_N tiene, por la proposición 5.3, parte lineal nilpotente.

Sea $(\mathcal{F}, E) : \omega = \sum_{i\in A} a_i \frac{dx_i}{x_i} + \sum_{i\notin A} a_i dx_i$ una foliación adaptada, con $E : \pi_{i\in A} x_i = 0$. El punto $P = (x_1, \dots, x_n) = (0, \dots, 0) \in \mathrm{Sing}(\mathcal{F})$ es llamado punto singular **presimple** si se cumple una de las siguientes condiciones:

a) $\nu(\mathcal{F}, E, P) = 0$,

b) $\nu(\mathcal{F}, E, P) = \mu(\mathcal{F}, E, P) = 1$ y existe i tal que a_i^1 (la parte lineal de a_i) no depende de las coordenadas x_i para todo $i \in A$.

Consideramos el subespacio

$$\mathcal{D}(\omega)(P) = \Big\{ D(P) : D \in \Theta_{X,P} \quad \text{y} \quad \omega(D) = 0 \Big\}$$

de $T_P X$. La codimensión de $\mathcal{D}(\omega)(P)$ en $T_P X$ es llamada el **tipo dimensional** de $(\mathcal{F}, E)$, y denotada en adelante por $\tau = \tau(\mathcal{F}, P)$.

Lema 5.5. *Supongamos que P es una singularidad presimple para $(\mathcal{F}, E)$ con $\nu_P(\mathcal{F}) = n - 1$. Entonces se tiene $n - 1 \leq e(E, P) \leq n$ y existen coordenadas formales $(x_1, \dots, x_n)$ que cumplen lo siguiente.*

- *Si* $e(E,P) = n-1$, $(\mathcal{F}, E)$ *está generada por* $\omega = dx_1 + \sum_{i=2}^{n} a_i \dfrac{dx_i}{x_i}$.
- *Si* $e(E,P) = n$, $(\mathcal{F}, E)$ *está generada por* $\omega = \dfrac{dx_1}{x_1} + \sum_{i=2}^{n} a_i \dfrac{dx_i}{x_i}$.

Además, existen $n-1$ *campos conmutativos tangentes a* $(\mathcal{F}, E)$*, los cuales son:*

- $a_i \dfrac{\partial}{\partial x_1} - x_i \dfrac{\partial}{\partial x_i}$, $i = 2, \ldots, n$; *cuando* $e(E,P) = n-1$ *y*
- $x_1 a_i \dfrac{\partial}{\partial x_1} - x_i \dfrac{\partial}{\partial x_i}$, $i = 2, \ldots, n,$; *cuando* $e(E,P) = n$.

donde $\nu_P(a_i) \geq 1$*, para todo* i.

Demostración. Supongamos que $e = e(E,P)$ no valga $n-1$ ni n. Entonces, al reordenar las variables si fuese necesario, podemos suponer

$$(\mathcal{F}, E) : \ \omega = a_1 \frac{dx_1}{x_1} + \cdots + a_e \frac{dx_e}{x_e} + a_{e+1} dx_{e+1} + \cdots + a_n dx_n,$$

donde $\mathrm{mcd}(a_1, \ldots, a_n) = 1$. De este modo, se obtiene

$$\begin{aligned} \mathcal{F} : \ \omega =& x_2 \cdots x_e a_1 dx_1 + \cdots + x_1 \cdots x_{e-1} a_e dx_e + \\ &+ x_1 \cdots x_e a_{e+1} dx_{e+1} + \cdots + x_1 \cdots x_e a_n dx_n. \end{aligned}$$

Por hipótesis se tiene $\nu_P(\mathcal{F}) = n-1$, y se cumple

$$\left\{ \begin{array}{rll} e - 1 + \nu_P(a_i) & \geq n-1, & \text{para } i = 1, \ldots, e \\ e + \nu_P(a_i) & \geq n-1, & \text{para } i = e+1, \ldots, n. \end{array} \right.$$

Dado que $e \neq n-1, e \neq n$ y además $e \leq n$ se sigue $e \leq n-2$, luego

$$\left\{ \begin{array}{rll} \nu_P(a_i) & \geq n - e \geq 2, & i = 1, \ldots, e \\ \nu_P(a_i) + 1 & \geq n - e \geq 2, & i = e+1, \ldots, n. \end{array} \right.$$

Esto implica

$$\mu(\mathcal{F}, E, P) = \text{mín} \left\{ \nu_P(a_1), \ldots, \nu_P(a_e), \nu_P(a_{e+1}) + 1, \ldots, \nu_P(a_n) + 1 \right\} \geq 2,$$

lo que contradice el hecho de que P sea una singularidad presimple.

Por otro lado, si $e(E,P)=n-1$, tendremos

$$(\mathcal{F},E):\ \omega = a_1\frac{dx_1}{x_1}+\cdots+a_{n-1}\frac{dx_{n-1}}{x_{n-1}}+a_n dx_n,$$

donde $\mathrm{mcd}(a_1,\ldots,a_n)=1$. En consecuencia, el saturado está generado por la 1-forma

$$\mathcal{F}:\Omega = x_2\cdots x_{n-1}a_1dx_1+\cdots+x_1\cdots x_{n-2}a_{n-1}dx_{n-1}+x_1\cdots x_{n-1}a_n dx_n.$$

De esta expresión y de la hipótesis $\nu_P(\mathcal{F})=n-1$, se sigue $\nu_P(a_i)\geq 1$, para $i=1,\ldots,n-1$, y $\nu_P(a_n)\geq 0$. Como la singularidad P es presimple, tenemos las siguiente opciones.

• Si $\nu_P(\mathcal{F},E,P)=0$, tan solo $\nu_P(a_n)=0$ es posible, es decir, a_n es unidad y podemos considerar $a_n=1$. En consecuencia, los $n-1$ campos tangentes a $(\mathcal{F},E)$ están dados por $x_i\dfrac{\partial}{\partial x_i}-a_i\dfrac{\partial}{\partial x_n}$, con $i=1,\cdots,n-1$.

• Si $\nu(\mathcal{F},E,P)=1=\mu(\mathcal{F},E,P)$, podemos considerar varias opciones. Si $\nu_P(a_n)=0$, resulta que a_n es unidad y el problema se reduce al caso anterior. Si $\nu_P(a_i)=1$, para algún $i=1,\ldots,n-1$, digamos $i=1$, un cambio lineal de coordenadas lleva a $\dfrac{\partial a_1}{\partial x_n}(0)=1$, y así se logra

$$a_1 = x_n+\sum_{|I|\geq 2}a_Ix^I+x_na',$$

con $\nu(a')\geq 1$, donde el segundo término no contiene x_n. Sea $I_0=(i_1^0,\ldots,i_{n-1}^0)=\text{mín}\{I:a_I\neq 0\}$, el mínimo con el orden lexicográfico graduado. Si $I_0=\infty$ se tiene $a_I=0$, para todo I, y en consecuencia x_n divide a a_1. Mientras que de satisfacerse $I_0<\infty$, mediante el cambio de coordenadas $x_n=x_n-a_{I_0}x^{I_0}$, reducimos la expresión a

$$a_1 = x_n-a_{I_0}x^{I_0}+\sum_{I\geq I_0}a_Ix^I+(x_n-a_{I_0}x^{I_0})a'.$$

Nuevamente en a' agrupamos los términos que contienen a x_n y los que no contienen a x_n para expresar $a' = \sum_{I>I_1} a'_I x^I + x_n a''$; obteniéndose

$$a_1 = x_n + \sum_{I>I_0} a_I x^I + x_n a'''.$$

Al continuar con este proceso, podremos suponer que x_n divide a a_1, es decir,

$$a_1 = x_n \widetilde{a}_1, \quad \text{con } \nu(\widetilde{a}_1) = 0.$$

De $\omega \wedge d\omega = 0$ se deriva que x_n divide a a_i para $i < n$, es decir, $a_i = x_n \widetilde{a}_i$. La foliación adaptada $(\mathcal{F}, E)$ está generada por

$$\omega = x_n \widetilde{a}_1 \frac{dx_1}{x_1} + \cdots + x_n \widetilde{a}_{n-1} \frac{dx_{n-1}}{x_{n-1}} + a_n dx_n.$$

Como $\tilde{a}_1$ es una unidad, nuevamente podemos suponer $\tilde{a}_1 = 1$. Luego $(\mathcal{F}, E)$ está igualmente generada por la 1-forma

$$(\mathcal{F}, E): \ \omega = \frac{dx_1}{x_1} + \widetilde{a}_2 \frac{dx_2}{x_2} + \cdots + \widetilde{a}_{n-1} \frac{dx_{n-1}}{x_{n-1}} + a_n \frac{dx_n}{x_n}.$$

Por lo tanto, $(\mathcal{F}, E)$ tiene $n-1$ campos tangentes dados por

$$x_1 a_i \frac{\partial}{\partial x_1} - x_i \frac{\partial}{\partial x_i},$$

con $i = 2, \ldots, n$, donde $a_i = \tilde{a}_j$, $j = 1, \ldots n-1$. Observemos que en este último caso formalmente se tiene $e(E, P) = n$.

Finalmente, si $e(E, P) = n$, la foliación adaptada $(\mathcal{F}, E)$ está generada por la 1-forma

$$(\mathcal{F}, E): \ \omega = a_1 \frac{dx_1}{x_1} + \cdots + a_n \frac{dx_n}{x_n},$$

donde $\mathrm{mcd}(a_1, \ldots, a_n) = 1$. Luego, el saturado $\mathcal{F}$ está generado por la 1-forma

$$\mathcal{F}: \Omega = x_2 \cdots x_n a_1 dx_1 + \cdots + x_1 \cdots x_{n-1} a_n dx_n.$$

Como se tiene $\nu_P(\mathcal{F}) = n-1$ para algún i, la serie a_i es una unidad. Si suponemos que esta es a_1, pondremos, por supuesto, directamente $a_1 = 1$. Por lo tanto, podemos expresar

$$(\mathcal{F}, E) :\ \omega = \frac{dx_1}{x_1} + a_2 \frac{dx_2}{x_2} + \cdots + a_n \frac{dx_n}{x_n}.$$

En conclusión los campos tangentes a $(\mathcal{F}, E)$ resultan ser

$$x_1 a_i \frac{\partial}{\partial x_1} - x_i \frac{\partial}{\partial x_i},\ \ i = 2, \ldots, n.$$

□

Teorema 5.6 (**Cano-Cerveau**). *Si P es una singularidad presimple para la foliación adaptada $(\mathcal{F}, E)$ en dimensión dos con $\nu_P(\mathcal{F}) = 1$, entonces existen coordenadas formales (x_1, x_2) en $\widehat{\mathcal{O}}_{X,P}$ tales que $(\mathcal{F}, E)$ está generada por alguna de las siguientes $1-$formas:*

A. $\omega = \lambda_1 \dfrac{dx_1}{x_1} + \lambda_2 \dfrac{dx_2}{x_2}$, *donde* $\lambda_1, \lambda_2 \in \mathbb{C}^*$;

B. $\omega = p_1 \dfrac{dx_1}{x_1} + p_2 \dfrac{dx_2}{x_2} + \varphi(x_1^{p_1} x_2^{p_2}) \dfrac{dx_2}{x_2}$, *donde* $p_1, p_2 \in \mathbb{Z}_{>0}$;

C. $\omega = dx_1 + (-p_1 x_1 + x_1^{p_1}) \dfrac{dx_2}{x_2}$, *donde* $p_1 \in \mathbb{Z}_{>0}$.

Demostración. Por el lema 5.5, el valor $e = e(\mathcal{F}, E)$ es 1 ó 2 y la foliación adaptada $(\mathcal{F}, E)$ está generada, dependiendo de e, por la 1-forma

$$\omega = dx_1 + a_2 \frac{dx_2}{x_2} \qquad \text{ó} \qquad \omega = \frac{dx_1}{x_1} + a_2 \frac{dx_2}{x_2}.$$

De este modo, existe un campo tangente a $(\mathcal{F}, E)$ dado

$$\begin{aligned} D_2 &= a_2 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2}, \quad \text{en el primer caso y por} \\ D_2 &= x_1 a_2 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2}, \quad \text{en el segundo.} \end{aligned}$$

Evidentemente, del campo D_2 se puede recuperar la foliación adaptada. Vía el teorema 5.4 existen coordenadas, que seguimos denotando por (x_1, x_2), en las que D_2 se descompone como suma de su parte semisimple y nilpotente, es decir,

$$D_2 = \lambda_2 x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} + \left(\sum_{\substack{m_1+m_2 \geq 1 \\ \lambda_2 m_1 - m_2 = \lambda_2}} a_{m_1 m_2} x_1^{m_1} x_2^{m_2} \right) \frac{\partial}{\partial x_1}.$$

Para determinar D_2, debemos hallar las soluciones del sistema

$$\lambda_2(m_1 - 1) = m_2, \quad \text{donde } m_1 + m_2 \geq 1. \tag{5.8}$$

Consideramos los siguientes casos.

Si $m_1 = 0$, se tiene $\lambda_2 = -m_2 \in \mathbb{Z}_{\leq 0}$. Con ello el sistema (5.8) arroja como única solución el par $(m_1, m_2) = (0, -\lambda_2)$. En consecuencia el campo vectorial resulta ser

$$D_2 = \lambda_2 x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} + a_{(0,-\lambda_2)} x_2^{-\lambda_2} \frac{\partial}{\partial x_1};$$

así $(\mathcal{F}, E)$ es de tipo **C** generada por

$$dx_1 - (\lambda_2 x_1 + a_{(0,-\lambda_2)} x_2^{-\lambda_2}) \frac{dx_2}{x_2}.$$

Si $m_1 = 1$, se tiene $m_2 = 0$, con λ_2 es arbitrario. Por la proposición 5.3, el par $(1, 0)$ no es solución del sistema (5.8). Luego, el campo es lineal, es decir, se tiene

$$D_2 = \lambda_2 x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2},$$

y la 1-forma que genera la foliación adaptada es de tipo **A** .

Si $m_1 \geq 2$, el número λ_2 ha de ser racional positivo:

$$\lambda_2 = \frac{p_2}{q} \in \mathbb{Q}_{\geq 0}.$$

Además, el campo se podrá escribir

$$D_2 = \lambda_2 x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} + x_1 \varphi(x_1^q x_2^{p_2}) \frac{\partial}{\partial x_2},$$

con $\nu_0(\varphi) \geq 1$. Luego, la forma que genera la foliación adaptada es de tipo **B**. $\square$

Teorema 5.7. *Supongamos que $\mathcal{G} \subset \widehat{\mathfrak{m}}\widehat{\Theta}_{X,P}$ sea una álgebra de Lie abeliana de dimensión finita. Entonces existen dos álgebras $\mathcal{G}_1$ y $\mathcal{G}_2$ de dimensión finita en $\widehat{\mathfrak{m}}\widehat{\Theta}_{X,P}$ tales que se satisface*
- *$\mathcal{G} \subset \mathcal{G}_1 \oplus \mathcal{G}_2$,*
- *si $D \in \mathcal{G}_1$ (respectivamente $D \in \mathcal{G}_2$), entonces D es semisimple (respectivamente nilpotente),*
- *$[\mathcal{G}_1, \mathcal{G}_2] = 0$.*

Demostración. Tomemos la parte semisimple D_S y nilpontente D_N de cada elemento $D \in \mathcal{G}$ para obtener las algebras

$$\mathcal{G}_1 = \{D_S : D \in \mathcal{G}\} \qquad \text{y} \qquad \mathcal{G}_2 = \{D_N : D \in \mathcal{G}\};$$

con esto quedan constatadas las dos primeras propiedades.

Para demostrar el tercer item, consideremos la adjunta del operador $D \in \widehat{\mathfrak{m}}\widehat{\Theta}_{X,P}$ dado por

$$\begin{array}{rccc} \mathrm{ad}_D : & \widehat{\mathfrak{m}}\widehat{\Theta}_{X,P} & \longrightarrow & \widehat{\mathfrak{m}}\widehat{\Theta}_{X,P} \\ & F & \longmapsto & [D, F]. \end{array}$$

Este operador lineal se descompone en su parte semisimple y nilpotente de manera única:

$$ad_D = \mathrm{ad}_D^S + \mathrm{ad}_D^N,$$

donde $[\mathrm{ad}_D^S, \mathrm{ad}_D^N] = [\mathrm{ad}_D^N, \mathrm{ad}_D^S]$. Más aún, tenemos que se cumple

$$\mathrm{ad}_D^S = \mathrm{ad}_{D_S} \qquad \text{y} \qquad \mathrm{ad}_D^N = \mathrm{ad}_{D_N}.$$

Como $\mathcal{G}$ es una álgebra conmutativa, para $D, D' \in \mathcal{G}$ tenemos

$$\mathrm{ad}_D(D') = [D, D'] = 0,$$

lo cual significa que D' es un autovector de ad_D con autovalor nulo. Por lo expuesto, tenemos la misma situación para $\mathrm{ad}_D^S = \mathrm{ad}_{D_S}$. Así tendremos

$$\mathrm{ad}_{D'}(D_S) = [D', D_S] = -[D_S, D'] = -\mathrm{ad}_{D_S}(D') = 0,$$

es decir, D_S será un autovector de $\mathrm{ad}_{D'}$. Luego, también es un autovector para $\mathrm{ad}_{D'}^S = \mathrm{ad}_{D'_S}$ y cumplirá

$$[D'_S, D_S] = \mathrm{ad}_{D'_S}(D_S) = 0.$$

Esto prueba que $\mathcal{G}_1$ es una álgebra de Lie abeliana.

Por último, de las tres últimas identidades se tiene directamente

$$[D_S, D'_N] = 0 \qquad \text{y} \qquad [D_N, D'_N] = 0,$$

y $\mathcal{G}_2$ resulta una algebra de Lie abeliana. $\square$

El siguiente resultado es consecuencia de la proposición 5.1.

Proposición 5.8. *Sea $\mathcal{G} \subset \widehat{\mathfrak{m}}\widehat{\Theta}_{X,P}$ una álgebra de Lie abeliana de dimensión finita y $(x'_1, \ldots x'_s)$, con $1 \leq s \leq n$, coordenadas formales tales que $D(x'_i) = \lambda'_i(D)x'_i$, con $\lambda'_i(D) \in \mathbb{C}$, $i = 1, \ldots, s$, para todo $D \in \mathcal{G}$. Entonces existe un sistema de coordenadas formales $(x_1, \ldots, x_n)$ de $\widehat{\mathcal{O}}_{X,P}$ tal que*

$$D = \sum_{i=1}^{n} \lambda_i(D) x_i \frac{\partial}{\partial x_i}, \qquad \text{para todo } D \in \mathcal{G},$$

donde $x_i = x'_i$ y $\lambda_i(D) = \lambda'_i(D)$ para $i = 1, \ldots, s$ y $D \in \mathcal{G}$.

Demostración. En cada paso de la recurrencia definida en la proposición 5.1 hay que aplicar un hecho conocido: una familia de operadores que conmutan entre sí se puede diagonalizar simultáneamente en una base donde cada una de ellas es diagonalizada. A partir de allí, la prueba sigue el rumbo de la proposición 5.1. $\square$

Sea $\mathcal{G} \subset \widehat{\mathcal{M}}\widehat{\Theta}_{X,P}$ un álgebra de Lie abeliana de dimensión finita. Por la proposición anterior, existen coordenadas formales $(x_1, \ldots, x_n)$ que linealizan simultaneamente los D_S para todo $D \in \mathcal{G}$.

Dada $D \in \mathcal{G}$ una derivación, denotemos por

$$\lambda(D) = (\lambda_1(D), \dots, \lambda_n(D))$$

los correspondientes autovalores. Por el teorema 5.7 tenemos

$$[D_S, D'_N] = 0, \text{ para todo } D, D' \in \mathcal{G}.$$

Esto significa que $D = D_S + D_N$ puede ser reescrito como

$$D = \sum_{j=1}^{n} \lambda_j(D) x_j \frac{\partial}{x_j} + \sum_{j=1}^{n} \sum_{\substack{|I| \geq 1 \\ \langle \lambda(D), I \rangle = \lambda_j(D),\, \forall\, D \in \mathcal{G}}} a_{I,j}(D) x^I \frac{\partial}{x_j}. \tag{5.9}$$

Teorema 5.9 (Cano). *Sea P una singularidad presimple para la foliación adaptada $(\mathcal{F}, E)$ y tal que $\nu_P(\mathcal{F}) = n - 1$, con tipo dimensional $\tau = \tau_P(\mathcal{F}, E)$. Entonces existen coordenadas formales $(x_1, \dots, x_n)$ tales que las componentes no dicríticas E_{nd} de E están contenidas en $\cup_{i=1}^{\tau}(x_i = 0)$ y $(\mathcal{F}, E)$ es generado por $\omega = 0$; acá ω resulta de uno de los siguientes tres tipos.*

A. *Existen $\lambda_i \in \mathbb{C}^*$, $i = 1, \dots, \tau$, tales que*

$$\omega = \sum_{i=1}^{\tau} \lambda_i \frac{dx_i}{x_i}.$$

B. *Existen $1 \leq k \leq \tau$, $p_1, \dots, p_k \in \mathbb{Z}_{>0}$, $(\alpha_2, \dots, \alpha_\tau) \in \mathbb{C}^{\tau-1} - \{0\}$, $\alpha_{k+1}, \dots, \alpha_\tau \in \mathbb{C}^*$, $\varphi \in \mathbb{C}[[t]]$, $\varphi(0) = 0$ tales que*

$$\omega = p_1 \frac{dx_1}{x_1} + \dots + p_k \frac{dx_k}{x_k} + \varphi(x_1^{p_1} \cdots x_k^{p_k}) \left(\sum_{i=2}^{\tau} \alpha_i \frac{dx_i}{x_i} \right).$$

C. *Existen $2 \leq k \leq \tau$, $p_2, \dots, p_k \in \mathbb{Z}_{>0}$, $(\alpha_2, \dots, \alpha_\tau) \in \mathbb{C}^{\tau-1} - \{0\}$, $\alpha_{k+1}, \dots, \alpha_\tau \in \mathbb{C}^*$ tales que*

$$\omega = dx_1 - x_1 \left(p_2 \frac{dx_2}{x_2} + \dots + p_k \frac{dx_k}{x_k} \right) + (x_2^{p_2} \cdots x_k^{p_k}) \left(\sum_{i=2}^{\tau} \alpha_i \frac{dx_i}{x_i} \right).$$

Demostración. Trabajamos por inducción sobre n.

Para $n = 2$, se tiene necesariamente $\tau = 2$ y el resultado es el teorema 5.6.

Supongamos entonces que el resultado se cumple para $n-1$. Si la dimensión es n, podemos suponer $\tau = n$, caso contrario tenemos un campo regular D tangente a $\mathcal{F}$, y al rectificar D se tendrá $\tau \leq n-1$ con lo que por la hipótesis inductiva obtenemos el resultado deseado. Por otro lado, la multiplicidad de $\mathcal{F}$ en el punto P es $\nu_P(\mathcal{F}) = n-1$. Del lema 5.5 tenemos $n-1 \leq e(E,P) \leq n$. Dependiente del valor de $e(E,P)$ se tienen las 1-formas

I. $\omega = dx_1 + \displaystyle\sum_{i=2}^{n} a_i \frac{dx_i}{x_i}$,

II. $\omega = \dfrac{dx_1}{x_1} + \displaystyle\sum_{i=2}^{n} a_i \frac{dx_i}{x_i}$

que generan la foliación adaptada $(\mathcal{F}, E)$. Luego, existen $n-1$ campos tangentes a $(\mathcal{F}, E)$ dados por:

$$\begin{aligned} D_j = &\quad a_j \frac{\partial}{\partial x_1} - x_j \frac{\partial}{\partial x_j}, \qquad j = 2, \ldots, n \quad \text{en el caso I} y \\ D_j = &\quad x_1 a_i \frac{\partial}{\partial x_1} - x_j \frac{\partial}{\partial x_j}, \quad j = 2, \ldots, n \quad \text{en el caso II.} \end{aligned}$$

Evidentemente, estos campos permiten recuperar la foliación adaptada. Al aplicar el teorema 5.7 al ágebra conmutativa de dimensión finita definida por

$$\mathcal{G} = \mathbb{C}D_2 + \cdots + \mathbb{C}D_n,$$

aparecen coordenadas formales donde cada D_i se expresa como la suma de su parte semisimple y nilpotente, con la parte semisimple lineal en estas coordenadas. Así, en estas nuevas coordenadas (que seguimos denotando por $(x_1, \ldots, x_n)$), se tendrá

$$D_j = \lambda_j x_1 \frac{\partial}{\partial x_1} - x_j \frac{\partial}{\partial x_j} + \left(\sum_{\substack{m_1 + \cdots + m_n \geq 1 \\ \lambda_i m_1 - m_i = \lambda_i \text{ para } i = 2, \ldots, n}} a^j_{m_1 \cdots m_n} x_1^{m_1} \cdots x_n^{m_n} \right) \frac{\partial}{\partial x_1},$$

para $j = 2, \dots, n$; obsérvese que el tercer término es nilpotente por la proposición 5.3. Para determinar estos campos debemos hallar soluciones del sistema

$$\lambda_i(m_1 - 1) = m_i, \;\; i = 2, \dots, n; \text{ sujeto a } m_1 + \cdots + m_n \geq 1. \tag{5.10}$$

Con m_1 en mente tenemos tres situaciones a considerar.

Si $m_1 = 1$, entonces el sistema (5.10) no tiene solución; es decir, los campos

$$D_j = \lambda_j x_1 \frac{\partial}{\partial x_1} - x_j \frac{\partial}{\partial x_j}, \;\; j = 2, \dots, n,$$

son lineales. Por lo tanto, la 1-forma que genera la foliación adaptada $(\mathcal{F}, E)$ es de tipo **A**.

Si $m_1 \geq 2$, los $\lambda_i = \dfrac{p_i}{q} \in \mathbb{Q}_{\geq 0}$, para $i = 2, \dots, n$, son racionales positivos. En este caso, los campos se escriben

$$D_j = \lambda_j x_1 \frac{\partial}{\partial x_1} - x_j \frac{\partial}{\partial x_j} + x_1 \varphi_i(x_1^q x_2^{p_2} \cdots x_n^{p_n}) \frac{\partial}{\partial x_1},$$

con $\nu_0(\varphi) \geq 1$. Por la condición de integrabilidad $\varphi_i = \alpha_i \varphi$, resulta que $(\mathcal{F}, E)$ es de tipo **B**.

Si $m_1 = 0$, se tendrá $\lambda_i = -m_i \in \mathbb{Z}_{\leq 0}, i = 2, \dots, n$. El sistema (5.10) tendrá así una única solución $(m_1, m_2, \dots, m_n) = (0, -\lambda_2, \dots, -\lambda_n)$. Los campos vectoriales quedan presentados cual

$$D_j = -m_i x_1 \frac{\partial}{\partial x_1} - x_j \frac{\partial}{\partial x_j} + \left(a^i_{(0, -\lambda_2, \cdots, -\lambda_n)} x_2^{-\lambda_2} \cdots x_n^{-\lambda_n} \right) \frac{\partial}{\partial x_1}.$$

En consecuencia $(\mathcal{F}, E)$ es de tipo **C** y

$$dx_1 + x_1 \left(p_2 \frac{dx_2}{x_2} + \cdots + p_k \frac{dx_k}{x_k} \right) + x_2^{p_2} \cdots x_k^{p_k} \left(\alpha_2 \frac{dx_2}{x_2} + \cdots + \alpha_n \frac{dx_n}{x_n} \right)$$

es la 1-forma que genera la foliación adaptada $(\mathcal{F}, E)$. $\square$

Sean las foliaciones generadas por las formas ω de tipo **A** y **B** como en el teorema 5.9. Un vector v definido por

$$v = \begin{cases} (\lambda_1, \dots, \lambda_\tau); & \text{en el caso } \mathbf{A}. \\ (\alpha_{k+1}, \dots, \alpha_\tau); & \text{en el caso } \mathbf{B}, \end{cases}$$

es llamado **vector residual** de la forma normal ω.

Sea $v = (v_1, \ldots, v_s)$ un vector de $\mathbb{C}^s$. Una s-upla $n = (n_1, \ldots, n_s) \in \mathbb{Z}^s_{\geq 0}$ es una **resonancia** de v si $\langle n, v\rangle = \sum_{i=1}^{s} n_i v_i = 0$. El vector v es **no resonante** si la única resonancia es $n = 0$.

Una singularidad presimple P de $(\mathcal{F}, E)$ es llamada **simple** cuando $(\mathcal{F}, E)$ es generada por una 1-forma normal de tipo **A** o **B** y el vector residual es no resonante.

6. Singularidades presimples con hojas cerradas

En esta última sección estudiaremos las foliaciones definidas en un entorno de una singularidad presimple donde todas las hojas son cerradas. Un tipo especial de estas foliaciones son las foliaciones con integral primera meromorfa alrededor de una singularidad presimple. Para este estudio revisaremos el paso de las singularidades presimples a las simples estudiadas por F. Cano [3] y por M. Fernández [5].

6.1. Singularidades de tipo C

En este apartado estudiaremos las singularidades presimples de tipo **C** de tipo dimensional τ, con $k \in \mathbb{N}$ tal que $2 \leq k \leq \tau$, como en el teorema 5.9 . Por simplicidad, solo trataremos el caso de tipo dimensional $\tau = 3$, lo que implica analizar los casos $k = 2$ y $k = 3$. Cuando la dimensión es mayor el estudio es análogo (ver [5]).

Si $k = 2$, la foliación adaptada $(\mathcal{F}, E)$ está definida por

$$\omega = dx_1 - p_2 x_1 \frac{dx_2}{x_2} + x_2^{p_2}\left(\alpha_2 \frac{dx_2}{x_2} + \alpha_3 \frac{dx_3}{x_3}\right). \tag{6.1}$$

Las singularidades y las separatrices de la foliación $\mathcal{F}$ son

$$\mathrm{Sing}(\mathcal{F}) = (x_1 = x_2 = 0) \cup (x_2 = x_3 = 0) \ \text{ y}$$

$$\mathrm{Sep}(\mathcal{F}) = (x_2 = 0) \cup (x_3 = 0).$$

Realicemos una explosión con centro en la recta $Y_1 : x_1 = x_2 = 0$. La variedad $(X^{(1)}, E^{(1)}, \pi^{(1)})$, donde $E^{(1)} = (\pi^{(1)})^{-1}(Y_1)$, así obtenida es cubierta por dos sistemas coordenadas, ver figura 1. Analicemos la transformada de la foliación en cada una de ellas

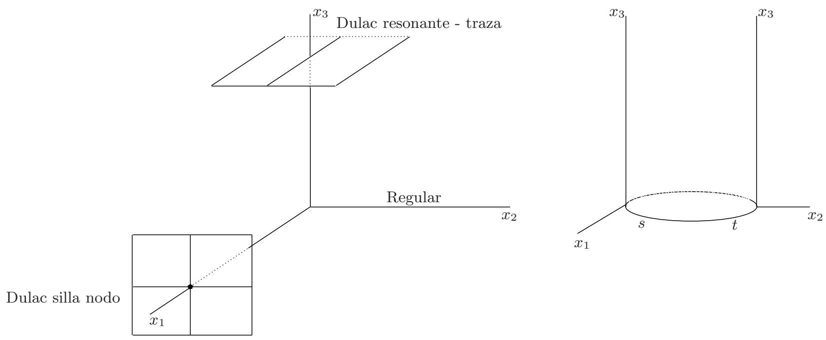

Figura 1: *Cortes transversales a los ejes coordenados en torno de una singularidad de tipo* **C** *y explosión con centro* Y_1.

En las coordenadas (t, x_2, x_3) de $X^{(1)}$, donde la explosión $\pi^{(1)}$ es determinada por la relación $x_1 = tx_2$, la transformada de la foliación $(\mathcal{F}, E)$ queda implícita por la $1-$forma

$$\frac{(\pi^{(1)})^*\omega}{x_2} = dt - (p_2 - 1)t\frac{dx_2}{x_2} + x_2^{p_2-1}\left(\alpha_2\frac{dx_2}{x_2} + \alpha_3\frac{dx_3}{x_3}\right),$$

y su tipo de singularidad coincide con en de la foliación (6.1), al cambiar $p_2 - 1$ por p_2.

En las coordenadas (x_1, s, x_3) de $X^{(1)}$, donde $\pi^{(1)}$ es determinada por la relación $x_2 = sx_1$, la transformada de la foliación adaptada queda definida por la 1-forma

$$-\frac{(\pi^{(1)})^*\omega}{x_1} = (p_2-1)\frac{dx_1}{x_1} + p_2\frac{ds}{s} + s^{p_2}x_1^{p_2-1}\left(-\alpha_2\frac{dx_1}{x_1} - \alpha_2\frac{ds}{s} - \alpha_3\frac{dx_3}{x_3}\right).$$

Si suponemos $p_2 > 1$ (el caso $p_2 = 1$ será tratado a continuación), entonces esta singularidad es presimple de tipo **B** con vector de resonancia $-\alpha_3 \neq 0$. Por definición, la singularidad es no resonante y, por consiguiente, es simple.

Supongamos que hemos realizado $p-1$ explosiones a partir de (6.1) con centro los ejes coordenados que aún no son simples. La transformada de la foliación está descrita por la $1-$forma

$$\omega = dx_1 - x_1\frac{dx_2}{x_2} + x_2\left(\alpha_2\frac{dx_2}{x_2} + \alpha_3\frac{dx_3}{x_3}\right).$$

Realicemos una explosión adicional con centro la recta $Y_2 : x_1 = x_2 = 0$ y determinemos cómo es la transformada de la foliación sobre la variedad $(X^{(2)}, E^{(2)}, \pi^{(2)})$, donde $E^{(2)} = (\pi^2)^{-1}(Y_2)$. En las coordenadas (t, x_2, x_3), donde $\pi^{(2)}$ localmente está dada por la relación $x_1 = tx_2$, se tiene

$$\frac{(\pi^{(2)})^*\omega}{x_2} = dt + \alpha_2\frac{dx_2}{x_2} + \alpha_3\frac{dx_3}{x_3},$$

de tipo dimensional 2; esto equivale a que $\alpha_2\dfrac{dx_2}{x_2} + \alpha_3\dfrac{dx_3}{x_3}$ sea simple en $(\mathbb{C}^2, 0)$.

En las coordenadas (x_1, s, x_3), donde $\pi^{(2)}$ localmente está dada por la relación $x_2 = sx_1$, tenemos

$$\begin{aligned}
\frac{(\pi^{(2)})^*\omega}{x_1} &= \alpha_2 s\frac{dx_1}{x_1} + (-1+\alpha_2 s)\frac{ds}{s} + \alpha_3 s\frac{dx_3}{x_3}\\
&= (-1+\alpha_2 s)\frac{ds}{s} + s\left(\alpha_2\frac{dx_1}{x_1} + \alpha_3\frac{dx_3}{x_3}\right)\\
&= \frac{ds}{s} + s(-1+\alpha_2 s)^{-1}\left(\alpha_2\frac{dx_1}{x_1} + \alpha_3\frac{dx_3}{x_3}\right)\\
&= \frac{ds}{s} + \varphi(s)\left(\alpha_2\frac{dx_1}{x_1} + \alpha_3\frac{dx_3}{x_3}\right), \quad \text{con} \quad \varphi(0) = 0.
\end{aligned}$$

Ahora debemos eliminar la resonancia $\dfrac{-n}{m} = \dfrac{\alpha_3}{\alpha_2} \in \mathbb{Q}_-$, con n, m enteros positivos. Para ello se toma $\alpha_3 = -n$ y $\alpha_2 = m$ con $m > n$. De todo esto logramos

$$\omega_1 = \frac{ds}{s} + \varphi(s)\left(m\frac{dx_1}{x_1} - n\frac{dx_3}{x_3}\right). \tag{6.2}$$

En adelante realizaremos explosiones con centro en líneas proyectivas (curvas compactas). Comenzamos con la línea proyectiva Y_3, dada localmente por las ecuaciones $x_1 = x_3 = 0$. La variedad $(X^{(3)}, E^{(3)}, \pi^{(3)})$, donde $E^3 = (\pi^{(3)})^{-1}(Y_3)$, obtenida por esta explosión es cubierta nuevamente por dos sistemas de coordenadas, ver figura 2. En las coordenadas (x_1, x_2, u), donde $\pi^{(3)}$ está dada por la relación $x_3 = ux_1$, se tiene

$$(\pi^{(3)})^*\omega_1 = \frac{ds}{s} + \varphi(s)\left((m-n)\frac{dx_1}{x_1} - n\frac{du}{u}\right),$$

la cual resulta resonante. Y en las coordenadas (v, x_2, x_3), donde $\pi^{(3)}$ está dada por la relación $x_1 = vx_3$, se tiene

$$(\pi^{(3)})^*\omega_1 = \frac{ds}{s} + \varphi(s)\left((m-n)\frac{dx_3}{x_3} + m\frac{dv}{v}\right),$$

la cual resulta no resonante.

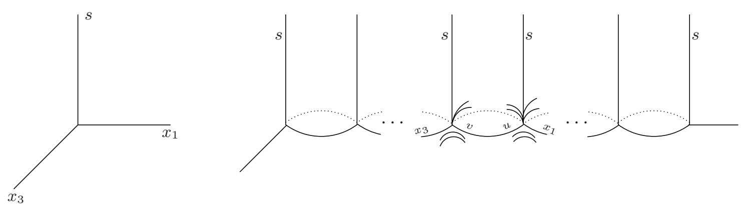

Figura 2: *Proceso de reducción de singularidades de (6.2).*

Continuando con este proceso, determinado por el algoritmo de la división de m y n, la foliación dada por (6.2) es transformada en la foliación definida por la $1-$forma

$$\omega_2 = \frac{ds}{s} + \varphi(s)\left(\frac{dx_1}{x_1} - \frac{dx_3}{x_3}\right). \tag{6.3}$$

Realicemos una explosión más con centro en la línea proyectiva Y_4, que en estas coordenadas está dada, localmente, por las ecuaciones $x_1 = x_3 = 0$, como en los casos anteriores. La variedad obtenida $(X^{(4)}, E^{(4)}, \pi^{(4)})$ necesita dos sistemas de coordenadas para ser descrita.

En las coordenadas (x_1, s, u), en donde $\pi^{(4)}$ es dada por la relación $x_3 = ux_1$, se tiene

$$(\pi^{(4)})^* \omega_2 = \frac{ds}{s} - \varphi(s) \frac{du}{u}.$$

Esta foliación es dicrítica de tipo dimensional 2 con una silla nodo sobre el divisor de ecuación $x_1 = 0$.

En las coordenadas (v, s, x_3), donde $\pi^{(4)}$ es definida por la relación $x_1 = vx_3$, se tiene

$$(\pi^{(4)})^* \omega_2 = \frac{ds}{s} + \varphi(s) \frac{dv}{v}.$$

Nuevamente, la foliación es dicrítica de tipo dimensional 2, que es una silla nodo sobre el divisor de ecuación $x_3 = 0$.

En resumen, el tipo de singularidad (6.1) no tiene todas sus hojas cerradas puesto que tiene una silla nodo (ver [1]).

Si $k = 3$, la foliación $(\mathcal{F}, E)$ está definida por

$$\omega = dx_1 - x_1 \left(p_2 \frac{dx_2}{x_2} + p_3 \frac{dx_3}{x_3} \right) + x_2^{p_2} x_3^{p_3} \sum_{i=2}^{3} \alpha_i \frac{dx_i}{x_i}. \tag{6.4}$$

La singularidad de $\mathcal{F}$ está formada por tres rectas

$$\operatorname{Sing}(\mathcal{F}) = (x_2 = x_3 = 0) \cup (x_1 = x_3 = 0) \cup (x_1 = x_2 = 0).$$

Las rectas $(x_1 = x_3 = 0)$ y $(x_1 = x_2 = 0)$ son singularidades de tipo traza (ver [4]), donde se tiene

$$\operatorname{Sep}(\mathcal{F}) = (x_2 = 0) \cup (x_3 = 0).$$

Realizamos una explosión con centro en $Y_1 : x_1 = x_2 = 0$ y analizamos la transformada de la foliación sobre la variedad inducida por esta explosión $(X^{(1)}, E^{(1)}, \pi^{(1)})$, donde $E^{(1)} = (\pi^{(1)})^{-1}(Y_1)$. Esta variedad queda descrita por dos sistemas de coordenadas. En las coordenadas (x_3, x_2, t), donde $\pi^{(1)}$ es determinada por la relación $x_1 = tx_2$, se tiene

$$\frac{(\pi^{(1)})^*\omega}{x_2} = dt - t\left((p_2-1)\frac{dx_2}{x_2} + p_3\frac{dx_3}{x_3}\right) + x_2^{p_2-1}x_3^{p_3}\left(\alpha_2\frac{dx_2}{x_2} + \alpha_3\frac{dx_3}{x_3}\right),$$

foliación que resulta del mismo tipo que la dada por (6.4) al cambiar p_2 por $p_2 - 1$. En las coordenadas (s, x_1, x_3), donde $\pi^{(1)}$ es determinada por la relación $x_2 = sx_1$, se obtiene la $1-$forma

$$-\frac{(\pi^{(1)})^*\omega}{x_1} = (p_2-1)\frac{dx_1}{x_1} - p_2\frac{dx_2}{x_2} + p_3\frac{dx_3}{x_3} + \\ + s^{p_2}x_1^{p_2-1}x_3^{p_3}\left(-\alpha_2\frac{ds}{s} - \alpha_2\frac{dx_1}{x_1} - \alpha_3\frac{dx_3}{x_3}\right),$$

la cual define una foliación con una singularidad presimple de tipo **B**. Si $p_2 > 1$ no hay vector residual y por lo tanto la singularidad será simple. Si $p_2 = 1$, entonces el vector residual es $-\alpha_2 \neq 0$ y al no tener resonancia, la singularidad es también simple.

Ahora realizamos una explosión con centro $Y_2 : \ x_1 = x_3 = 0$ (ver figura 3) y analizamos la transformada de la foliación sobre la variedad $(X^{(2)}, E^{(2)}, \pi^{(2)})$ inducida por esta explosión. En las coordenadas (s, x_2, x_3), donde $\pi^{(2)}$ es determinada por $x_1 = sx_3$, de (6.4) se obtiene

$$\frac{(\pi^{(2)})^*\omega}{x_3} = ds - s\left((p_3-1)\frac{dx_3}{x_3} + p_2\frac{x_2}{x_2}\right) + x_2^{p_2}x_3^{p_3-1}\left(\alpha_2\frac{dx_2}{x_2} + \alpha_3\frac{dx_3}{x_3}\right),$$

foliación en la que se ha cambiado p_3 por $p_3 - 1$. En las coordenadas (s, x_2, x_3), donde $\pi^{(2)}$ está definida por la relación $x_3 = tx_1$, se tiene gracias a (6.4) la siguiente expresión

$$-\frac{(\pi^{(2)})^*\omega}{x_1} = (p_3-1)\frac{dx_1}{x_1} - p_2\frac{dx_2}{x_2} + p_3\frac{dt}{t} + \\ + x_1^{p_3-1}x_2^{p_3}t^{p_3}\left(-\alpha_3\frac{dx_1}{x_1} - \alpha_2\frac{dx_2}{x_2} - \alpha_3\frac{dt}{t}\right).$$

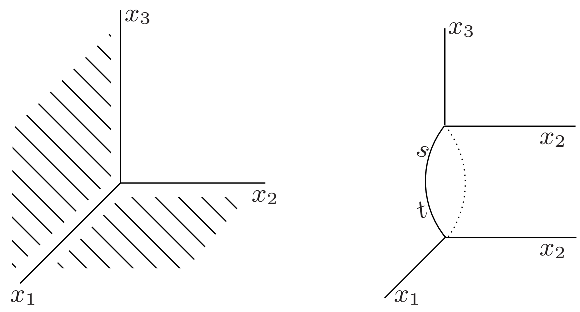

Figura 3: *Explosión con centro* Y_2.

Esta singularidad es de tipo **B**. Si $p_1 > 1$ no hay vector residual y por consiguiente es una singularidad simple. Cuando $p_1 = 1$, el vector residual es $-\alpha_2 \neq 0$, no resonante, y en consecuencia también simple.

Supongamos que hemos realizado $p_3 - 1$ explosiones con centro el eje x_3 y $p_2 - 1$ explosiones con centro el eje x_2 a la foliación inducida por la $1-$forma (6.4). La foliación transformada por estas explosiones queda definida por la $1-$forma

$$\omega = dx_1 - x_1 \left(\frac{dx_2}{x_2} + \frac{dx_3}{x_3} \right) + x_2 x_3 \left(\alpha_2 \frac{dx_2}{x_2} + \alpha_3 \frac{dx_3}{x_3} \right).$$

Si realizamos una explosión adicional con centro el eje x_3, esta explosión genera una variedad $(X^{(3)}, E^{(3)}, \pi^{(3)})$, donde $X^{(3)}$ es cubierta por dos sistemas coordenadas. Analicemos la foliación transformada en cada una de ellas.

En las coordenadas (t, x_2, x_3), donde $\pi^{(3)}$ está dada por la relación $x_1 = t x_2$, obtenemos

$$\omega_1 = dt - t \frac{dx_3}{x_3} + x_3 \left(\alpha_2 \frac{dx_2}{x_2} + \alpha_3 \frac{dx_3}{x_3} \right).$$

Observe que ω_1 es equivalente a (6.1). Luego se sigue como en el caso

$k = 2$. Es decir, se realiza una explosión adicional con centro el eje x_3 y a continuación se elimina la resonancia que aparece en una componente dicrítica de la foliación inducida.

En las coordenadas (x_1, s, x_3), donde $\pi^{(3)}$ está definida por la relación $x_2 = sx_1$, se tiene

$$\begin{aligned}
(\pi^{(3)})^*\omega &= dx_1 - x_1\left(\frac{dx_1}{x_1}+\frac{ds}{s}+\frac{dx_3}{x_3}\right)+\\
&+ \quad x_1sx_3\left(\alpha_2\left(\frac{dx_1}{x_1}+\frac{ds}{s}\right)+\alpha_3\frac{dx_3}{x_3}\right)\\
&= -x_1\left(\frac{ds}{s}+\frac{dx_3}{x_3}\right)+x_1sx_3\alpha_2\left(\frac{dx_1}{x_1}+\frac{ds}{s}\right)\\
&= \frac{ds}{s}+\frac{dx_3}{x_3}+\alpha_2sx_3\left(\frac{dx_1}{x_1}+\frac{ds}{s}\right).
\end{aligned}$$

Así, el vector residual es 1 y es, por lo tanto no resonante. Por consiguiente la singularidad es simple. Así, la foliación definida por (6.4) no puede tener todas sus hojas cerradas dado que admite una silla nodo.

Concluimos del análisis que las singularidades de tipo **C** no tienen todas sus hojas cerradas (ver [1]).

6.2. Singularidades de tipo B

En este apartado analizaremos las singularidades de tipo **B** que tienen todas sus hojas cerradas. En este caso, la foliación está definida por la $1-$forma

$$\omega = \sum_{i=1}^{k} p_i\frac{dx_i}{x_i} + \psi\left(x_1^{p_1}\cdots x_k^{p_k}\right)\left(\sum_{j=2}^{\tau}\alpha_i\frac{dx_i}{x_i}\right),$$

donde $p_1, \dots p_k \in \mathbb{Z}_{>0}$, $(\alpha_2, \dots, \alpha_\tau) \in \mathbb{C}^{\tau-1} - \{0\}$, $\alpha_{k+1}, \dots, \alpha_\tau \in \mathbb{C}^*$, $\varphi \in \mathbb{C}[[t]]$, $\varphi(0) = 0$. Estas foliaciones tienen singularidades y separatrices dadas por las ecuaciones

$$\begin{aligned}\mathrm{Sing}(\mathcal{F}) &= \bigcup_{i=1}^{\tau}(x_1 = \ldots = x_{i-1} = x_{i+1} = \ldots = x_\tau = 0),\\ \mathrm{Sep}(\mathcal{F}) &= \bigcup_{i=1}^{\tau}(x_i = 0).\end{aligned}$$

Para $k = \tau = 3$, tenemos

$$\omega = p_1\frac{dx_1}{x_1} + p_2\frac{dx_2}{x_2} + p_3\frac{dx_3}{x_3} + \psi\,(x_1^{p_1}x_2^{p_2}x_3^{p_3})\left(\alpha_2\frac{dx_2}{x_2} + \alpha_3\frac{dx_3}{x_3}\right).$$

Como se tiene $p_1, p_2, p_3 > 0$, la singularidad no admite vector residual, y es simple.

Para $k = 2, \tau = 3$ tenemos

$$\omega = p_1\frac{dx_1}{x_1} + p_2\frac{dx_2}{x_2} + \psi\,(x_1^{p_1}x_2^{p_2})\left(\alpha_2\frac{dx_2}{x_2} + \alpha_3\frac{dx_3}{x_3}\right).$$

Como se tiene $p_1, p_2 > 0$, el vector residual es $\alpha_3 \neq 0$ y no hay resonancia: es simple. Observemos que en este caso los ejes x_1 y x_2 son sillas nodos de tipo dimensional dos.

Para $k = 1, \tau = 3$ se tiene

$$\omega = p_1\frac{dx_1}{x_1} + \psi\,(x_1^{p_1})\left(\alpha_2\frac{dx_2}{x_2} + \alpha_3\frac{dx_3}{x_3}\right).$$

Como $p_1 > 0$, el vector residual es (α_2, α_3) y la resonancia surge cuando se tiene

$$-\frac{n}{m} = \frac{\alpha_2}{\alpha_3} \in \mathbb{Q}^-, \quad n, m > 0.$$

De este modo, hay que continuar el algoritmo de la división aplicado a n y m. Al explotar únicamente el eje x_1 $(x_2 = x_3 = 0)$, en el penúltimo paso logramos

$$p_1\frac{dx_1'}{x_1'} + \psi(x_1^{p_1})\left(\frac{dx_2'}{x_2'} - \frac{dx_3'}{x_3'}\right).$$

Una explosión más con centro el eje x_1 lleva a una variedad (X', E', π), con E' un divisor dicrítico. En las coordenadas (x_2', t, x_3'), la aplicación

π' está dada por $x_2 = tx_3$. La foliación transformada está definida por la $1-$forma

$$p_1 \frac{dx_1'}{x_1'} + \psi(x_1^{p_1}) \frac{dt}{t},$$

esto es, una silla-nodo.

En las coordenadas (x_2', s, x_3'), la aplicación π queda determinada por la relación $x_3 = sx_2$. La foliación transformada está definida por la $1-$forma

$$p_1 \frac{dx_1'}{x_1'} + \psi(x_1^{p_1}) \frac{ds}{s},$$

que también representa una silla-nodo.

En resumen, las singularidades de tipo **B** con $\tau = k$, son foliaciones que pueden tener sus hojas cerradas; en los demás casos siempre tenemos sillas-nodos y por lo tanto, no será posible que todas sus hojas sean cerradas (ver [1]).

6.3. Singularidades de tipo A

En este caso, la foliación está dada por la $1-$forma

$$\omega = \sum_{i=1}^{\tau} \lambda_i dx_i = \lambda_1 \frac{dx_1}{x_1} + \lambda_2 \frac{dx_2}{x_2} + \lambda \frac{dx_3}{x_3}, \ \lambda_i^* \in \mathbb{C}.$$

Solo analizaremos el caso $\tau = 3$ y supondremos además que es resonante. Consideremos los $\mathbb{Q}-$espacios vectoriales V_0 y W_0 definidos por

$$V_0 = \left\{ r \in \mathbb{Q}^3 : r_1\lambda_1 + r_1\lambda_2 + r_3\lambda_3 = 0 \right\},$$

$$W_0 = \left\{ t \in \mathbb{Q}^3 : \sum_{j=1}^{3} r_j t_j = 0, \ \text{ para todo } \ r \in V_0 \right\}.$$

Observemos que la condición $\dim V_0 = 0$ equivale a que el vector residual $(\lambda_1, \lambda_2, \lambda_3)$ sea no resonante. Así, tenemos

$$\dim V_0 + \dim W_0 = 3 \qquad \text{y} \qquad 1 \leq \dim V_0 \leq 2.$$

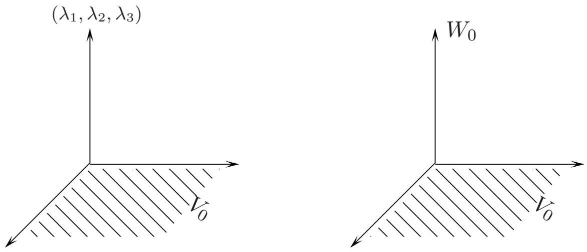

Figura 4: $\mathbb{Q}-$*espacios vectoriales* V_0, W_0, *a partir de* $(\lambda_1, \lambda_2, \lambda_3)$.

Analicemos los dos posibles casos: $\dim V_0 = 1$ y $\dim V_0 = 2$.

Cuando $\dim V_0 = 1$, consideremos $r^0 = (r_1, r_2, r_3)$ una resonancia de $(\lambda_1, \lambda_2, \lambda_3)$, con $\mathrm{mcd}(r_1, r_2, r_3) = 1$. Veamos el comportamiento al realizar una explosión con centro $Y_0 = T_{ij} = (x_i = 0) \cap (x_j = 0)$. Al fijar dos índices i, j con $r_i, r_j > 0$, sin perdida de generalidad consideremos $i = 1$, $j = 2$ y $r_1 > r_2$. La explosión π con centro Y_0, genera la variedad (X', E', π), donde $E' = \pi^{-1}(Y_0)$. Esta variedad está descrita por dos sistemas de coordenadas (ver figura 5).

En las coordenadas (t, x_2, x_3), donde π está dada por la relación $x_1 = tx_2$, la transformada de la foliación queda definida por la $1-$forma

$$\begin{aligned}(\pi)^*\omega &= \lambda_1\Big(\frac{dt}{t} + \frac{dx_2}{x_2}\Big) + \lambda_2\frac{dx_2}{x_2} + \lambda_3\frac{dx_3}{x_3}\\ &= \lambda_1\frac{dt}{t} + (\lambda_1 + \lambda_2)\frac{dx_2}{x_2} + \lambda_3\frac{dx_3}{x_3}.\end{aligned}$$

Una resonancia (s_1, s_2, s_3) del vector residual $(\lambda_1, \lambda_1 + \lambda_3, \lambda_3)$ es tal que $s_1\lambda_1 + s_2(\lambda_1 + \lambda_2) + s_3\lambda_3 = 0$, que es lo mismo que escribir $(s_1 + s_2)\lambda_1 + s_2\lambda_2 + s_3\lambda_3 = 0$. Así, podemos suponer $\mathrm{mcd}(s_1, s_2, s_3) = 1$. Como $\dim V_0 = 1$, se tiene $nr_1 = s_1 + s_2$, $nr_2 = s_2$ y $nr_3 = s_3$ para algún $n \in \mathbb{Z}$, $n \geq 0$. Dado que ambas resonancias tienen máximo común

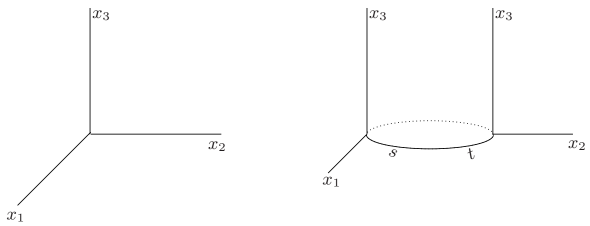

Figura 5: *Explosión con centro* $Y_0 = T_{ij}$.

divisor igual a 1, se tiene

$$s_1 = r_1 - r_2, s_2 = r_2, s_3 = r_3.$$

Por otro lado, al tenerse $r_1 > r_2$, hay sitio para resonancias siempre y cuando queden sujetas a

$$s_1 + s_2 + s_3 < r_1 + r_2 + r_3.$$

En las coordenadas (x_1, s, x_2), donde π está definida por la relación $x_2 = x_1 s$, la transformada de la foliación está determinada por la $1-$forma

$$\begin{aligned}(\pi')^*\omega &= \lambda_1 \frac{dx_1}{x_1} + \lambda_2\Big(\frac{ds}{s} + \frac{dx_1}{x_1} + \lambda_3 \frac{dx_3}{x_3}\Big)\\ &= (\lambda_1 + \lambda_2)\frac{dx_1}{x_1} + \lambda_2 \frac{ds}{s} + \lambda_3 \frac{dx_3}{x_3}.\end{aligned}$$

Sea (s_1, s_2, s_3) una resonancia de $(\lambda_1 + \lambda_2, \lambda_2, \lambda_3)$, esto es, valores que cumplen $s_1(\lambda_1 + \lambda_2) + s_2\lambda_2 + s_3\lambda_3 = 0$, lo que equivale a escribir $s_1\lambda_1 + (s_2 + s_1)\lambda_2 + s_3\lambda_3 = 0$. Como $\dim V_0 = 1$, de $s_1 = nr_1$, $s_1 + s_2 = nr_2$ y $s_3 = nr_3$ se pasa a

$$s_2 = n(r_2 - r_1).$$

Ahora, como se cumple $s_2 \geq 0$ y $r_2 - r_1 < 0$, necesariamente se tiene $n = 0$. Luego $(\lambda_1 + \lambda_2, \lambda_2, \lambda_3)$ es no resonante y por lo tanto, la singularidad es simple.

Supongamos ahora $\dim(V_0) = 2$. Entonces existe $\alpha \in \mathbb{C}^*$ tal que

$$\alpha(\lambda_1, \lambda_2, \lambda_3) = (p, -q, -n),$$

donde $p, q \in \mathbb{Z}^+$. En efecto, sea $\{(r_1, r_2, r_3), (s_1, s_2, s_3)\}$ una base de V_0. Una menor de la matriz, formada por los vectores de la base de V_0, es diferente de cero. Supongamos que esta menor es $r_1 s_3 - s_1 r_3 \neq 0$. Así, al multiplicar por s_1 y por r_1 obtenemos $r_1\lambda_1 + r_2\lambda_2 + r_3\lambda_3 = 0$ y $s_1\lambda_1 + s_2\lambda_2 + s_3\lambda_3 = 0$, respectivamente. En consecuencia, arribamos a

$$\frac{\lambda_3}{\lambda_2} = \left(\frac{s_1 r_1 - r_1 s_2}{s_1 r_3 - r_1 s_3}\right) = \frac{p_3}{p_2} \in \mathbb{Q}',$$

con $p_2, p_3 \in \mathbb{Z}$. Luego, al multiplicar por $\dfrac{p_2}{\lambda_2}$ la ecuación $r_1\lambda_1 + r_2\lambda_2 + r_3\lambda_3 = 0$, obtenemos

$$r_1 \frac{\lambda_1}{\lambda_2} p_2 + r_2 p_2 + r_3 p_3 = 0; \tag{6.5}$$

Así, $\dfrac{\lambda_1}{\lambda_2} = \dfrac{p_1'}{p_2'} \in \mathbb{Q}$ y al multiplicar (6.5) por p_2' llegamos a $r_1 p_1 + r_2 p_2 + r_3 p_3 = 0$. Por lo tanto $\alpha = \dfrac{p_2 p_2'}{\lambda_2} \in \mathbb{C}^*$ satisface $\alpha(\lambda_1, \lambda_2, \lambda_3) = (p_1, p_2, p_3) \in \mathbb{Z}^3$.

Ahora, por la existencia de una resonancia, las tres componentes no pueden ser positivas (o negativas) simultáneamente. Podemos asumir que al multiplicar por el signo adecuado dos son positivas y una negativa, digamos $p_3 = p \in \mathbb{Z}^+, p_2 = -q$ y $p_3 = -n$ con $q, n \in \mathbb{Z}^+$. En esta situación, realizamos una explosión con centro $x_1 = x_2 = 0$ o $x_1 = x_3 = 0$. El vector residual es $(p - q, -q, -n)$ o $(p, q - p, -n)$, respectivamente. En ambos casos la suma de las componentes del vector residual es menor que la suma de las componentes del vector $(p, -q, -n)$. Este proceso no se puede repetir indefinidamente.

Resumimos lo trabajado en esta sección en dos teoremas.

Teorema 6.1 (Cano, Fernández)**.** *Las singularidades presimples tras un número finito de explosiones con centro permitidos en curvas regulares se transforman en singularidades simples.*

Teorema 6.2. *Las singularidades presimples de tipo* **C** *y las tipo* **B** *con* $\tau \neq k$ *no tienen todas sus hojas cerradas.*

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Abstract

In dimension 2 as in 3, singularity reduction of holomorphic foliations of codimension 1 have two phases: first, from the given singularity to the presimple singularity; second, form the presimple singularities to the simple ones. In this article we establish the normal forms of presimple singularities according to the papers [2], [4] and [3]. Also, we characterize presimple singularities with closed leafs.

Keywords: Foliations, singularities, singularity reduction

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On the temporal discretizations of convection dominated convection-diffusion equations in time-dependent domain

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Abstract

This paper presents the numerical analysis of a convection dominated scalar equation with different time discretizations in time-dependent domains. The implicit Euler, Crank-Nicolson and backward-difference methods are used for the temporal discretization. The time-dependent domain is handled by the arbitrary Lagrangian-Eulerian (ALE) approach. In particular, the non-conservative form of the ALE scheme is considered. The Streamline Upwind Petrov-Galerkin (SUPG) finite element method is used for spatial discretization. It is shown that the stability of the fully discrete solution, irrespective of the temporal discretization, is only conditionally stable. The dependence of the numerical solution on the stabilization parameter δ_k is also studied. It is seen that the Crank-Nicolson scheme is less dissipative than the implicit Euler and the backward difference method. Moreover, the backward difference scheme is more sensitive to the stabilization parameter δ_k than the other time discretizations.

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Keywords: Convection-diffusion-reaction equation, SUPG stabilization, Geometric conservation Law (GCL), time-dependent domain, arbitrary Lagrangian-Eulerian approach, temporal discretizations.

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1 Introduction

Solution of a transient convection-diffusion-reaction equation in a time-dependent domain is highly demanded in many applications. The scalar variable can be a temperature or a concentration or a density of a species. Numerical approximation of the solution of a scalar partial differential equation becomes more challenging when the equation is convection dominated. Further, the computations become more complex when the domain contains moving boundaries, e.g., fluid-structure interactions (FSI) problems. In addition to a stabilized numerical method, an efficient approach is necessary to handle the mesh movement in time-dependent domains.

In general, the standard Galerkin finite element approximation induces spurious oscillations in the numerical solution of a convection dominated convection-diffusion equation. Therefore, stabilization schemes, as for instance, the streamline upwind Petrov-Galerkin (SUPG) [1, 3, 19], the local projection stabilization [11, 7], the continuous interior penalty method [4, 5], the subgrid scale modeling [8], the orthogonal subscales method [12, 13], have been proposed and analyzed in the literature. Each of these methods has its own advantages and disadvantages. Nevertheless, almost all stabilization methods introduce an unknown numerical parameter that controls the stability. The optimal choice of this parameter is an open problem [10, 19, 15, 16]. Almost all these stabilization methods have been studied only for PDEs in stationary domains.

In this paper, the SUPG finite element scheme for the computation of transient convection-diffusion equation in time-dependent domains is presented. Further, the arbitrary Lagrangian-Eulerian (ALE) approach is used to handle the moving boundaries of the time-dependent domain. The ALE formulation introduces a convection type mesh velocity term into the model equation, and hence it alters the overall convective field of the problem. Nevertheless, the model problem can still be convection dominated and can have boundary/interior layers even after reformulating into an ALE form.

Conservative and non-conservative forms are the two variants of the ALE approach. The stability estimates for the conservative ALE-SUPG discretization with the backward Euler and the Crank-Nicolson methods have been derived in [14]. In this paper, we consider the non-conservative ALE form. Further, backward Euler, Crank-Nicolson, and backward difference (BDF2) methods are considered for the temporal discretization. The non-conservative ALE-SUPG formulation avoids the necessity of the geometric conservation law (GCL). As it has been discussed in [6], GCL mainly boils down to a quadrature formula for the exact time integration of the term containing the mesh velocity introduced from Reynolds identity. Though we considered the non-conservative form, and the GCL is not needed, still we get only the conditionally stable estimates, which reflects the issue with the numerical computation of mesh velocity term $\mathbf{w}$ in the analysis. The stability estimates for the implicit Euler, Crank-Nicolson, and backward difference time discretizations with inconsistent SUPG for non-conservative ALE form of the convection-diffusion-reaction equation in time-dependent domains are derived. It can be seen from the numerical results that the behavior of the solutions of conservative and non-conservative ALE forms depends on the deformation of the domain. If the deformation of the domain or the mesh velocity is large, the solution is very much influenced by the ALE form. Contrarily, the difference between the solutions of different ALE forms is almost negligible when the deformation of the domain is small.

The paper is organized as follows. In Section 2, transient convection-diffusion equation in a time-dependent domain and its ALE formulation are given. The spatial discretization using the SUPG finite element method is also presented in this section. Further, the stability of the semi-discrete (in space) problem is derived. Section 3 is devoted to the stability estimates of the fully discrete problem with implicit Euler, Crank-Nicolson, and backward difference (BDF2) discretization in time. Numerical results are presented in Section 4. Finally, a brief summary is presented in Section 5.

2 Statement of the problem

Let Ω_t be a time-dependent bounded domain in R^d, $d = 2, 3$, with Lipschitz boundary $\partial\Omega_t$ for each $t \in [0, \mathrm{T}]$. Here, T is a given end time. Consider a transient convection-diffusion-reaction equation: find $u(t, x) : (0, \mathrm{T}] \times \Omega \to \mathbb{R}$ subject to

$$\begin{aligned} \frac{\partial u}{\partial t} - \epsilon \Delta u + \mathbf{b} \cdot \nabla u + cu &= f && \text{in } (0, \mathrm{T}] \times \Omega_{\mathrm{t}}, \\ u &= 0 && \text{on } [0, \mathrm{T}] \times \partial\Omega_{\mathrm{t}}, \\ u(0, x) &= u_0(x) && \text{in } \Omega_0. \end{aligned} \tag{2.1}$$

Here $u(t, x)$ is an unknown scalar function, $u_0(x)$ is a given initial data, ϵ is the diffusivity constant, $\mathbf{b}(t, x)$ is the convective flow velocity, $c(t, x)$ is a reaction function, and $f(x)$ is a given source term with $f \in L^2(\Omega_t)$.

2.1 ALE formulation

Let $\hat{\Omega}$ be a reference domain. The reference domain $\hat{\Omega}$ can simply be the initial domain Ω_0 or the previous time-step domain when the deformation of the domain is large. Let $\mathcal{A}_t$ be a family of bijective ALE mappings, which, at each time $t \in (0, \mathrm{T}]$, maps a point Y of a reference domain $\hat{\Omega}$ to a point on the current domain Ω_t, given by

$$\mathcal{A}_t : \hat{\Omega} \to \Omega_t, \qquad \mathcal{A}_t(Y) = x(Y, t), \qquad t \in (0, \mathrm{T}).$$

Further, for any time $t_1, t_2 \in [0, \mathrm{T}]$, the ALE mapping between two time levels will be given by

$$\mathcal{A} : \Omega_{t_1} \to \Omega_{t_2} \qquad\qquad \mathcal{A}_{t_1,t_2} = \mathcal{A}_{t_2} \circ \mathcal{A}_{t_1}^{-1}.$$

We assume that Ω_t is bounded with Lipschitz continuous boundary for each $t \in [0, \mathrm{T}]$. For a function $g \in C^0(\Omega_t)$ on the Eulerian frame, we define the corresponding function $\hat{g} \in C^0(\hat{\Omega})$ on the ALE frame as

$$\hat{g} : \hat{\Omega} \times (0, \mathrm{T}) \to \mathbb{R}, \qquad \hat{g} = g \circ \mathcal{A}_t, \qquad \text{with} \qquad \hat{g}(Y, t) = g(\mathcal{A}_t(Y), t).$$

The temporal derivative on the ALE frame is defined as

$$\frac{\partial g}{\partial t}\Big|_Y : \Omega_t \times (0,\mathrm{T}) \to \mathbb{R}, \qquad \frac{\partial g}{\partial t}\Big|_Y (x,t) = \frac{\partial \hat{g}}{\partial t}(Y,t), \qquad Y = \mathcal{A}_t^{-1}(x).$$

We apply the chain rule to the time derivative of $g \circ \mathcal{A}_t$ on the ALE frame to get

$$\frac{\partial g}{\partial t}\Big|_Y = \frac{\partial g}{\partial t}(x,t) + \frac{\partial x}{\partial t}\Big|_Y \cdot \nabla_x g = \frac{\partial g}{\partial t} + \frac{\partial \mathcal{A}_t(Y)}{\partial t} \cdot \nabla_x g = \frac{\partial g}{\partial t} + \mathbf{w} \cdot \nabla_x g,$$

where the domain velocity $\mathbf{w}$ is defined as

$$\mathbf{w}(x,t) = \frac{\partial x}{\partial t}\Big|_Y (\mathcal{A}_t^{-1}(x), t).$$

By using the relation in the model problem (2.1), we get

$$\frac{\partial u}{\partial t}\Big|_Y - \epsilon \Delta u + (\mathbf{b} - \mathbf{w}) \cdot \nabla u + cu = f. \tag{2.2}$$

This equation is the ALE counterpart of the model equation (2.1). The difference between equations (2.1) and (2.2) is the additional domain velocity in the ALE form that accounts for the deformation of the domain.

2.2 Variational form

In this section, the finite element variational form of the ALE equation (2.2) is derived. Let

$$V = \Big\{ v \in H_0^1(\Omega_t), \;\; v : \Omega_t \times (0,\mathrm{T}] \to \mathbb{R}, \;\; \mathrm{v} = \hat{\mathrm{v}} \circ \mathrm{A}_\mathrm{t}^{-1}, \; \hat{\mathrm{v}} \in \mathrm{H}_0^1(\hat{\Omega}) \Big\}$$

be the solution space for equation (2.2). Multiplying equation (2.2) with a test function $v \in V$ and applying integration by parts to the higher order derivative term, the variational form of the equation (2.2) becomes: *find $u \in V$ such that*

$$\left(\frac{\partial u}{\partial t}\Big|_Y, \; v \right) + (\epsilon \nabla u, \; \nabla v) + ((\mathbf{b} - \mathbf{w}) \cdot \nabla u, \; v) + (cu, \; v) = (f, \; v), \quad v \in V, \tag{2.3}$$

for given $\mathbf{b}$, $\mathbf{w}$, c, u_0, Ω_0, f, *and for all* $t \in (0, \mathrm{T}]$. Of course, $(\cdot,\ \cdot)$ denotes the L^2-inner product in Ω_t.

Further, the notations for L^2-inner product, the norm and the semi-norm, $(\cdot,\cdot)_t$, $\|\cdot\|_{0,t}$ and $|\cdot|_{1,t}$, respectively over Ω_t, which will be followed throughout the paper, are given as

$$(u, v)_t = \int_{\Omega_t} uv\, dx, \quad \|v\|^2_{0,t} = (v, v)_t, \quad |v|^2_{1,t} = (\nabla v, \nabla v)_t\,, \text{ for } u, v \in V.$$

2.3 SUPG finite element space discretization

The numerical analysis for the standard Galerkin solution of (2.3) can be found in [2, 6, 9]. It has been shown that the stability estimate is independent of the domain velocity. However, the Galerkin approximation suffers instabilities for convection dominant scalar equations of type (2.3). In order to overcome this issue, we use the SUPG stabilization for the considered model problem with ALE equation (2.3). Note that the convective velocity in the ALE form is "$(\mathbf{b} - \mathbf{w})$", whereas the convective velocity for problems in time-independent domains is "$\mathbf{b}$".

Let $\mathcal{T}_{h,0}$ be a triangulation of the domain Ω_0. For each $t \in (0, \mathrm{T}]$, $\mathcal{T}_{h,t}$ denote the family of shape regular triangulations of Ω_t into simplices obtained by triangulating the time-dependent domain Ω_t. We denote the diameter of the cell $K \in \mathcal{T}_{h,t}$ by $h_{K,t}$ and the global mesh size in the triangulated domain $\Omega_{h,t}$ by $h_t = \max\{h_{K,t} \ : \ K \in \mathcal{T}_{h,t}\}$. Suppose $V_h \subset V$ is a conforming finite element (finite dimensional) space. Let $\phi_h = \phi_i(x)$, $i = 1, 2, ..., \mathcal{N}$, be the finite element basis functions of V_h. The discrete finite element space V_h is then defined by

$$V_h = \left\{ v_h \in C(\overline{\Omega_t}) : \ v_h|_{\partial\Omega_t} = 0; \ v_h|_K \in P_k(K) \right\} \subset H^1_0(\Omega_t),$$

where P_k is the set of polynomials of degree less than or equal to k for discretization of the ALE mapping in space. We next define the discrete ALE mapping $\mathcal{A}_{h,t}(Y)$ and the mesh velocity $\mathbf{w}_h$ in space. We use the Lagrangian finite element space

$$\mathcal{L}^k(\hat{\Omega}) = \left\{ \psi \in H^k(\hat{\Omega}) : \psi|_K \in P_k(\hat{K}) \text{ for all } \hat{K} \in \hat{\Omega}_h \right\},$$

Using the linear space, we define the semidiscrete ALE mapping in space for each $t \in [0, \mathrm{T})$ by

$$\mathcal{A}_{h,t} : \hat{\Omega}_h \to \Omega_{h,t}. \tag{2.4}$$

Further, the semidiscrete (continuous in time) mesh velocity $\mathbf{w}_h(t, Y) \in \mathcal{L}^1(\hat{\Omega})^d$ in the ALE frame for each $t \in [0, \mathrm{T})$ is defined by

$$\hat{\mathbf{w}}_h(t, Y) = \sum_{i=1}^{\mathcal{M}} \mathbf{w}_i(t)\psi_i(Y); \quad \mathbf{w}_i(t) \in \mathbb{R}^d.$$

Here $\mathbf{w}_i(t)$ denotes the mesh velocity of the i^{th} node of simplices at time t, and $\psi_i(Y)$, $i = 1, 2, ..., \mathcal{M}$, are the basis functions of $\mathcal{L}^1(\hat{\Omega}_h)$. We then define the semidiscrete mesh velocity in the Eulerian frame as

$$\mathbf{w}_h(t, x) = \hat{\mathbf{w}}_h \circ \mathcal{A}_{h,t}^{-1}(x).$$

Using the above finite element spaces and applying the inconsistent SUPG finite element discretization to the ALE form (2.2), the semidiscrete form in space reads as follows. *For a given* $u_h(x, 0) = u_{h,0}$*,* $\mathbf{b}$*,* $\mathbf{w}_h$*,* c*,* $\Omega_{h,0}$ *and* f*, find* $u_h(t, x) \in V_h$ *such that, for all* $t \in (0, \mathrm{T}]$*, we have*

$$\left(\frac{\partial u_h}{\partial t}\bigg|_Y, v_h \right)_t + a_{SUPG}(u_h, v_h)_t - \int_{\Omega_{h,t}} \mathbf{w}_h \cdot \nabla u_h \; v_h \; dx = \\ = \int_{\Omega_{h,t}} f v_h \; dx + \sum_{K \in \mathcal{T}_{h,t}} \delta_K \int_K f \; (\mathbf{b} - \mathbf{w}_h) \cdot \nabla v_h \; dK, \tag{2.5}$$

where

$$a_{SUPG}(u, v)_t = \epsilon(\nabla u, \nabla v)_t + (\mathbf{b} \cdot \nabla u, v)_t + (cu, v)_t + \\ + \sum_{K \in \mathcal{T}_{h,t}} \delta_K(-\epsilon\Delta u + (\mathbf{b} - \mathbf{w}_h) \cdot \nabla u + cu, (\mathbf{b} - \mathbf{w}_h) \cdot \nabla v)_K. \tag{2.6}$$

Here δ_K is the local stabilization parameter, whose value depends on the mesh size and the convective velocity field. Further, $u_{h,0} \in V_h$ is defined as the L^2-projection of the initial value u_0 onto V_h.

Next we briefly describe the inverse inequality used in our subsequent analysis, given as

$$||\Delta u_h||_{0,K} = c_{inv} h_K^{-1} |u_h|_{1,K}, \text{ for all } \quad u_h \in V_h, \tag{2.7}$$

where c_{inv} is a constant.

Lemma 1. Coercivity of $a_{SUPG}(\cdot,\cdot)$*: Assume that there exists a constant* μ *such that*

$$\left(c - \frac{1}{2}\nabla \cdot \mathbf{b}\right)(x) \geq \mu > 0, \quad \textit{for all } x \in \Omega_t. \tag{2.8}$$

Let the discrete form of assumption (2.8) *be satisfied. Assume that the SUPG parameter satisfies*

$$\delta_K \leq \frac{\mu}{2||c||^2_{K,\infty}}, \qquad \delta_K \leq \frac{h_K^2}{2\epsilon c_{inv}^2}, \tag{2.9}$$

where c_{inv} *is a constant used in the inverse inequality; see equation* (2.7)*. Then, the SUPG bilinear form satisfies*

$$a_{SUPG}(u_h, u_h)_t \geq \frac{1}{2}|||u_h|||_t^2,$$

where the mesh dependent norm is defined as

$$|||u|||_t^2 = \left(\epsilon |u|^2_{1,t} + \sum_{K\in\mathcal{T}_{h,t}} \delta_K ||(\mathbf{b} - \mathbf{w}_h)\cdot\nabla u||^2_{0,K} + \mu||u||^2_{0,t}\right).$$

Proof. The coercivity of the bilinear form follows from [14, Lemma 1]. □

Lemma 2. Stability of the semi-discrete problem*: Let the discrete version of* (2.8) *and the assumption* (2.9) *on* δ_K *hold true. Then, the* *solution of problem* (2.5) *satisfies*

$$||u_h(t)||^2_{0,t} + \frac{1}{2}\int_0^T |||u_h(t)|||^2_t dt \leq ||u_h(0)||^2_{0,t} + \frac{2}{\mu}\int_0^T ||f(t)||^2_{0,t}\ dt$$
$$+ 2\int_0^T \sum_{K\in\mathcal{T}_{h,t}} \delta_K ||f(t)||^2_{0,K}\ dt.$$

Proof. Take $v_h = u_h$ in equation (2.5) and consider the relations

$$\int_{\Omega_{h,t}} \frac{\partial u_h}{\partial t}\bigg|_Y u_h\ dx = \frac{1}{2}\left(\frac{d}{dt}||u_h||^2_{0,t} - \int_{\Omega_{h,t}} u_h^2 \nabla\cdot\mathbf{w_h} dx\right)$$

and

$$\int_{\Omega_{h,t}} \mathbf{w_h}\cdot\nabla u_h\ u_h\ dx = -\frac{1}{2}\int_{\Omega_{h,t}} u_h^2\ \nabla\cdot\mathbf{w}_h\ dx.$$

Use the Cauchy-Schwarz inequality to bound the right hand side terms:

$$|(f,u_h)| = \left|\left(\frac{f}{\mu^{1/2}}, \mu^{1/2}u_h\right)\right| \leq \frac{1}{\mu}||f||^2_{0,t} + \frac{1}{4}\mu||u_h||^2_{0,t}$$

and

$$\left|\sum_{K\in\mathcal{T}_{h,t}} \delta_K(f,(\mathbf{b}-\mathbf{w}_h)\cdot\nabla u_h)_K\right| \leq \sum_{K\in\mathcal{T}_{h,t}} \delta_K||f||^2_{0,K} +$$
$$\frac{1}{4}\sum_{K\in\mathcal{T}_{h,t}} \delta_K||(\mathbf{b}-\mathbf{w}_h)\cdot\nabla u_h)||^2_{0,K}.$$

Now with the coercivity of bilinear form, the stability estimate for the semi-discrete problem can be derived. Here the stability properties are not affected by the domain velocity field in the semi-discrete problem (2.5). However, we may not expect the same result to hold true for the fully discrete case. □

3 Fully discrete scheme

In this section the stability estimates for the fully discrete ALE-SUPG form are derived. We consider the first order implicit Euler, the second order modified Crank-Nicolson, and the backward-difference (BDF2) method for the temporal discretization.

Consider the partition of the time interval $[0, \mathrm{T}]$ as $0 = t^0 < t^1 < \cdots < t^N = \mathrm{T}$ into N equal sub-intervals. Let us denote the uniform time step by $\Delta t = \tau^n = t^n$ - t^{n-1}, $1 \leq n \leq N$. Further, let u_h^n be the approximation of $u(t^n, x)$ in $V_h \subset H_0^1(\Omega_{t^n})$, where Ω_{t^n} is the deforming domain at time $t = t^n$.

We first discretize the ALE mapping in time using a linear interpolation. We denote the discrete ALE mapping by $\mathcal{A}_{h,\Delta t}$, and define it for every $\tau \in [t^n, t^{n+1}]$ by

$$\mathcal{A}_{h,\Delta t}(Y) = \frac{\tau - t^n}{\Delta t}\mathcal{A}_{h,t^{n+1}}(Y) + \frac{t^{n+1} - \tau}{\Delta t}\mathcal{A}_{h,t^n}(Y),$$

where $\mathcal{A}_{h,t}(Y)$ is the time continuous ALE mapping exhibited in (2.4). Since the ALE mapping is discretized in time using a linear interpolation, we obtain the discrete mesh velocity

$$\hat{\mathbf{w}}_h^{n+1}(Y) = \frac{\mathcal{A}_{h,t^{n+1}}(Y) - \mathcal{A}_{h,t^n}(Y)}{\Delta t} = \frac{x_h^{n+1} - x_h^n}{\Delta t} \tag{3.1}$$

as a piecewise constant function in time. We define the mesh velocity on the Eulerian frame as

$$\mathbf{w}_h^{n+1} = \hat{\mathbf{w}}_h^{n+1} \circ \mathcal{A}_{h,\Delta t}^{-1}(x).$$

Further, the integral u_h^n on a domain Ω_{t^s}, with $t^s \neq t^n$, is written through the ALE mapping as

$$\int_{\Omega_{t^s}} u_h^n \; dX = \int_{\Omega_{t^s}} u_h^n \circ \mathcal{A}_{t^n,t^s} \; dX.$$

3.1 Discrete ALE-SUPG with implicit Euler time discretization method

Applying the backward Euler time discretization to the semi-discrete problem (2.5), the discrete form reads

$$\left(\frac{u_h^{n+1}-u_h^n}{\Delta t}, v_h\right)_{t^{n+1}} + a_{SUPG}^{n+1}(u_h^{n+1}, v_h)_t - \int_{\Omega_{t^{n+1}}} \mathbf{w}_h^{n+1}\cdot\nabla u_h^{n+1}\; v_h\; dx$$
$$= \int_{\Omega_{t^{n+1}}} f^{n+1} v_h\; dx + \sum_{K\in\mathcal{T}_{h,t^{n+1}}} \delta_K \int_K f^{n+1}\;(\mathbf{b}-\mathbf{w}_h^{n+1})\cdot\nabla v_h\; dK, \tag{3.2}$$

where

$$a_{SUPG}^{n+1}(u_h, v_h) = \epsilon(\nabla u_h, \nabla v_h)_{t^{n+1}} + (\mathbf{b}\cdot\nabla u_h, v_h)_{t^{n+1}} + (cu_h, v_h)_{t^{n+1}} +$$
$$\sum_{K\in\mathcal{T}_{t^{n+1}}} \delta_K(-\epsilon\Delta u_h + (\mathbf{b}-\mathbf{w}_h^{n+1})\cdot\nabla u_h + cu_h,\; (\mathbf{b}-\mathbf{w}_h^{n+1})\cdot\nabla v_h)_K.$$

Lemma 3. Stability estimate for non-conservative ALE-SUPG form with implicit Euler method: *Let the discrete version of* (2.8) *and the assumption* (2.9) *on* δ_K *hold true. Further, assume* $\delta_K \leq \frac{\Delta t}{4}$*. Then the solution of problem* (3.2) *satisfies*

$$\|u_h^{n+1}\|^2_{0,t^{n+1}} + \frac{\Delta t}{2}\sum_{i=1}^{n+1} |||u_h^i(t)|||^2_{t^i} \leq$$
$$\leq \left((1+\Delta t\alpha_2^0)\|u_h^0\|^2_{0,t^0} + \Delta t\sum_{i=1}^{n+1}\left(\frac{2}{\mu}+\frac{\Delta t}{2}\right)\|f^i(t)\|^2_{0,t^i}\right)\cdot$$
$$\exp\left(\Delta t\sum_{i=1}^{n+1}\frac{\alpha_1^i+\alpha_2^i}{1-\Delta t(\alpha_1^i+\alpha_2^i)}\right),$$

where α_1^n *and* α_2^n *are defined as in the proof of this lemma.*

Proof. Substituting $v_h = u_h^{n+1}$ in the discrete form (3.2) and applying integration by parts to the mesh velocity integral term, we get

$$\left(\frac{u_h^{n+1}-u_h^n}{\Delta t}, u_h^{n+1}\right)_{t^{n+1}} + a_{SUPG}^{n+1}(u_h^{n+1}, u_h^{n+1}) + \frac{1}{2}\int_{\Omega_{t^{n+1}}} \nabla\cdot \mathbf{w}_h^{n+1}|u_h^{n+1}|^2\ dx$$
$$= \int_{\Omega_{t^{n+1}}} f^{n+1}u_h^{n+1}dx + \sum_{K\in\mathcal{T}_{t^{n+1}}} \delta_K \int_K f^{n+1}\ (\mathbf{b}-\mathbf{w}_h^{n+1})\cdot\nabla u_h^{n+1}dK.$$

Using the coercivity of the bilinear form a_{SUPG}, the Cauchy-Schwarz inequality yields

$$\begin{aligned}
&||u_h^{n+1}||_{0,t^{n+1}}^2 + \frac{\Delta t}{2}|||u_h^{n+1}|||_{t^{n+1}}^2 \\
&\quad\le -\frac{1}{2}\Delta t\int_{\Omega_{t^{n+1}}} \nabla\cdot\mathbf{w}_h^{n+1}|u_h^{n+1}|^2\ dx + \frac{1}{2}||u_h^n||_{0,t^{n+1}}^2 \\
&\quad+ \frac{1}{2}||u_h^{n+1}||_{0,t^{n+1}}^2 + \frac{\Delta t}{4}\sum_{K\in\mathcal{T}_{h,t^{n+1}}} \delta_K||(\mathbf{b}-\mathbf{w}_h^{n+1})\cdot\nabla u_h^{n+1}||_{0,K}^2 \\
&\quad+ \Delta t\sum_{K\in\mathcal{T}_{h,t^{n+1}}} \delta_K||f^{n+1}||_{0,K}^2 + \frac{\Delta t}{\mu}||f^{n+1}||_{0,t^{n+1}}^2 + \Delta t\frac{\mu}{4}||u_h^{n+1}||_{0,t^{n+1}}^2.
\end{aligned}$$

From the Reynolds identity, we have

$$||u_h^n||_{0,t^{n+1}}^2 = ||u_h^n||_{0,t^n}^2 + \int_{t^n}^{t^{n+1}}\int_{\Omega_t} \nabla\cdot\mathbf{w}_h|u_h^n|^2\ dx\ dt,$$

and we get

$$\begin{aligned}
&||u_h^{n+1}||_{0,t^{n+1}}^2 + \frac{1}{2}\Delta t|||u_h^{n+1}|||_{t^{n+1}}^2 \le \int_{t^n}^{t^{n+1}}\int_{\Omega_t} \nabla\cdot\mathbf{w}_h|u_h^n|^2\ dx\ dt \\
&\quad - \frac{\Delta t}{2}\int_{\Omega_{t^{n+1}}} \nabla\cdot\mathbf{w}_h^{n+1}|u_h^{n+1}|^2\ dx + ||u_h^n||_{0,t^n}^2 + \Delta t\frac{2}{\mu}||f^{n+1}||_{0,t^{n+1}}^2 \\
&\quad + 2\Delta t\sum_{K\in\mathcal{T}_{h,t^{n+1}}} \delta_K||f^{n+1}||_{0,t^{n+1}}^2.
\end{aligned}$$

Let

$$\mathcal{A}_{t_n,t_{n+1}} = \mathcal{A}_{t_{n+1}} \circ\ \mathcal{A}_{t_n}^{-1}$$

be the ALE mapping between Ω_{t^n} and $\Omega_{t^{n+1}}$, and $J_{\mathcal{A}_{t_n,t_{n+1}}}$ be its Jacobian. Then we have

$$
\begin{aligned}
&||u_h^{n+1}||^2_{0,t^{n+1}} + \frac{1}{2}\Delta t|||u_h^{n+1}|||^2_{t^{n+1}} \\
&\qquad \leq \Delta t||\nabla\cdot\mathbf{w}_h(t^{n+1})||_{\infty,t^{n+1}}||u_h^{n+1}||^2_{0,t^{n+1}} + \Delta t\frac{2}{\mu}||f^{n+1}||^2_{0,t^{n+1}} \\
&\qquad + \left(1+\Delta t \sup_{t\in(t^n,t^{n+1})}||J_{\mathcal{A}_{t_n,t_{n+1}}}\nabla\cdot\mathbf{w}_h||_{\infty,t}\right)||u_h^n||^2_{0,t^n} \\
&\qquad + \Delta t\sum_{K\in\mathcal{T}_{h,t^{n+1}}}\delta_K||f^{n+1}||^2_{0,t^{n+1}}.
\end{aligned}
$$

Further, using the notation

$$\alpha_1^n = ||\nabla\cdot\mathbf{w}_h(t^n)||_{\infty,t^n},$$

$$\alpha_2^n = \sup_{t\in(t^n,t^{n+1})}||J_{\mathcal{A}_{t_n,t_{n+1}}}\nabla\cdot\mathbf{w}_h||_{\infty,t},$$

the above equation can be written as

$$
\begin{aligned}
&||u_h^{n+1}||^2_{0,t^{n+1}} + \frac{1}{2}\Delta t|||u_h^{n+1}|||^2_{t^{n+1}} \leq \\
&\qquad \leq \Delta t\alpha_1^{n+1}||u_h^{n+1}||^2_{0,t^{n+1}} + (1+\Delta t\alpha_2^n)\,||u_h^n||^2_{0,t^n} + \\
&\qquad \Delta t\frac{2}{\mu}||f^{n+1}||^2_{0,t^{n+1}} + 2\Delta t\sum_{K\in\mathcal{T}_{h,t^{n+1}}}\delta_K||f^{n+1}||^2_{0,t^{n+1}}.
\end{aligned}
$$

Summing over the index as $i = 0, 1, 2, \ldots, n$, and using the assumptions on δ_K, we arrive to

$$
\begin{aligned}
||u_h^{n+1}||^2_{0,t^{n+1}} + \frac{1}{2}\Delta t \sum_{i=1}^{n+1} |||u_h^i(t)|||^2_{t^i} \leq& \\
\leq \Delta t \alpha_1^{n+1} ||u_h^{n+1}||^2_{0,t^{n+1}} + \Delta t \sum_{i=1}^{n} (\alpha_1^i + \alpha_2^i) ||u_h^i||^2_{0,t^i} + (1 + \Delta t \alpha_2^0) ||u_h^0||^2_{0,t^0}& \\
+ 2\Delta t \sum_{i=1}^{n+1} \sum_{K \in \mathcal{T}_{h,t^i}} \delta_K ||f^n||^2_{0,t^i} + \Delta t \sum_{i=1}^{n+1} \frac{2}{\mu} ||f^i||^2_{0,t^i}& \\
\leq \Delta t \sum_{i=1}^{n+1} (\alpha_1^i + \alpha_2^i) ||u_h^i||^2_{0,t^i} + (1 + \Delta t \alpha_2^0) ||u_h^0||^2_{0,t^0}& \\
+ \sum_{i=1}^{n+1} \left(\frac{2\Delta t}{\mu} + \frac{\Delta t^2}{2} \right) ||f^i||^2_{0,t^i}.&
\end{aligned}
$$

We now apply Gronwall's lemma to get

$$
\begin{aligned}
\|u_h^{n+1}\|^2_{0,t^{n+1}} + \frac{\Delta t}{2} \sum_{i=1}^{n+1} |||u_h^i(t)|||^2_{t^i} \leq& \\
\leq \left[(1 + \Delta t \alpha_2^0) \|u_h^0\|^2_{0,t^0} + \Delta t \sum_{i=1}^{n+1} \left(\frac{2}{\mu} + \frac{\Delta t}{2} \right) \|f^i\|^2_{0,t^i} \right]& \\
\exp \left(\Delta t \sum_{i=1}^{n+1} \frac{\alpha_1^i + \alpha_2^i}{1 - \Delta t(\alpha_1^i + \alpha_2^i)} \right).&
\end{aligned}
$$

The above stability estimate is stable provided we have

$$
\Delta t < \frac{1}{\alpha_1^n + \alpha_2^n} = \left(\|\nabla \cdot \mathbf{w}_h(t^n)\|_{\infty,t^n} + \sup_{t \in (t^n, t^{n+1})} \|J_{A_{t^n,t^{n+1}}} \nabla \cdot \mathbf{w}_h\|_{\infty,t} \right)^{-1}
$$

□

3.2 Discrete ALE-SUPG with Crank-Nicolson time discretization

We now consider the modified Crank-Nicolson method which is basically Runge-Kutta method of order two. For an equation

$$\frac{du(t)}{dt} = f(u(t), t), \quad t > 0 \text{ and } u(0) = u_0, \tag{3.3}$$

with the modified Crank-Nicolson method, we get

$$u^{n+1} - u^n = \Delta t f\left(\frac{u^{n+1} + u^n}{2}, t^{n+\frac{1}{2}}\right).$$

Lemma 4. Stability estimates for the non-conservative ALE-SUPG form applying Crank-Nicolson method: *Let the discrete version of* (2.8) *and the assumption* (2.9) *on δ_K hold true. Further, assume $\delta_K \leq \frac{\Delta t}{4}$. Then the solution obtained from the Crank-Nicolson time discretization satisfies*

$$\begin{aligned}
&\|u_h^{n+1}\|^2_{0,t^{n+1}} + \frac{\Delta t}{4}\sum_{i=0}^{n} |||(u_h^{i+1} + u_h^i)|||^2_{t^{i+1/2}} \leq \\
&\quad \leq \left((1 + \Delta t\beta_2^0)\|u_h^0\|^2_{0,t^0} + \Delta t\sum_{i=0}^{n}\left(\frac{2}{\mu} + \Delta t\right)\|f^{i+1/2}\|^2_{0,t^{i+1/2}}\right) \cdot \\
&\qquad \exp\left(\Delta t\sum_{i=0}^{n}\frac{\beta_1^i + \beta_2^i}{1 - \Delta t(\beta_1^i + \beta_2^i)}\right).
\end{aligned}$$

Proof. Applying the modified Crank-Nicolson time discretization to the semi-discrete equation (5) we get

$$\begin{aligned}
&\left(\frac{u_h^{n+1} - u_h^n}{\Delta t}, v_h\right)_{t^{n+1}} + a^{n+1/2}_{SUPG}\left(\frac{u_h^{n+1} + u_h^n}{2}, v_h\right) - \int_{\Omega_{t^{n+1/2}}} \mathbf{w}_h \cdot \nabla\left(\frac{u_h^{n+1} + u_h^n}{2}\right) v_h dx \\
&\quad = \int_{\Omega_{t^{n+1/2}}} f^{n+1/2} v_h \, dx + \sum_{K\in\mathcal{T}_{h,t^{n+1/2}}} \delta_K \int_K f^{n+1/2} \, (\mathbf{b} - \mathbf{w}_h) \cdot \nabla v_h \, dK.
\end{aligned}$$

Testing the above equation with $v_h = (u_h^{n+1} + u_h^n)$, and using the relations

$$(u_h, u_h + v_h) = \frac{1}{2}||u_h||^2 + \frac{1}{2}||u_h + v_h||^2 - \frac{1}{2}||v_h||^2$$

and

$$||u_h^n||^2_{0,t^{n+1}} = ||u_h^n||^2_{0,t^n} + \int_{t^n}^{t^{n+1}} \int_{\Omega_t} \nabla \cdot \mathbf{w}_h |u_h^n|^2 \; dx \; dt,$$

the first term can be rewritten as

$$\begin{aligned}
&\int_{\Omega_{t^{n+1}}} u_h^{n+1}(u_h^{n+1} + u_h^n) \; dx - \int_{\Omega_{t^{n+1}}} u_h^n(u_h^{n+1} + u_h^n) \; dx \\
&= \frac{1}{2}||u_h^{n+1}||^2_{0,t^{n+1}} + \frac{1}{2}||u_h^{n+1} + u_h^n||^2_{0,t^{n+1}} - \frac{1}{2}||u_h^n||^2_{0,t^{n+1}} - \frac{1}{2}||u_h^n||^2_{0,t^{n+1}} \\
&\quad - \frac{1}{2}||u_h^{n+1} + u_h^n||^2_{0,t^{n+1}} + \frac{1}{2}||u_h^{n+1}||^2_{0,t^{n+1}} \\
&= ||u_h^{n+1}||^2_{0,t^{n+1}} - ||u_h^n||^2_{0,t^{n+1}} \\
&= ||u_h^{n+1}||^2_{0,t^{n+1}} - ||u_h^n||^2_{0,t^n} - \Delta t \int_{\Omega_{t^{n+1/2}}} \nabla \cdot \mathbf{w}_h |u_h^n|^2 \; dx.
\end{aligned}$$

Using the above relation, the coercivity of the bilinear form and the Cauchy-Schwarz inequality for the right hand side terms, the expression can be written as

$$\begin{aligned}
&||u_h^{n+1}||^2_{0,t^{n+1}} - ||u_h^n||^2_{0,t^n} - \Delta t \int_{\Omega_{t^{n+1/2}}} \nabla \cdot \mathbf{w}_h |u_h^n|^2 dx + \frac{\Delta t}{4} |||u_h^{n+1} + u_h^n|||^2_{0,t^{n+1/2}} \\
&\leq \frac{\Delta t}{2} \int_{\Omega_{t^{n+1/2}}} \big(\mathbf{w}_h \cdot \nabla(u_h^{n+1} + u_h^n)\big) \; (u_h^{n+1} + u_h^n) \; dx + \frac{\Delta t}{\mu} ||f^{n+1/2}||^2_{0,t^{n+1/2}} \\
&+ \frac{\mu \Delta t}{8} ||u_h^{n+1} + u_h^n||^2_{0,t^{n+1/2}} + \Delta t \sum_{K \in \mathcal{T}_{h,t^{n+1/2}}} \delta_K ||f^{n+1/2}||^2_{0,K} \\
&+ \frac{\Delta t}{8} \sum_{K \in \mathcal{T}_{h,t^{n+1/2}}} \delta_K ||(\mathbf{b} - \mathbf{w}_h) \cdot \nabla(u_h^{n+1} + u_h^n)||^2_{0,K}.
\end{aligned}$$

Absorbing the right hand side terms into the SUPG norm and using integration by parts for the mesh velocity term we get

$$\begin{aligned}
&||u_h^{n+1}||^2_{0,t^{n+1}} + \frac{\Delta t}{8}|||(u_h^{n+1} + u_h^n)|||^2_{t^{n+\frac{1}{2}}} \\
&\quad \le \Delta t \int_{\Omega_{t^{n+\frac{1}{2}}}} \nabla \cdot \mathbf{w}_h |u_h^n|^2 \, dx - \frac{\Delta t}{4} \int_{\Omega_{t^{n+\frac{1}{2}}}} \nabla \cdot \mathbf{w}_h |u_h^{n+1} + u_h^n|^2 \, dx \\
&\qquad + ||u_h^n||^2_{0,t^n} + \frac{\Delta t}{\mu} ||f^{n+\frac{1}{2}}||^2_{0,t^{n+\frac{1}{2}}} + \Delta t \sum_{K \in \mathcal{T}_{h,t^{n+\frac{1}{2}}}} \delta_K ||f^{n+\frac{1}{2}}||^2_{0,K} \\
&\quad \le \Delta t \int_{\Omega_{t^{n+\frac{1}{2}}}} \nabla \cdot \mathbf{w}_h \left(|u_h^n|^2 - \frac{1}{4}|u_h^{n+1} + u_h^n|^2 \right) \, dx \\
&\qquad + ||u_h^n||^2_{0,t^n} + \frac{\Delta t}{\mu} ||f^{n+\frac{1}{2}}||^2_{0,t^{n+\frac{1}{2}}} + \Delta t \sum_{K \in \mathcal{T}_{h,t^{n+\frac{1}{2}}}} \delta_K ||f^{n+\frac{1}{2}}||^2_{0,K} \\
&\quad \le \Delta t \int_{\Omega_{t^{n+\frac{1}{2}}}} \nabla \cdot \mathbf{w}_h \left(|u_h^n|^2 + |u_h^{n+1}|^2 \right) \, dx + ||u_h^n||^2_{0,t^n} \\
&\qquad + \frac{\Delta t}{\mu} ||f^{n+\frac{1}{2}}||^2_{0,t^{n+\frac{1}{2}}} + \Delta t \sum_{K \in \mathcal{T}_{h,t^{n+\frac{1}{2}}}} \delta_K ||f^{n+\frac{1}{2}}||^2_{0,K}.
\end{aligned}$$

Using the ALE map and its Jacobian, the equation becomes

$$\begin{aligned}
&||u_h^{n+1}||^2_{0,t^{n+1}} + \frac{\Delta t}{4}|||(u_h^{n+1} + u_h^n)|||^2_{t^{n+\frac{1}{2}}} \\
&\le \Delta t ||\nabla \cdot \mathbf{w}_h||_{\infty,t^{n+\frac{1}{2}}} ||u_h^{n+1}||^2_{0,t^{n+\frac{1}{2}}} + \Delta t ||\nabla \cdot \mathbf{w}_h||_{\infty,t^{n+\frac{1}{2}}} ||u_h^n||^2_{0,t^{n+\frac{1}{2}}} \\
&+ \frac{\Delta t}{\mu} ||f^{n+\frac{1}{2}}||^2_{0,t^{n+\frac{1}{2}}} + \Delta t \sum_{K \in \mathcal{T}_{h,t^{n+\frac{1}{2}}}} \delta_K ||f^{n+\frac{1}{2}}||^2_{0,K}.
\end{aligned}$$

Further, with the notation

$$\begin{aligned}
\beta_1^{n+1} &= \left|\left| J_{\mathcal{A}_{t_{n+1},\ t_{n+\frac{1}{2}}}} \right|\right|_{\infty,t^{n+1}} ||\nabla \cdot \mathbf{w}_h||_{\infty,t^{n+\frac{1}{2}}}, \\
\beta_2^n &= \left|\left| J_{\mathcal{A}_{t_n,\ t_{n+\frac{1}{2}}}} \right|\right|_{\infty,t^n} ||\nabla \cdot \mathbf{w}_h||_{\infty,t^n},
\end{aligned}$$

the inequality turns out to be

$$\begin{aligned}||u_h^{n+1}||^2_{0,t^{n+1}} &+ \frac{\Delta t}{4}|||(u_h^{n+1}+u_h^n)|||^2_{t^{n+1/2}}\\ &\leq \Delta t\beta_1^{n+1}||u_h^{n+1}||^2_{0,t^{n+1}} + (1+\Delta t\beta_2^n)||u_h^n||^2_{0,t^n}\\ &\quad + \frac{\Delta t}{\mu}||f^{n+1/2}||^2_{0,t^{n+1/2}} + 2\Delta t\sum_{K\in\mathcal{T}_{t^{n+1/2}}}\delta_K||f^{n+1/2}||^2_{0,K}.\end{aligned}$$

Summing over $i = 0, 1, 2, \ldots, n$, and using the assumption on δ_k, we obtain

$$\begin{aligned}&||u_h^{n+1}||^2_{0,t^{n+1}} + \frac{\Delta t}{4}\sum_{i=0}^{n}|||(u_h^{i+1}+u_h^i)|||^2_{t^{i+1/2}}\\ &\leq \Delta t\beta_1^{n+1}||u_h^{n+1}||^2_{0,t^{n+1}} + \Delta t\sum_{i=1}^{n}(\beta_1^i+\beta_2^i)||u_h^i||^2_{0,t^i}\\ &+ (1+\Delta t\beta_2^0)||u_h^0||^2_{0,t^0} + \sum_{i=0}^{n}\left(\frac{\Delta t}{\mu}||f^{i+1/2}||^2_{0,t^{i+1/2}} + \frac{\Delta t^2}{2}||f^{i+1/2}||^2_K\right)\\ &\leq \Delta t\sum_{i=1}^{n+1}(\beta_1^i+\beta_2^i)||u_h^i||^2_{0,t^i} + (1+\Delta t\beta_2^0)||u_h^0||^2_{0,t^0}\\ &\quad + \Delta t\sum_{i=0}^{n}\left(\frac{2}{\mu}+\frac{\Delta t}{2}\right)\|f^{i+1/2}\|^2_{0,t^{i+1/2}}.\end{aligned}$$

Finally, using Grownwall's lemma, we get

$$\begin{aligned}\|u_h^{n+1}\|^2_{0,t^{n+1}} &+ \frac{\Delta t}{4}\sum_{i=0}^{n}|||(u_h^{i+1}+u_h^i)|||^2_{t^{i+1/2}}\\ &\leq \left((1+\Delta t\beta_2^0)\|u_h^0\|^2_{0,t^0} + \Delta t\sum_{i=0}^{n}\left(\frac{2}{\mu}+\Delta t\right)\|f^{i+1/2}\|^2_{0,i+1/2}\right)\cdot\\ &\qquad \exp\left(\Delta t\sum_{i=1}^{n+1}\frac{\beta_1^i+\beta_2^i}{1-\Delta t(\beta_1^i+\beta_2^i)}\right).\end{aligned}$$

The estimate is stable provided we have

$$\Delta t < \frac{1}{\beta_1^n + \beta_2^n} = \Big(\Big|\Big| J_{\mathcal{A}_{t_n,\ t_{n-1/2}}} \Big|\Big|_{\infty,t^n} ||\nabla \cdot \mathbf{w}_h||_{\infty,t^{n-1/2}} + \Big|\Big| J_{\mathcal{A}_{t_n,t_{n+1/2}}} \Big|\Big|_{\infty,t^n} ||\nabla \cdot \mathbf{w}_h||_{\infty,t_{n+1/2}} \Big)^{-1}.$$

Thus, the scheme is only conditionally stable with the time step restriction given above. □

3.3 Discrete ALE-SUPG with backward-difference (BDF2) time discretization

We now consider the backward difference method of order two for temporal discretization. For the equation (3.3), the backward-difference method gives

$$\frac{3}{2}u^{n+1} - 2u^n + \frac{1}{2}u^{n-1} = \Delta t\ f(u^{n+1}, t^{n+1}).$$

Lemma 5. Stability estimates for the non-conservative ALE-SUPG form applying backward-difference formula: *Let the discrete version of* (2.8) *and assumption* (2.9) *on* δ_K *hold true. Further, assume* $\delta_K \leq \frac{\Delta t}{4}$*. Then the solution satisfies*

$$||u_h^{n+1}||^2_{0,t^{n+1}} + ||2u_h^{n+1} - u_h^n||^2_{0,t^n} + \Delta t \sum_{i=1}^{n+1} |||u_h^i(t)|||^2_{t^i} \leq$$

$$\leq \left((1 + \Delta t \alpha_2^0) \|u_h^0\|^2_{0,t^0} + ||2u_h^1 - u_h^0||^2_{0,t^1} + \Delta t \sum_{i=1}^{n+1} \left(\frac{2}{\mu} + \frac{\Delta t}{2} \right) \|f^i\|^2_{0,t^i} \right)$$

$$\exp\left(\Delta t \sum_{i=1}^{n+1} \frac{2\alpha_1^i + \alpha_2^i}{1 - \Delta t(2\alpha_1^i + \alpha_2^i)} \right).$$

Proof. Applying the backward difference temporal discretization to the semi-discrete equation (5) with the test function u_h^{n+1}, we get

$$\begin{aligned}\left(\frac{3}{2}u_h^{n+1} - 2u_h^n + \frac{1}{2}u_h^{n-1}, u_h^{n+1}\right)_{\Omega_{t^{n+1}}} &+ \Delta t\ a_{SUPG}^{n+1}(u_h^{n+1}, u_h^{n+1})_t \\ - \frac{\Delta t}{2}\int_{\Omega_{t^{n+1}}} \mathbf{w}_h^{n+1}\cdot\nabla((u_h^{n+1})^2)\ dx &= \Delta t \int_{\Omega_{t^{n+1}}} f^{n+1}u_h^{n+1}\ dx \\ &+ \Delta t \sum_{K\in\mathcal{T}_{t^{n+1}}} \delta_K \int_K f^{n+1}\ \left(\mathbf{b} - \mathbf{w}_h^{n+1}\right)\cdot\nabla u_h^{n+1}\ dK.\end{aligned}$$

The first term can be written as

$$\begin{aligned}\left(\frac{3}{2}u_h^{n+1} - 2u_h^n + \frac{1}{2}u_h^{n-1}, u_h^{n+1}\right)_{\Omega_{t^{n+1}}} &= \frac{1}{4}\Big(||u_h^{n+1}||^2_{0,t^{n+1}} - ||u_h^n||^2_{0,t^n} \\ + ||2u_h^{n+1} - u_h^n||^2_{0,t^{n+1}} - ||2u_h^n - u_h^{n-1}||^2_{0,t^n} &+ ||u_h^{n+1} - 2u_h^n + u_h^{n-1}||^2_{0,t^{n+1}} \\ + \int_{t^n}^{t^{n+1}}\int_{\Omega_t} \nabla\cdot\mathbf{w}_h^{n+1}(t)\Big[(u_h^n)^2 + \Big(u_h^n - u_h^{n-1}\Big)^2\Big]\Big).\end{aligned} \tag{3.4}$$

We substitute equation (3.4) for the first term and use the same estimates as we worked out in previous sections so that the fully discrete equation becomes

$$\begin{aligned}&\frac{1}{4}\left(||u_h^{n+1}||^2_{0,t^{n+1}} + ||2u_h^{n+1} - u_h^n||^2_{0,t^{n+1}} + ||u_h^{n+1} - 2u_h^n + u_h^{n-1}||^2_{0,t^{n+1}}\right) \\ &+ \frac{\Delta t}{4}|||u_h^{n+1}|||^2_{t^{n+1}} \le \frac{1}{4}\left(||u_h^n||^2_{0,t^n} + ||2u_h^n - u_h^{n-1}||^2_{0,t^n}\right) \\ &+ \frac{1}{4}\int_{t^n}^{t^{n+1}}\int_{\Omega_t} \nabla\cdot\mathbf{w}_h^{n+1}(t)\Big[(u_h^n)^2 + (u_h^n - u_h^{n-1})^2\Big] + \frac{2\Delta t}{\mu}||f^{n+1}||^2_{0,t^{n+1}} \\ &+ \frac{\Delta t}{2}||\nabla\cdot\mathbf{w}_h^{n+1}||_{\infty,t_{n+1}}||u_h^{n+1}||^2_{0,t^{n+1}} + 2\Delta t\sum_{K\in\mathcal{T}_{t^{n+1}}}\delta_K||f^{n+1}||^2_K.\end{aligned}$$

Summing over $i = 0, 1, 2, \ldots, n$, and with the same notations as in the implicit Euler case, we get

$$
\begin{aligned}
&||u_h^{n+1}||^2_{0,t^{n+1}} + ||2u_h^{n+1} - u_h^n||^2_{0,t^{n+1}} + ||u_h^{n+1} - 2u_h^n + u_h^{n-1}||^2_{0,t^{n+1}} \\
&+ \Delta t \sum_{i=0}^{n} |||u_h^{i+1}|||^2_{t^{i+1}} \leq 2\Delta t \alpha_1^{n+1} ||u_h^{n+1}||^2_{0,t^{n+1}} + ||u_h^0||^2_{0,t^0} + ||2u_h^1 - u_h^0||^2_{0,t^1} \\
&+ \Delta t \alpha_2^0 ||u_h^0||^2_{0,t^0} + \Delta t \sum_{i=1}^{n} (2\alpha_1^i + \alpha_2^i) ||u_h^i||^2_{0,t^i} + \\
&+ 4\Delta t \sum_{i=1}^{n} \left(\frac{2}{\mu} + \frac{\Delta t}{2}\right) \|f^{i+1}\|^2_{0,t^{i+1}} \\
&\qquad \leq (1 + \Delta t \alpha_2^0) ||u_h^0||^2_{0,t^0} + ||2u_h^1 - u_h^0||^2_{0,t^1} + \Delta t \sum_{i=1}^{n+1} (2\alpha_1^i + \alpha_2^i) ||u_h^i||^2_{0,t^i} \\
&\qquad\qquad + 4\Delta t \sum_{i=1}^{n+1} \left(\frac{2}{\mu} + \frac{\Delta t}{2}\right) \|f^i\|^2_{0,t^i}.
\end{aligned}
$$

By using Grownwall's lemma we get the stability estimate provided we have

$$
\Delta t < \frac{1}{\sup_{n \in [0,N]} (2\alpha_1^n + \alpha_2^n)}.
$$

Here, we can see the stability estimate is again only conditionally stable with the time step restriction depending on the ALE map. □

4 Numerical results

Numerical results obtained with the proposed numerical scheme for the considered model problem are presented in this section. We first consider a standard example without convection and compare the standard Galerkin solutions with different time steps Δt and the SUPG solution with varying δ_0, which is a numerical parameter in $\delta_K (= \delta_0 h_{K,t} / |\mathbf{w}|_{\infty,t})$. We next consider a PDE in a time-dependent domain with boundary and interior layers. In this example, we study the influence of different time discretizations on the stabilized solution of the PDE. The system of matrix resulting from the considered numerical scheme is solved by using the direct solver UMFPACK [17]. All computations are performed using our in-house finite element package, ParMooN, see [18] for more details.

The piecewise quadratic finite elements are used for the spatial discretization. The first order backward Euler, the second order Crank-Nicolson, and the backward-difference(BDF2) are used for the temporal discretization. Numerical solutions obtained with the standard Galerkin and the SUPG method are presented.

4.1 Example 1

The purpose of this example is to study the influence of the time-step, Δt, the stabilization parameter, δ_0, on the stabilized ALE solution obtained with different time-discretization methods. Moreover, we consider a problem with zero convection. Nevertheless, the ALE formulation introduces a convective mesh velocity in the PDE, see equation (2.2). Therefore, the effects of the SUPG stabilization depend solely on the mesh movement.

Let the scalar time-dependent equation be

$$\begin{aligned} \frac{\partial u}{\partial t} - \epsilon\Delta u &= 0 && \text{in } \Omega_t \times (0, \mathrm{T}], \\ u &= 0 && \text{on } \partial\Omega_t \times [0, \mathrm{T}], \\ u|_{t=0} &= 1600\ Y_1(1 - Y_1)\ Y_2(1 - Y_2) && \text{in } \Omega_0, \end{aligned}$$

where $\epsilon = 0.01$ and $\Omega_0 = (0,1)^2$ is the initial (reference) domain, $\hat{\Omega}$. Further, the position of the time-dependent domain $x \in \Omega_t$ at time t is ruled by

$$x(Y,t) = \mathcal{A}_t(Y) \text{ such that} \left\{ \begin{array}{l} x_1 = Y_1(2 - \cos(20\pi t)) \\ x_2 = Y_2(2 - \cos(20\pi t)), \end{array} \right. \qquad (Y_1, Y_2) \in \hat{\Omega}.$$

In computations, the piecewise linear interpolation in time is used to obtain the new position of the domain, that is, $x(Y, s) \in \Omega_s$ for $s \in$

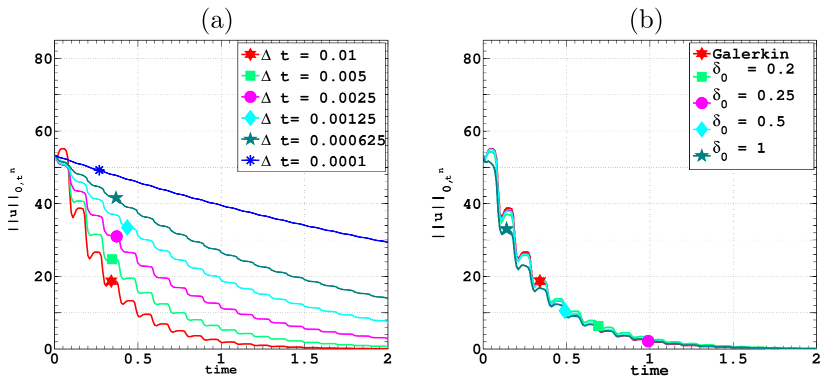

Figure 1: *L^2-norm of the solution obtained with implicit Euler time discretization, (a) Galerkin solution with different time-steps Δt, (b) SUPG solution for $\Delta t = 0.01$ with different stabilization parameters δ_0.*

$[t^n, t^{n+1}]$ given by

$$x(Y,s) = \frac{s-t^n}{\Delta t} x\left(Y, t^{n+1}\right) + \frac{t^{n+1}-s}{\Delta t} x\left(Y, t^n\right).$$

Hence, the domain velocity is obtained as

$$\mathbf{w}(Y,s) = \frac{x\left(Y, t^{n+1}\right) - x\left(Y, t^n\right)}{\Delta t}.$$

For the considered displacement, the square domain will expand and shrink over a period of time, but the origin will remain intact.

We triangulate the square domain with $8,192$ triangles. Further, piecewise linear finite elements P_1 are used for the spatial discretization, which results in $4,225$ degrees of freedom (dof). The standard Galerkin and the SUPG solutions obtained with the backward Euler time discretization for different Δt and δ_0 with $\Delta t = 0.01$, respectively, are depicted in Figure 1. As expected, the backward Euler solution is

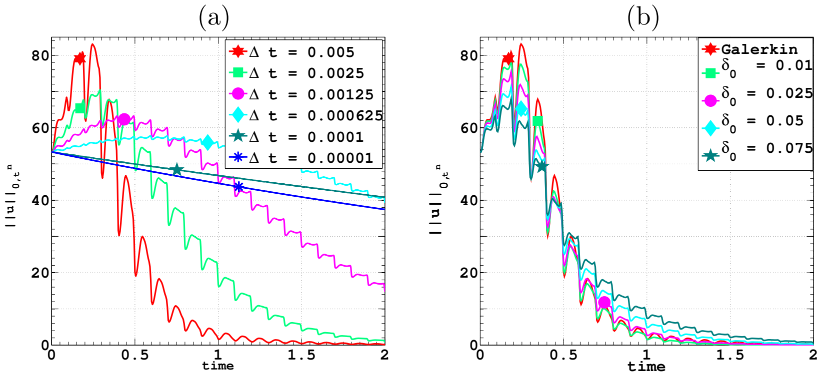

Figure 2: *L^2-norm of the solution obtained with Crank-Nicolson time discretization, (a) Galerkin solution with different time-steps Δt, (b) SUPG solution for $\Delta t = 0.005$ with different stabilization parameters δ_0.*

more diffusive and the diffusivity effect reduces when the time step is reduced, see Figure 1 (a). In addition to the diffusivity effect, a periodic oscillation in the solution due to the mesh movement is observed when the time step is large. These oscillations reduce with smaller time steps, and the solution becomes monotone. This behavior supports our stability analysis that the time step depends on the mesh velocity, see Lemma 3. Figure 1 (b) presents the SUPG solutions for different δ_0, where the oscillatory case $\Delta t = 0.01$ from Figure 1 (a) is considered. This parametric study is performed in order to see the effects of SUPG discretization. Though the Euler solution is already more diffusive, the artificial diffusive effect of SUPG is not observed with the chosen time step $\Delta t = 0.01$. Since piecewise linear finite elements are used, SUPG induces diffusion only in the streamline (mesh velocity) direction.

Next, the standard Galerkin and the SUPG solutions obtained with the Crank-Nicolson time discretization for different Δt and δ_0 with $\Delta t =$

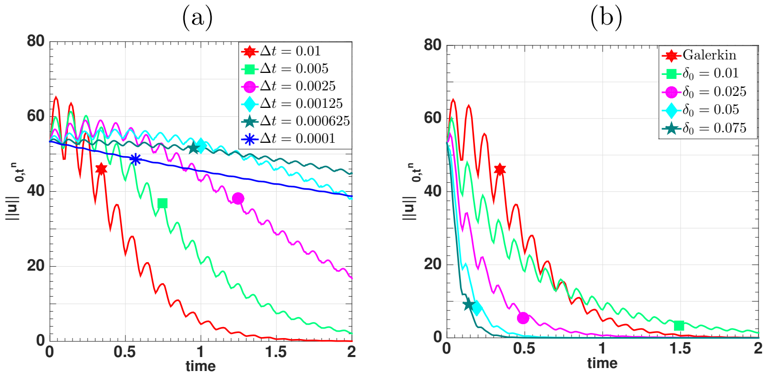

Figure 3: *L^2-norm of the solution obtained with backward difference time discretization (BDF2), (a) Galerkin solution with different time-steps Δt, (b) SUPG solution for $\Delta t = 0.01$ with different stabilization parameters δ_0.*

0.005, respectively, are depicted in Figure 2. Recall that the Crank-Nicolson method is only A-stable and not strongly A-stable even in a stationary domain. In addition, the time step depends on the mesh velocity, see Lemma 4. Thus the influence of the mesh velocity is very high in the Crank-Nicolson solution, especially when Δt is large, see Figure 2 (a). The Galerkin solution is oscillatory except for the cases $\Delta t = 0.0001$ and $\Delta t = 0.00001$. The reason for high oscillations in the Crank-Nicolson method can be due to the assembling of the stiffness matrix on the $\Omega_{t^{n+1/2}}$ domain, which induces more numerical error. Further, the SUPG discretization suppresses the oscillations when the value of δ_0 is increased. More interestingly, the SUPG solution become less diffusive compared to the Galerkin solution, see Figure 2 (b).

Finally, the backward-difference (BDF2) solution with the standard Galerkin and the SUPG discretizations for different Δt and δ_0 with $\Delta t = 0.01$, respectively, are depicted in Figure 3. Though there are oscillations in the BDF2 solution due to the mesh velocity, oscillations are less, compared to the Crank-Nicolson solution, see Figure 3 (a) and Figure 2 (a). Moreover, the influence of the SUPG parameter is very high and the solutions become more diffusive, see Figure 3 (b), when the value of δ_0 is increased, which is the inherent property of the SUPG discretization.

4.2 Example 2

The purpose of this example is to study the effect of different time discretizations and the ALE formulations (conservative, non-conservative ALE forms) on the SUPG solution of advection dominated PDEs in time-dependent domain. Thus, a typical fluid-structure interaction (FSI) problem, in which a flow passing through a rectangular structure (beam) that deforms with time, is considered. A predefined adaptive mesh with high resolution in the vicinity of the deforming structure is considered. Nevertheless, the mesh away from the structure is comparatively coarser.

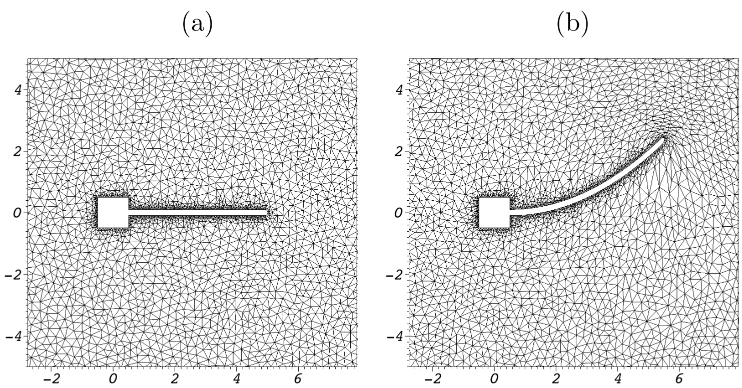

Figure 4: *The schematic diagram of the domain with unstructured triangular mesh as illustrated in Example 2. (a) Beam at the initial position and (b) beam at its maximum amplitude.*

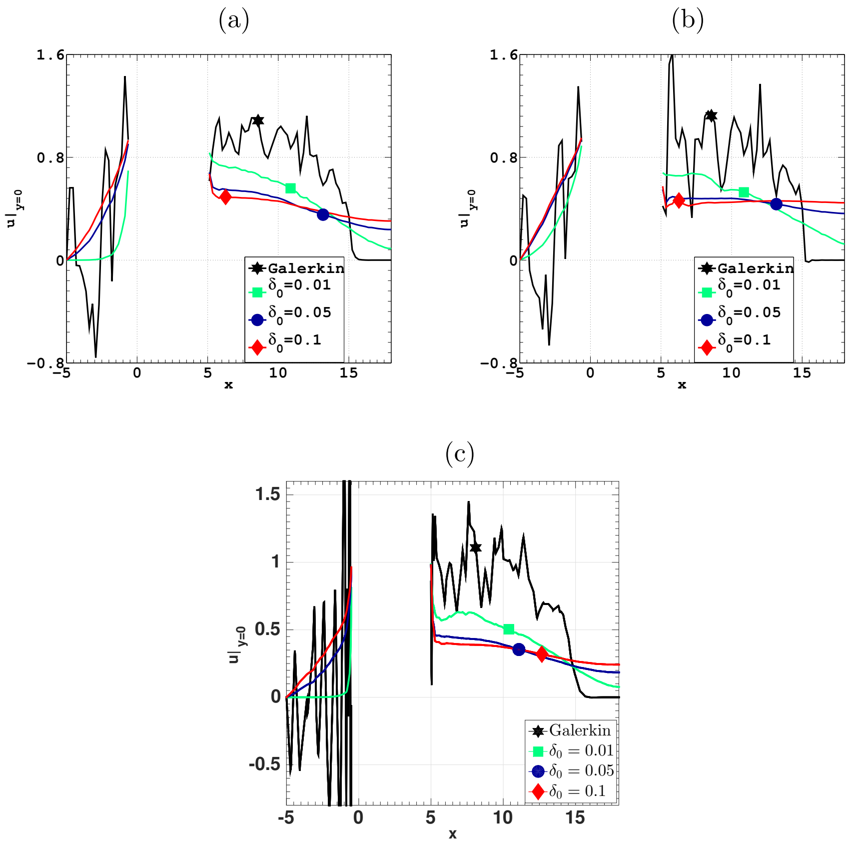

Figure 5: *Galerkin and SUPG solutions with different values of δ_0 over the line $y = 0$ in Example 2. (a) Implicit Euler, (b) Crank-Nicolson, and (c) backward difference (BDF2).*

Further, the tip of the beam is considered to be semi-circular to do away with the singularities that might occur due to the sharp corners. Let

$$\Omega_0^S = \{(-0.5, 0.5) \times (-0.5, 0.5)\} \,\cup\, \{(0.5, 4.5) \times (-0.03, 0.03)\}$$

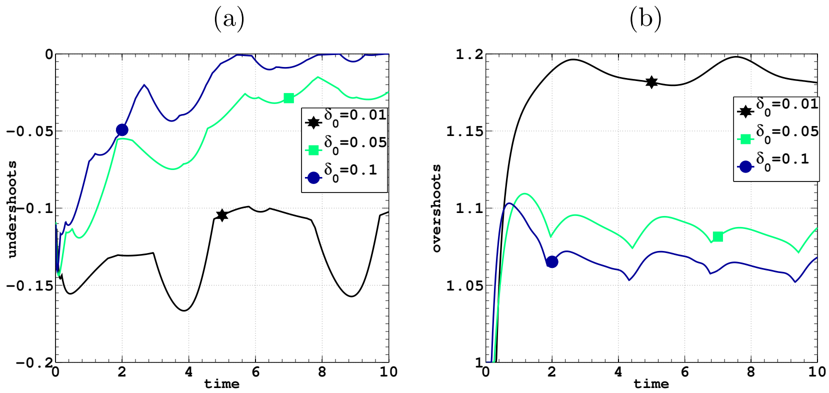

Figure 6: *The observed undershoots/overshoots with implicit Euler time discretization in the SUPG solution for different values of δ_0: (a) undershoots and (b) overshoots.*

be the rectangular structure (beam) at time $t = 0$. The two-dimensional channel that excludes the beam Ω_0^S is defined by

$$\Omega_0 = \{(-5, 18) \times (-5, 5)\} \setminus \bar{\Omega}_0^S.$$

Further, we define $\Gamma_D = \{-5\} \times (-5, 5)$ as the Dirichlet boundary and $\Gamma_N = \partial\Omega_t \setminus \Gamma_D$ as the Neumann boundary. Moreover, we consider the initial domain as the reference domain for the ALE mapping, that is, $\hat{\Omega} = \Omega_0$. Now, the coordinates of the point in time-dependent domain $(x_1, x_2) \in \Omega_t$ are defined by

$$x(Y,t) = \mathcal{A}_t(Y) \begin{cases} x_1 = Y_1 + 0.05(0.25\ d\ \tan\theta - Y_2\ \sin\theta) \\ x_2 = Y_2 + 0.05d, \end{cases} \quad (Y_1, Y_2) \in \hat{\Omega},$$

where the sinusoidal movement of the beam is prescribed by

$$d = 0.75(x_1 - 0.5)^2 \sin(2\pi t/5), \quad \theta = \tan^{-1}\left(\frac{x_2}{x_1 - 0.5}\right).$$

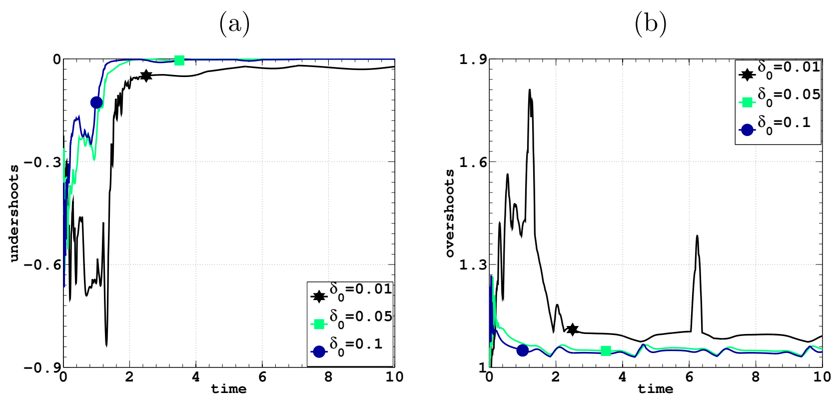

Figure 7: *The observed undershoots/overshoots with Crank-Nicolson time discretization in the SUPG solution for different values of δ_0: (a) undershoots and (b) overshoots.*

We now solve a scalar (energy) equation

$$\frac{\partial u}{\partial t} - \epsilon \Delta u + \mathbf{b} \cdot \nabla u = 0, \qquad \text{in } \Omega_t \times (0, \mathrm{T}],$$

with $\epsilon = 10^{-6}$ and $\mathbf{b}(x_1, x_2) = (1, 0)^T$. Further, we impose the homogeneous Neumann condition on Γ_N, and set

$$u_D(x_1, x_2) = \begin{cases} 1 & \text{on } \partial\Omega_t^S, \\ 0 & \text{else,} \end{cases}$$

on the Dirichlet boundary. Boundary and interior layers in the vicinity and the wake region, respectively, of the solid beam will occur due to a small diffusive constant. Since the solid beam deforms (up and down) periodically, the position of the boundary and the interior layers will also change with time. The computations are performed up to the dimensionless time $T = 10$ with a time step $\Delta t = 0.01$. Further, an elastic-solid

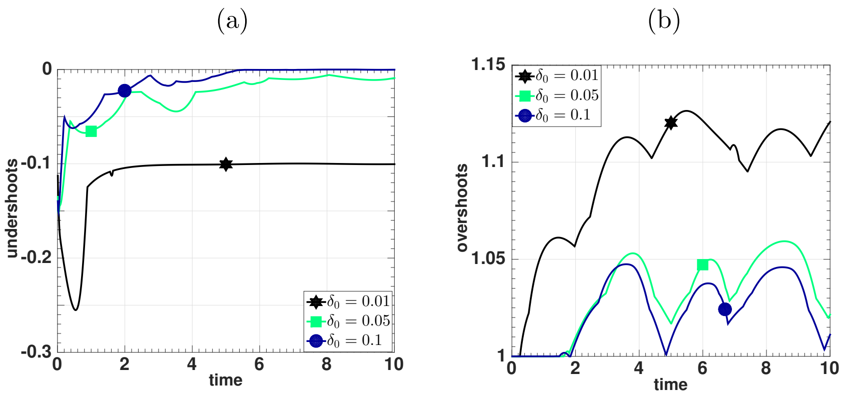

Figure 8: *The observed undershoots/overshoots with backward difference time discretization in the SUPG solution for different values of δ_0: (a) undershoots and (b) overshoots.*

mesh update technique is used to handle the mesh movement. At each time step, we first compute the displacement of the beam and then solve the linear elastic equation in Ω_{t^n} to compute the inner points displacement by considering the displacement on $\partial\Omega^S_{t^{n+1}}$ as the Dirichlet value. This elastic update technique avoids the re-meshing during the entire simulation, see Figure 4.

In order to understand the behavior of the Galerkin and SUPG solutions with different values of δ_0, numerical solutions at the middle of the channel, that is, at $y = 0$, are depicted in Figure 5 for different time discretizations. As expected, the Galerkin solutions obtained with (a) implicit Euler, (b) Crank-Nicolson, and (c) backward difference (BDF2) discretizations contain spurious oscillations and instabilities, see the Galerkin solutions in Figure 5, whereas the oscillations are suppressed in the SUPG solution. Moreover, the boundary layer is approximated sharply by any smaller value of δ_0.

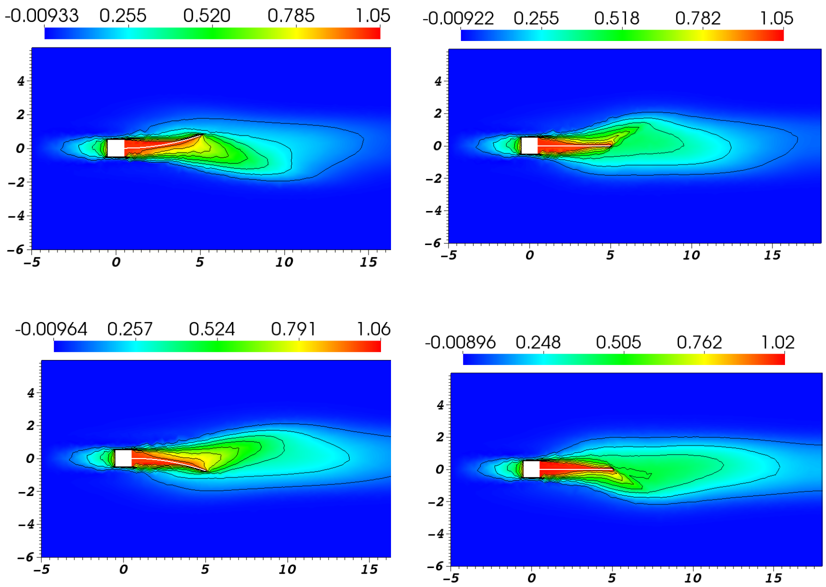

Figure 9: *The sequence of solutions obtained with the SUPG* $\delta_0 = 0.05$ *method for implicit Euler time discretization at different instances* $t = 6.3, 7.5, 8.8, 10$.

Next, to get an insight into the robustness and effectiveness of the SUPG stabilization, the undershoots and the overshoots in the SUPG solution for different values of δ_0 are examined in this example. An array of computations with (i) $\delta_0 = 0.01$, (ii) $\delta_0 = 0.05$, and (iii) $\delta_0 = 0.1$ are performed, and the results are presented in Figure 6, 7, and 8. We can observe that the undershoots and overshoots are lesser in the implicit Euler method than the other Crank-Nicolson, and backward-difference schemes at the initial period of time until $t = 2$ (note that the scaling of figures are different). Though the oscillations in the SUPG solution obtained with the Crank-Nicolson scheme are more frequent at the initial

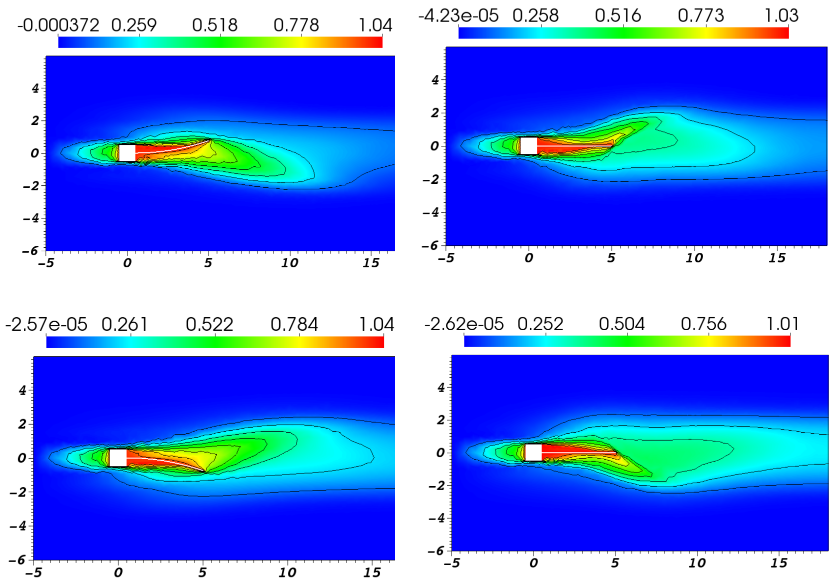

Figure 10: *The sequence of solutions obtained with the SUPG* $\delta_0 = 0.05$ *method for Crank-Nicolson time discretization at different instances* $t = 6.3, 7.5, 8.8, 10$.

stage, for us until $t = 2$, the undershoots and overshoots at the later stage are minimum compared to the other two cases, see Figure 7. Further, the undershoots and overshoots in the SUPG solutions with $\delta_0 = 0.05$ and (ii) $\delta_0 = 0.1$ are similar in all time discretizations. Nevertheless, a smaller value of δ_0 is preferred in order to avoid the smearing effect, which will be discussed next.

Though the SUPG approximation almost suppressed the spurious oscillations in the numerical solution, there are very small overshoots and undershoots (around 10%) for the chosen $\delta_0 = 0.05$. One could reduce these undershoots and overshoots by increasing the value of δ_0 further.

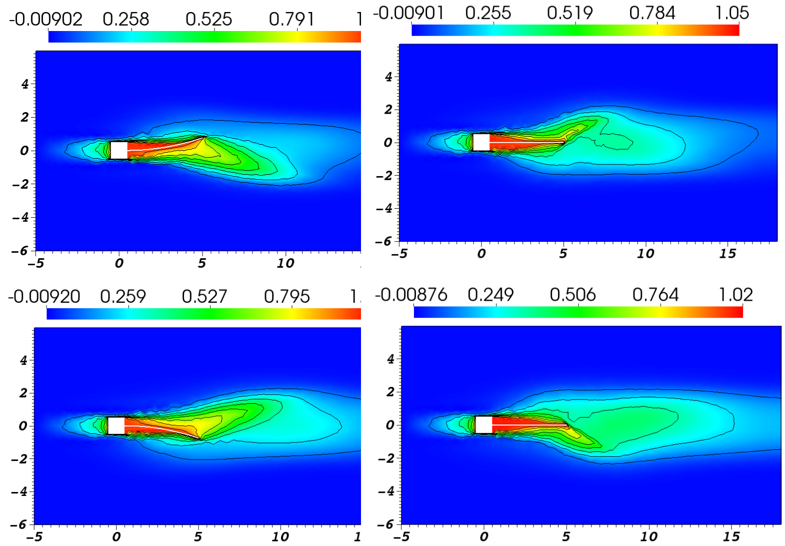

Figure 11: *The sequence of solutions obtained with the SUPG* $\delta_0 = 0.05$ *method for backward difference time discretization at different instances* $t = 6.3, 7.5, 8.8, 10$.

However, it will smear the solution. This is a well-known behavior of the SUPG method in stationary domains. Thus, the optimal choice of the stabilization parameter varies with numerical examples, and it is an open problem, see [20]. Plots in Figure 5 show the smearing effect of the boundary layer. In order to see the smearing effect in the entire domain, a sequence of SUPG solution profiles obtained with $\delta_0 = 0.05$ at different instances $t = 6.3, 7.5, 8.8, 10$ are depicted in Figures 9, 10, and 11. In these plots, the range of values in the caption for the respective time instances among all time discretizations is kept the same for better comparison. We can observe from these plots that the solutions obtained with the backward Euler and BDF2 discretizations are more diffusive

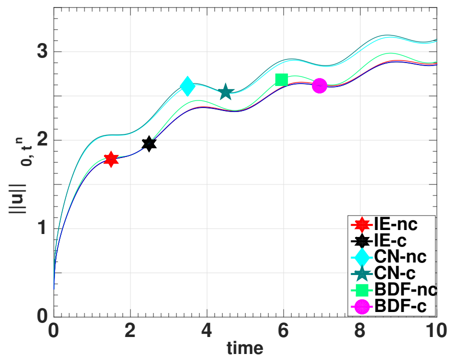

Figure 12: *The L^2-norm of the solution for Example 2 with all the time discretizations for both the conservative and non-conservative case.*

than the solution of the Crank-Nicolson discretization.

Finally, to study quantitatively the stabilization effects on the solution, the variation in the total energy of the system over the period of time in different time discretizations are plotted in Figure 12. In addition to different discretizations, solutions obtained with both the conservative and the non-conservative ALE forms are also compared in Figure 12.

Since the backward Euler and BDF2 solutions are more diffusive than the Crank-Nicolson solution, the total energy in the system is slightly less due to the homogeneous Dirichlet boundary condition on the side wall. Nevertheless the variation is very small (≈ 0.2). More interestingly, the difference between the conservative and the non-conservative solutions obtained with different time discretization is negligible in this numerical example. Note that the domain velocity is not very high in this particular example. Therefore, either the conservative or non-conservative scheme can be used. The non-conservative is preferred for domains with small deformation, since a reaction type mesh velocity term and the GCL condition can be avoided.

5 Summary

The ALE-SUPG finite element scheme for convection dominated transient convection-diffusion equation in time-dependent domains is presented in this paper. Further, the influence of the backward Euler, the Crank-Nicolson, and the backward difference (BDF2) temporal discretizations on the solution of the non-conservative ALE-SUPG finite element method is investigated. It is observed that the Crank-Nicolson scheme induces high oscillations in the numerical solution compared to the implicit Euler and the backward difference time discretizations. Further, the difference between the solutions obtained with the conservative and non-conservative ALE forms highly depends on the deformation of the domain. The difference is significant when the deformation of the domain is large, whereas it is negligible in domains with small deformation. Since the deformation in most of the FSI problems is moderate, as the volume of the domain does not vary too much and the divergence of mesh velocity calculation can be avoided, the non-conservative ALE form is preferred. Next, the influence of the SUPG stabilization parameter on the solution is presented. It is observed that the Crank-Nicolson scheme is less dissipative than the implicit Euler and the backward difference method. Moreover, the backward difference scheme is more sensitive to the stabilization parameter δ_k than the other time discretizations.

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Resumen: El presente artículo desarrolla el análisis numérico de una ecuación escalar con convección dominada y distintas discretizaciones temporales en dominios dependientes del tiempo. Para la discretización temporal se hará uso de los métodos en reversa de Euler, el de Crank-Nicolson y otros métodos de diferencias finitas en reversa. El dominio dependiente del tiempo es tratado desde un enfoque lagrangiano-euleriano arbitrario (ALE). Particularmente, consideramos la forma no conservativa del enfoque ALE. Además, empleamos el método de Petrov-Galerkin (SUPG) para discretización espacial. Se prueba que la estabilidad de la solución completamente discreta, independiente de la discretización temporal, es solo condicionalmente estable. Además, se estudia la dependencia de la solución numérica respecto al parámetro estabilizador δ_k. Se corrobora que el esquema Crank-Nicolson es menos disipativo que el método implícito de Euler y el método de diferencias en reversa. Más aun, el esquema de diferencias en reversa resulta más sensible al parámetro estabilizador δ_k que otras discretizaciones temporales.

Palabras clave: Ecuación reacción-convección-difusión, estabilización SUPG, ley de conservación geométrica (GCL), dominio dependiente del tiempo, enfoque lagrangiano-euleriano arbitrario, discretizaciones temporales.

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