Mathematics, Synthetic Unity and the Constitutive *A Priori*

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Abstract: Starting from the emphasis on the Kantian notion of the original synthetic unity of apperception as the basic function on which the production of synthetic judgment lays, in this paper I will defend the idea of the constitutive *a priori* of experience under the guise of a synthetic principle that is a similar, if essentially formal, version of it; namely, the concepts of transformation and group of transformations. Simultaneously, I shall try to prove that Kant based this particular conception on mathematics generally, and synthetic geometry specifically.
Key words: Epistemology, Kant, constitutive *a priori*, mathematics

Introduction

There can be no doubt that, from Kant´s point of view, to know an object means that a synthetic unity has been enacted in the multiplicity of intuition. The main task of the *Critique of Pure Reason* is to understand this mechanism and examine it in each of its conditions. It aims to show how the various fundamental forms of knowledge –sensation and pure intuition, the categories of pure understanding and the ideas of pure reason– work together, and how their interaction determines the theoretic conformation of reality. As for this determination, it is not drawn from the object; rather, it entails an act of spontaneity of the understanding.

Once we have breached the first stages of the transcendental investigations, the elucidation of the pure form of sensibility, and entered the realm of the analysis of understanding, Kant leads us to the reductive assumption that the fundamental function responsible for the constitutive synthesis of the objects of experience is the original synthetic unity of apperception through figurative synthesis[1]. In terms of this unifying principle,

[1] In the footnote to B134, Kant states, in a well-known phrase, that: "The synthetic unity of apperception is therefore the highest concept on which all use of understanding depends, including logic as a whole and, in conformity with it, transcen-

we can say that the pure intuitions of space and time, insofar as they are pure concepts of understanding, are different aspects or manifestations of the basic form of the unifying synthetic function. The original synthetic unity of apperception and the figurative synthesis thus comprise the fundamental creative activity of thought to Kant, by virtue of which every empiric or mathematic object of knowledge is progressively generated.

What is truly new to the Kantian vantage is what might be called the cosmo-image of theoretic knowledge that does not appear to us as "given", a finished product that is somehow pressed on us by the nature of things, but that is, in fact, the result of the free constitution of the mind, which is never arbitrary, and entirely subject to laws. Where are we to find the key to this idea of constitution, fundamental as it is to Kantian epistemology?

The importance acquired by mathematics during the XVIth century, both through the constitution of the scientific image and the construction of philosophical theories, is well-known. In my opinion, the idea for the constitutive *a priori* was formulated by Kant on the basis of a constructive model afforded him by geometry and by synthetic geometry in particular. We can, indeed, place Kant amongst a certain line of thinkers who rejected the Cartesian analytic methods and defended those of ancient geometry as the paradigm for the construction of the spatial. This method became extended and was applied not only to geometry but also to the rest of mathematics and to the general theory of definition. Among the philosophers adopting this approach are Hobbes, Spinoza and Leibniz. What these thinkers strove for was not only to import geometric methods into the core of epistemology, but to show us something deeper: namely, that the same conceptual function operating in the generation of mathematical thinking operates in the construction of empiric knowledge.

Thus, it seems to us that we must seek the key to the constitutive *a priori* of Kantian philosophy in this procedure of synthetic construction which has its roots in ancient geometry. Furthermore, if we were to interpret the Kantian constitutive *a priori* in this light, that is, on the basis of the notion of synthetic function, we might then be able to make sense of a certain later philosophy, the emphasis of which was in the constitutive role of certain formal principles which, despite being formal, served as veritable synthetic principles for the elements conforming experience. On these grounds, and by virtue of a new appeal to mathematics, we could formulate a fresh conception of the constitutive *a priori.*

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dental philosophy. What is more, this faculty is understanding itself". (Kant, I., *Critique of Pure Reason*, Madrid: Alfaguara, 1988; from this point: *CPR*).

The objective of this work is to show that, just like mathematics and geometry served as inspirations to the Kantian formulation of his concept of the constitutive *a priori*, they may yet serve us for this very aim. Because of this, I shall divide the present article into three main parts: in the first, I shall contrast the analytic and synthetic methods in geometry, focusing especially on the latter; the second section will be dedicated to surveying the Kantian notions of synthetic unity of apperception and figurative synthesis, under the precept that both are inspired in the basic tenets of synthetic geometry; finally, I will make use of the third section to suggest that the constitutive *a priori* that Kant formulated on the model of synthetic geometry can find a new formulation in the mathematical concept of group.

1. Synthetic and analytic geometry

Ancient geometry distinguishes between two methods. On the one hand is a method stemming from the assumption that desired results can be attained; as in when we suppose we have succeeded in performing a desired construction in the current sense of the word. Thence, and from the vantage of these suppositions, we argue “backwards”, so to speak, towards the very conditions that make our construction possible, and the ways in which it can be made. This is the analytic method, which has sometimes been attributed to Plato, but was not explicitly and systematically exploited until Descartes, whose very name is drawn from the “analytic” approach in question. The other method is the synthetic one. Its application consists in trying to enact the desired result through the actual performance of constructions and, what’s more important, this construction proceeds on the basis of simple elements corresponding to a fixed set of rules. What distinguishes both methods is, quite broadly speaking, that the analytic method does not perform actual constructions, while the synthetic one does.

The classical paradigm for the use of the synthetic method in geometry is found in the *Elements* of Euclides, which begin by presenting 23 definitions –thus settling most of the basic terms–, five postulates and five common notions. The postulates permit the execution of certain geometric constructions: joining two points with a line, tracing a circle with any radius and center, etc. As for the common notions, they are permissible deductions or rules of inference which can be applied outside mathematics: that things which equal the same thing also equal one another; that if equals are added to equals, then the wholes are equal, and so forth.

What is the structure of a proposition in Euclides’ geometry? First, there is the enunciation of a general proposition. For example, proposition

20 of the *Elements* reads: "In any triangle the sum of any two sides is greater than the remaining one"[2]. This part of the proposition was called the πρότᾶσις. But Euclides never proceeds singly on the basis of an enunciation: for every proposition, he demonstrates the way in which it is possible to construct a "specimen" of the figure enunciated in the πρότᾶσις. This construction is equivalent to a true proof of the proposition at hand. Thus, proposition 20 continues by saying: "Let ΑΒΓ be a triangle. I say that in the triangle ΑΒΓ the sum of any two sides is greater than the remaining one, that is, the sum of ΒΑ and ΑΓ is greater than ΒΓ, the sum of ΑΒ and ΒΓ is greater than ΑΓ, and the sum of ΒΓ and ΓΑ is greater than ΑΒ"[3]. This part of an Euclidean proposition was known as the ἒκϑεσς or exposition (the Latin translation of which was *expositio*).

The exposition or ἒκϑεσς is closely related to the next or third part, of an Euclidean proposition: the *auxiliary construction*. This part was often termed the *preparation* or the *organization* (κατασκεή), and involved declaring that the figure constructed in the exposition had to be completed through the tracing of some additional lines, points and circles. In our example, the preparation reads as follows: "Draw ΒΑ through to the point Δ, and make ΑΔ equal to ΓΑ. Join ΑΓ"[4].

The construction was followed by the ἀπόδειξις, or proof proper. No further constructions were performed. The proof involved a series of inferences concerning the figure that had been introduced in the exposition and completed in the auxiliary construction. These inferences made use of axioms, previous propositions and the properties of the figure that could be inferred from the way in which it had been drawn.

After having reached the desired conclusion about a particular figure, Euclides returned once again to the general enunciation, by saying, for example: "Therefore *in any triangle the sum of any two sides is greater than the remaining one*"[5].

It follows that, in synthetic geometry, the basic activity on which any demonstration rests is the construction which, in the sophists and in Plato, as well as in Euclides, had to be conducted through the use of two essential instruments: the ruler and the compass. Constructions with a ruler and a compass implied the following assumptions: (I) that any two points can be joined by a line segment; (II) that any line segment can be extended into a line; and (III) that a circle can be traced with any center and radius. These

[2] Euclides, *Elements of geometry*, Madrid: Gredos, 2000, p. 43.
[3] *Ibid.*
[4] *Ibid.*
[5] *Ibid,* p. 44

three assumptions, usually attributed to Plato, correspond to the first three of the five Euclidean postulates. The two remaining postulates are: (IV) any two straight angles are equal; and (V) that, if a straight line falls on two straight lines and makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, will meet on that side on which the angles that amount to less than the two right angles are. These two postulates, and the last one notably, have posed problems to mathematicians. Since Euclides avoided using the fifth postulate until it was necessary for the proof of proposition 29 (of Book I), it seems reasonable to assume that he, too, felt somewhat uncertain about it. But what I would like to highlight here is that the five Euclidian postulates include assumptions which are at the ground of geometric constructions with a ruler and compass.

This emphasis on constructions is also evident in the first three propositions of Book I: "Proposition I. To construct an equilateral triangle on a given finite straight line… Proposition II. To place a straight line equal to a given straight line with one end at a given point…. Proposition III. To cut off from the greater of two given unequal straight lines a straight line equal to the less."[6]. The constructions of lines and figures are not mere material contrivances with didactical aims; they actually help to determine their existence. The genre of existence can only be clearly grasped when they are distinguished clearly from one another and contained within a fixed circle of content. Thus, each geometric shape is invested with an isolate, unvarying character.

As is known, the geometry of the XVIIth y XVIIIth centuries was deeply influenced by analytic methods. Dissatisfied with the approaches of synthetic geometry, Descartes and Fermat triggered a movement amongst mathematicians which led them to prefer coordinates and calculus when dealing with geometric matters.

Descartes took mathematics to the very heart of his philosophy. His attempt at reconstructing knowledge from unmovable grounds led him to seek out a new method that would allow him to obtain certain and trustworthy knowledge. This method was the analytic one, precisely.

In Ancient Greece, what was normally submitted to analysis was a geometric configuration exemplified by a figure. The several parts of analysis ranged from a geometric object to another or perhaps from a number of these objects to another; with the stages from a geometric object to another being measured on the basis of their interdependence within the frame es-

[6] *Ibid,* pp. 19-23.

tablished by the remainder of the configuration. In studying these interrelations, an analyst "analyzed" the configuration at hand in an almost literal manner which, befitting common sense, would find him separating it into its parts. Descartes and his contemporaries chose to generalize and developed the notion of "analysis" as the decomposition of a configuration. In the practice of the geometers of Ancient Greece, these interrelations were typically perceived as simple equivalences between the different lines and angles of the figure. By means of a greater display of algebraic methods at the hands of Descartes' immediate predecessors, these interdependencies became progressively more pliable, to the point that, in Cartesian analytic geometry, any polynomial dependency could be geometrically represented. Indeed, Descartes came to stress this algebraic characteristic of a wide variety of different kinds of geometric interdependencies in his geometry.

And it is worth noting how he expresses himself on this matter. Since he did not dispose of any general concept for function (functional dependency); in referring to it, he had to speak of "comparisons". All in all, his purpose was clear: Descartes actually affirms that "all knowledge that is not acquired through the pure and simple intuition of an isolated object is acquired through the comparison of two or more objects amongst themselves"[7].

Despite the deep influence of analytic geometry, synthetic geometry also had its apologists during the XVIIth and XVIIIth centuries. Interest in this geometric method was enhanced by the demands of artists and architects, who needed an appropriate technique and set of conventions to represent three-dimensional figures on a plane. The first two important names involved in this particular pursuit were those of Desargues and Pascal, who were quickly joined by Monge, Poncelet and Von Staudt. Desargues and Pascal visualized conic sections (the circle, the ellipse, the parabola and the hyperbola) as projections of circles, discovered other properties of cones and prepared the foundations for projective synthetic geometry. Monge made important contributions to the question on imaginary points, developed by Poncelet and Von Staudt[8]. What is typical to these new developments in synthetic geometry is that the mathematicians partaking in them did not return to the primitive use of diagrams; rather, they reinterpreted their use in the light of the exact results that analytic geometry had already proven. In effect, the step taken forward by Poncelet involved not taking any par-

[7] Descartes, R., *Rules for the Direction of the Mind*, México D.F.: Porrúa, 2001, p. 163.
[8] A detailed history of these developments can be found in Chapter II of my doctoral dissertation, *De lo sintético a priori a lo a priori formal constitutivo. La geometría y la evolución de lo a priori de Kant a Carnap*, México D.F.: UNAM, 2005 (unpublished).

ticular diagram as an object for the study of geometry, but as a complex sign the components of which could be manipulated without considering the particularities derived from their visualizable features, thus being able to obtain the general properties of figures.

What must be underscored is that, despite the increasing levels of formalization to which synthetic geometry was taken, the use of constructions remained as its distinctive feature. One last example should help prove this. Allow me to consider the construction of the projective planes and space of three-dimensions. In projective synthetic geometry the undefined elements are points and lines. The projective plane and the projective space of three-dimensions can then be built on the basis of the postulates of existence and incidence[9].

Postulates of incidence[10]:

P1. *If A and B are distinct points, there is at least one line on both A and B.*

P2. *If A y B are distinct points, there is not more than one line on both A and B.*

P3. *If A, B and C are points not all on the same line, and D and E are distinct points such that B, C, and D are on a line and C, A, and E are on a line, there is a point F such that A, B and F are on a line and D, E and F are on a line.*

Postulates of existence:

P4. *There exists at least one line.*

P5. *There are at least three distinct points on every line.*

P6. *Not all points are on the same line.*

P7. *Not all points are on the same plane.*

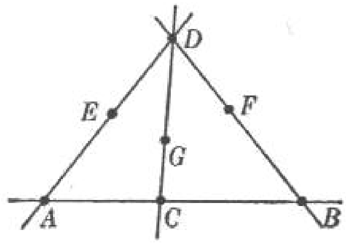
P8. *If S3 is a three-dimensional space, every point is on S3.*

Let us now construct a representation for postulates P1-P8. We shall represent a point with the capital letters A, B, C…; a line, with two capital letters representing the different points on the line, AB, AC, BC… or by a simple lowercase letter such as a, b, c…; and a plane by three capital letters representing the three non-colineal points of the plane ABC, BCF… or by a lowercase Greek letter ἀ, β… π…

[9] Obviously, these postulates can be extended in such a way that an n-space can be defined in terms of them.

[10] The term "incidence" is used to indicate the property that a line is on a point or that a point is on a line.

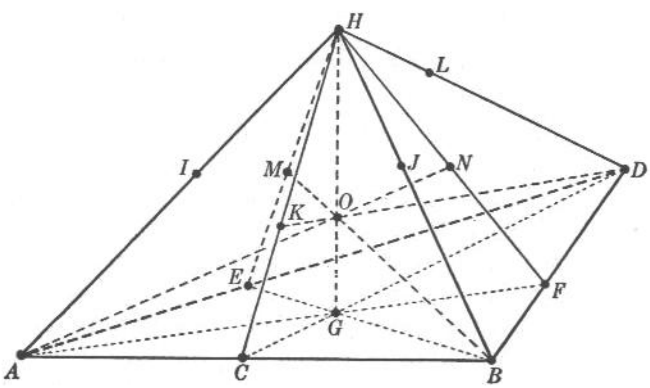
Figure 1:



There is a line, let us call it m, for postulate P4; and for postulate P5, this line contains at least three distinct points, which we shall call A,B and C. Also, there is a point D for postulate P6 that is not on line m. Then, for postulate P1 there are lines DA, DB, DC and possibly some others joining point D to the other points on m. Furthermore, for postulate P5, each of these lines contains another point, let us call them E, F and G, as shown by Figure 1. Generally, for any AB and any point D that is not on AB, the totality of points on the lines which join D to the points on AB is termed *projective plane* ABD.

Meanwhile, postulate P7 shows there is a point H that is not on plane ABD. By postulate P5, each of the lines connecting H to points on ABD contains at least three distinct points, as in Figure 2. Generally, for any projective plane ABD and any point H that is not a point on ABD, the totality of points on the lines connecting H to points on ABD is called *projective three-dimensional space* ABDH.

Figure 2:



As stated, we part from a few primitive elements and a handful of rules to construct the universe of geometric figures that make up this sort of geometry.

2. Geometry, synthetic unity and a priori synthesis

Throughout the history of philosophy, mathematics –and geometry in particular– have been more than mere objects of curiosity. The best-known list of those who placed this branch of mathematics at the center of their ruminations spans the Pythagoreans, runs through Plato, and reaches Descartes and Spinoza. This list could be further extended if we were to include such XVIIth and XVIIIth century thinkers as Leibniz, Hobbes and Kant, as well as Helmholtz, Poincaré, Cassirer, Husserl and Carnap in the XIXth and XXth centuries. What all of these philosophers have in common is not that they concerned themselves with geometry as an object of study, but that they considered it a veritable model for all knowledge.

It has already been mentioned how Descartes applied the analytic method of geometry which he himself developed to theoretical and epistemological queries. The cases of Leibniz, Hobbes and Kant are an expression of that very tendency, with the difference that they found amongst the methods of synthetic geometry a concept they could generalize and graft onto their notions of the production of human knowledge, namely, the concept of construction.

In Hobbes[11] we are met with the statement that the laws of the mechanism controlling nature are nothing but the concrete expression of geometric rules. That is why he claims that: “If whomever investigating natural philosophy should fail to consider geometry as the principle for his investigation, he shall investigate in vain; and those who write or expound on her, and do not know her, abuse her readers and her listeners”[12]. And by the principle of geometric investigation one must not understand anything other than the *constructive* method, which he calls the *compositive* method.

As for Spinoza, besides his famous attempt to found ethics on a geometric foundation[13], he also attempted to extend the principles of this discipline, particularly those of synthetic geometry, to the totality of knowledge. In his *Tractatus de intellectus emmendatione*, and pointedly in his doctrine of definition[14], he claims that for a definition to be perfect it should convey the most intimate essence of a thing, that is, it should not include anything that is not among its essential qualities. And for such a definition

[11] Refer especially to: Hobbes, T., *Hobbes: An Anthology*, Barcelona: Peninsula, 1987.
[12] *Ibid.* p. 112
[13] His *Ethics Demonstrated in Geometric Order* was erected on the model of Euclides' *Elements.*
[14] In sections 51 and onwards.

to be possible, the intellect must strive to forge internal causes that agree with its construction. In Spinoza's words: "To form part of the concept of the sphere, for example, I decide to conceive of a cause, namely, a semicircle spinning around its center and the sphere being engendered through rotation"[15]. In other words, for Spinoza concepts must not be exposed as complete, quiescent forms, which we must settle to assimilate. The unconditional guarantee of their proof can only be obtained once we have penetrated into the law of their becoming: into their genesis.

In this very context of the doctrine of definition, Leibniz, who –like the Port Royal logicians– distinguished between nominal and real definitions, sustains the latter are those in which "we must be careful to establish that the thing is possible"[16], something that can only be done by proceeding to construct the object in question. Leibniz says: "The concept of circle advanced by Euclides –that is, the figure described by the movement of a straight line in a plane around a fixed point– affords us with a real definition, insofar as it is clear that such a figure is possible"[17]. To his mind, a real definition provides a rule by means of which he can build an object from a collection of qualities whilst proving, at the same time, that the object's possible. According to Leibniz, this art of construction, or –as he calls it– this art of "synthesis" or "combination" is the qualitative science of objects, of the similar and the dissimilar, and it is "precisely what the geometers do when proving and arranging propositions according to their mutual dependencies"[18].

Finally, Kant's interest in geometric matters can be traced back to his youth, to the works in which he tries to solve the disputes between Newton and Leibniz on the nature of space[19], and it remained at the center of his thoughts for all his life. In the *Dissertatio* (1770), he has no qualms in affirming that geometry is "the prototype for all sensible knowledge"[20], and that, like pure mathematics, "it is the organ of all intuitive and distinct

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15 Spinoza, B., *On the improvement of the understanding*, Madrid: Tecnos, 1989, p. 36.

16 Leibniz, G.W., "Of Universal Synthesis and Analysis; or, of the Art of Discovery and of Judgment", en: Parkinson, G.H.R. (ed.), *Leibniz. Philosophical Writings*, London: Rownian and Littlefield, p. 12.

17 *Ibid.*

18 *Ibid.* p. 10

19 Among these works we may mention: *Thoughts on the True Estimation of Living Forces* (1746), *A new elucidation on the first principles of metaphysical cognition* (1755), *Universal Natural History and Theory of the Heavens* (1755), *Physical monadology* (1756).

20 Kant, I., *On the Form and Principles of the Sensible and the Intelligible World,* Madrid: CSIC, 1996, p. 13.

knowledge; and since its objects are not just the formal principles of all intuition, but *original intuitions* unto themselves, they give us a knowledge that is true to the highest degree, and which is also the model of supreme evidence for the other forms of knowledge"[21]. Why is it, then, that geometry serves as a *model* for the evidence of other forms of knowledge?

Allow me to begin by considering the known Kantian distinction between analytic and synthetic judgments. According to the elucidation Kant provides for it in the introduction to the CPR, an analytic judgment is obtained through the analysis or breakdown of a concept, with the observations wrought from the analysis serving as predicates for it. He also terms such judgments to be "explicative", insofar as all they do is to *display* or to *develop*[22] the content of the concept of subject. As he will write later (in the section on the "Supreme principle of all analytic judgments" in A150-B190), the principle governing the development of analytic judgments is the principle of contradiction which leads to the affirmation of the content of the subject; in other words, this principle warns us not to place as predicate to the judgment anything that might contradict the concept of subject.

As for synthetic judgments, which Kant also calls "extensive", the crux is found precisely in the fact that the predicate of such judgments *adds* certain determinations to the concept of subject: determinations which cannot be found in it through conceptual analysis. A genuine composition, a synthesis between heterogeneous objects, occurs. In the section on "The supreme principle of all synthetic judgments" (A154-B194), Kant states that: "…the supreme principle of all synthetic judgments is that every object complies to the conditions necessary for unity which synthesize the diversity of an intuition into a possible experience"[23]; or, to phrase it differently, that any object we consider is conformed in compliance with a synthetic reunion of features that appear in intuition. An object is, in this sense, a construction made up from simple characters, in agreement with a rule. Where must we find, according to Kant, the key to this rule of unity, and with it, the possibility for synthetic judgments? To this, he replies: "It is thus in the internal sense, in the imagination and in apperception that one must seek the possibility of synthetic judgments"[24].

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[21] *Ibid.*, p. 16.

[22] The Latin root for the word "explain" may help us understand the underlying idea. The verb *explico, explicare* is a compound of the basic verb *plico, plicare* and the preposition *ex*. *Plico* means to fold, to bend, and when joined to the preposition *ex*, it would mean to fold towards the outside or display, or also develop.

[23] *CPR*, A158-B197.

[24] *Ibid.*, A155-B194. I should not wish to enter a debate here, but I must remark my discrepancy with J.A. Coffa, who, at the start of his already famous *The Semantic*

According to Kant, the complex of representations –what we would call an object– cannot be given to us through the senses, nor is it the product of its pure form, that is to say, that neither time nor space provides us with this combination. Instead, it obeys a principle of understanding, "it is an act of spontaneity of the faculty of representing""[25]. This conjunction, which may be pure or empiric, sensible or intellectual, is what goes by the name of *synthesis* in Kantian terminology (B130). Likewise, the complex integrated by the idea of a diversity that is subjected to a function of synthetic collection is what Kant terms "combination". But this *combination* has one more element to it which sets it apart from mere synthesis, and that is the concept of unity. Kant says: "To combine is to represent the *synthetic* unity of the diverse"[26]; *combining* does not merely involve randomly synthesizing a series of notes, it actually implies the synthetic collection of certain diversity, accompanied by unity, in a synthesis. In every judgment in which certain elements are synthesized according to concepts there is a combination *sensu strictu*, as the synthesis that operates in this is accompanied by unity.

But what is this unity? It means that in each act of perception I recognize the representations teeming to my conscience as *my own*: I remit them to a basic conscience that unifies and connects them. This conscience, the "I think", not only enables the synthesis of intuitions making up every perceptive experience, but also the unity of these experiences in such a way that they comprise a unitary complex I acknowledge as belonging to me. To phrase it differently, the "I think" is an original, synthetic formal principle[27] responsible not only for the constitution of our individual perceptual experiences, but also for their conjunction into an orderly, stable, consistent totality; indeed, it is the principle responsible for the objectivity in our vision of our world. Kant says: "Since it is given *a priori*, the synthetic unity of the diverse in intuitions constitutes the groundwork for

Tradition from Kant to Carnap: To the Vienna Station, Cambridge: Cambridge University Press, 1991, interprets Kant as sustaining that the basis for synthetic judgments is intuition. It is on this emphasis that Coffa grounds his thesis on the semantic tradition which, in his opinion, collects the results of the crisis to which the geometric revolution of the XIXth century led intuition, and with it, the idea of the constitutive *a priori*. In my doctoral dissertation, already mentioned in footnote 8, I meticulously rebate Coffa's interpretation.

[25] *CPR*, B130.

[26] *Ibid,* B131

[27] In referring to the "I think" as a *formal* principle, I am echoing the Kantian distinction between pure or original and empiric apperception, since Kant defines the first as that which should accompany every representation but cannot be accompanied by any other representation in its turn. Likewise, Kant´s affirmation that our understanding is not the producer of its own content is crucially important, insofar as it caters to its typically synthetic nature (*cf. ibid,* B136).

the identity of apperception itself, which precedes the whole of my determined thought *a priori*"[28].

This fundamental faculty for the linkage of representations is what permits the construction of synthetic judgments and hence, the construction of knowledge. To know, in Kant's point of view, is to build an object through the synthetic reunion of features and insert it within a consistent totality. And it is not casual that Kant exemplifies this point with a case drawn from geometry: "To know something in space –a line, for example– one has to trace it and synthetically effect a certain combination of the given variety, so that the unity of this act is, at one time, the unity of conscience (in the concept of a line) and the means through which one knows an object (a determined space)" (B138). The constructive procedure we have signaled as being typical to synthetic geometry is clearly exposed. It could be said that the *given variety* to which Kant refers are the points, and that the *determined combination* stands in the first two axioms of Euclides. What is interesting is that in continuing this passage, Kant will emphatically add that: "the synthetic unity of conscience is thus an objective condition for all knowledge"[29]. In my opinion, Kant raises the synthetic method of geometry to a universal principle for the construction of knowledge.

But let us look into another important component Kant mentions as being central to the construction of synthetic judgments: imagination. In the section entitled "The application of the categories to objects of the senses in general" of the second edition, Kant distinguishes between intellectual and figurative syntheses. The first refers to the unity of the objects of intuition in general, as related to the unity of apperception. The second refers to the application of this very unity of apperception –and with it, of the categories–, concretely to the diversity of sensible intuition *a priori*. This synthesis is, in his words: "the condition to which all the objects of our intuition must necessarily submit"[30]. Kant calls this the "transcendental synthesis of the imagination".

One may immediately wonder what the imagination has to do with this, and Kant argues too obscurely to glean a clear idea from it. Perhaps we could begin by trying to shape an idea of the synthetic process that Kant is conceiving, and then seeing why imagination is important to it. With this in mind, let us consider the example Kant himself suggests in B154, where he says: "This we perceive always amongst ourselves. We cannot think of a

[28] *Ibid,* B134
[29] *Ibid.*
[30] *Ibid,* B151

line without *tracing* it in our thoughts, nor on a circle without *describing* it, nor can we represent the three dimensions of space without *constructing* three perpendicular lines from the same starting point". As I see it, it is not casual that the example is taken from geometry, nor is it a coincidence that the verb *construct* and other italicized terms, entailing the idea of construction, appear. That is why I shall direct my attention to how Kant conceives the production of geometric knowledge and see if the key to his figurative synthesis can be found there.

In the "General observations on the Transcendental Aesthetic", Kant asks the following about the propositions of geometry: "where do we find such propositions and on what does our understanding rest in order to arrive to such absolutely necessary and universally valid truths?"[31]. The answer is that knowledge of this sort can only be obtained from two sources: intuitions or concepts. And since both are given *a priori* and *a posteriori*, we must consider each of these options. It is clear that, as Kant seems to believe (and no one in the XVIIIth century would have thought differently[32]), geometry is a discipline conformed by universal and necessary truths, and could not draw its propositions from empiric concepts or their corresponding intuitions, since this would turn geometry into a hopelessly empiric discipline. The only options left standing are that geometric knowledge is derived from pure concepts, or from *a priori* intuitions. The first of these possibilities is discarded by appealing to the definition of analycity and to the impossibility of explaining a judgment of geometry on the basis of such an idea. Thus, the only remaining option is that the propositions of geometry derive from pure intuitions. But here, Kant posits a distinction, by saying: "But, what kind of pure intuition would this be: an *a priori*, or an empirical one?"[33]. In my opinion, a pure empiric intuition would be the immediate representation of a particular moment in time, or a particular, inactual space: for example, the representation I may have at the very instant in which I am writing this paper of the university classroom where I impart a course on the *Critique of Pure Reason.* It involves a repre-

[31] *Ibid*, B64-A47

[32] In the XIXth century, instead, the emergence of non-Euclidean geometries led some to think that geometry might be empirical. Gauss tried to directly determine, through the ordinary triangulation performed with topographical instruments, if the sum of the angles of a great triangle is equal to the sum of two straight angles or not. He performed his measurements on a triangle formed by three mountains, the Brocken, the Hoger and the Hagen, as well as the Isenberg, the sides of which measure 69, 85 and 107 kilometers respectively. Needless to say, he did not detect any 180º deviation within the margin of error, which must have disappointed him.

[33] *CPR*, A48-B65

sentation that is independent from the experience of the case, but not from all experience. And this is possible through the intervention of the faculty of imagination, even if it is a purely reproductive one. Likewise, it would be a representation that is not universally valid, but valid in a purely subjective way.

On the other hand, an *a priori* intuition would be a representation emerging as a necessity of the pure form of sensibility, independent of past or present sensible features, albeit not completely independent of every form of sensible representation. Let us not forget that, for Kant, geometry, and mathematical knowledge in general, is knowledge through concept *construction*, that is, a knowledge in which it is indispensable to present the *a priori* intuition corresponding to the concept. According to Kant, this is a representation which, insofar as it is an intuition, represents a singular object. However, in the measure in which a concept is constructed there, said representation must express the properties common to such representations; in other words, it must have a universal validity. In A714-B742 he says: "To give a case, I construct a triangle representing either the object which corresponds to this concept by means of mere imagination, in pure intuition, or on paper, in empiric intuition, but in both cases completely *a priori*, without the model of experience". Imagination, which is in charge of the construction itself, proceeds not in merely in a reproductive fashion, but in an essentially productive one, because, as Kant affirms, it does *not take its model from experience*; it abstracts the specifically sensible circumstances that conform a concrete geometric figure (the magnitude of its sides and angles, for example) highlighting the universal properties that the particular representation is exemplifying. To put it in terms that will later be clearer, geometric figures constructed in pure intuition constitute representations of the unvarying properties that characterize a certain type of figures.

Let us now return to the figurative synthesis. Maybe now the definition that Kant gives of the imagination as "the faculty of representing an object of the intuition even *when it is not present*"[34] makes more sense. In the figurative synthesis, imagination is the only faculty that can supply sensible variety *a priori* for the understanding to perform its synthesis, ruled by the categories, through the synthetic unity of apperception. However, it is the imagination itself that –according to Kant–conducts the synthesis, even as it is ruled by the categories, and this is justified because we are dealing with the application of the categories to sensibility, and not of a synthesis merely conceived within the category (intellectual synthesis).

[34] *Ibid,* B151

Once again, geometry helps illustrate the case: "The mathematics of extension (geometry) and its axioms are based in this successive synthesis of the productive imagination"[35].

So it seems that the figurative synthesis proceeds –as regards the whole of synthetic knowledge– under the aegis of concept construction that takes place in mathematics and in pure geometry, particularly. The productive imagination –that which does not heed the *model of experience*– is not only capable of presenting the corresponding intuitions *a priori*, it actually *constructs* the objects of knowledge in compliance with a rule dictated by the understanding. This is how the idea of construction, the basic principle of synthetic geometry, becomes the basic tenet of synthetic knowledge. And this is the reason why I have been speaking of synthetic judgments without distinguishing –as Kant does– between synthetic *a priori* and synthetic *a posteriori* judgments. For the principle that makes them possible is exactly the same, with the difference that, in the first case, constructions occur *a priori* and, in the second, *a posteriori*. Kant makes this point clear in section 6 of his *Prolegomena*, where he writes that: "This observation regarding the nature of mathematics leads us to the first and foremost condition of its possibility, namely: there must be at its ground some intuition in which it can display all of its concepts *in concreto* and yet, *a priori*, by constructing them, it might be said…because just as empiric intuition makes it possible, without any difficulty, that the concept we conform of an object of intuition is synthetically amplified through the experience of new predicates set forth by intuition itself, this can also be done by pure intuition, with only this difference: that, in the last case, synthetic judgments had to be *a priori*, true and apodictic; while in the first, they need be only *a posteriori* and empirically certain"[36].

I should like to conclude this exposition by underlining a point that I did not make explicit in the previous section, but which will be of critical importance for the problem I am faced with: from Kant's perspective, a distinction should be made between the active principle of the constitution of experience –the original synthetic unity of apperception through figurative synthesis–, and the rules which dictate the criteria for unity that this function obeys –the structure of pure intuition and the categories. The relevance of making this distinction will become more readily apparent in what follows.

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[35] *Ibid.*, B204.
[36] Kant, I., *Prolegomena to any future metaphysics*, México D.F.: Porrúa, 1991, p. 41.

3. Group theory and the constitutive a priori

In the previous section, we examined Kant's conception according to which the constitutive function of knowledge rests on a basic tenet of unification bearing strong resemblances to the method of synthetic geometry that we described in section 1. Just as in the case we then considered –the construction of the projective plane and space from primitive elements, in compliance with the axioms of incidence and existence– the Kantian constitution that is operated by the understanding stems from the *a priori* variety of the sensibility and, according to *a priori* rules, constructs any synthetic judgment. Thus, for Kant, the constitution of experience proceeds through the combination of a rule and an active principle, axiom or concept and the synthetic unity of apperception[37]. According to the specifications Kant bestowed on these axioms and concepts and the reach of his determination on the constitutive function, the result of such a combination is the construction of a world with an Euclidean and Newtonian structure[38].

However, since the XIXth century until today, both the formal and the factual sciences have experienced such great changes that they led us to question the model for explaining knowledge put forth by Kant. First, there was the emergence of non-Euclidean geometries, which tumbled the Kantian notion of intuition, and then, there was the theory of relativity, which defined a radically new frame for time-space, affecting the content of the table of categories. It looked as if little of the Kantian epistemology could be saved. Several authors in the XIXth and XXth century alike, however, tried to rescue at least part of the Kantian spirit while acknowledging the profound changes that the scientific revolutions implied.

The challenge consisted in separating the strictly constitutive function –the active principle of synthesis– from the rules conditioning the types of synthesis to be performed. In other words, the challenge laid (and lies) in retaining something that did not have the form of a particular principle or enunciation as a basic constitutive function, insofar as these could be transformed, replaced or rejected.

[37] In *CPR*, A116, Kant makes it clear that what he there terms as "pure apperception" must be presupposed before we even speak of intuitions, since these only make sense by virtue of the necessary connection of the ones and the others in a single conscience.

[38] We can clearly not blame Kant for not having looked further than what the science of his time allowed. This led him to try to prove the unicity of our conceptual schema. Part of the efforts of the Analytic lay in trying to prove that the world, such as we experience it, is possible because of the conjunction between a single and determined complex of subjective conditions.

During the XIXth century, some philosophers related to mathematics, such as Helmholtz and Poincaré[39], began to exploit the virtues of a fundamental concept: that of *group of transformations*. With the eventual translation of this concept to the core of geometry by Klein in what was called the Erlangen program or the *geometric invariant theory*, other philosophers like Cassirer and Husserl[40] saw a powerful tool in it, honed to seek a clear function of synthesis as separate from further specifications.

These notions were not sufficiently valued in their time, even if, in my mind, they contained elements that we might use today to best articulate the broad conception of the constitutive *a priori*[41]. The main principle of this position is that, in cognition, we generally perform *a search and constitution of invariants*; with "search and constitution" meaning the application of a basic principle of synthesis that we can identify with the mathematical concept of group transformations –which obeys certain rules determined by certain orders of invariants, that is, the shape of experience– over a given variety. The responsible principle of synthesis, together with the fundamental concepts setting the rule for constitution, are *a priori*, attributes of what is known as rationality.

As it would exceed the limits and pretensions of this essay to provide a theory of experience that makes use of the concepts of groups of transformations and invariants, I shall, firstly, proceed, to expose the concept of group of transformations, to later demonstrate the application of this con-

[39] For an exposition on how Poincaré made use of this concept, see my "Idealización, constitución y convención en la filosofía de la geometría de Henri Poincaré", in: Txapartegi, E. (comp.), *Los objetos de la ciencia. El mundo que la ciencia construye*, Córdoba: Brujas, 2005.

[40] For an elucidation of the conceptions of these philosophers, albeit in relation to the influence they exerted on Carnap's early thinking, refer to my article "La teoría de grupos y el espacio intuitivo en *Der Raum* de Rudolf Carnap", soon to be published in *Diánoia*.

[41] In recent years several interesting conceptions of the *a priori* based on Kant or in more traditional rationalist schemes have arisen. Most of them, however, have been developed within the semantic tradition, or –to use an expression by L. Bonjour– have moved within *cuasiempiricist* realms. The closer stances to what I am defending are those by M. Friedman and A. Richardson, albeit the first prefers to speak of semantic constitution and the second of a constitution of practices, whereas I prefer to speak –in a more traditional sense- of the constitution of experience. For a critique to the positions of Friedman and Richardson, see my "Reconsiderando a Friedman, Richardson y lo *a priori* constitutivo", in: *Ideas y Valores*, 131 (2006). For a comprehensive panorama of contemporary conceptions of the *a priori*, see Boghosian, P.A. and C. Peacocke, *New Essays on the A Priori*, Oxford: Clarendon Press, 2000.

cept –jointly with the constitution of invariants– is essential to at least three areas of cognition: mathematics, physics and perception[42].

The concepts of *transformation* and *group of transformations* are owed to two prominent British mathematicians, Arthur Cayley (1821-1895) and James Sylvester (1814-1897). But before we give a precise definition of transformation, we should explain the idea of mapping[43].

Definition: A mapping of a set A in a set B is a conjoining of elements from A and a subset of B, so that every element of A is matched with exactly one element of the subset of B, and every element of that subset of B is matched with at least one element of A. The special case of the mapping of set A into set B for which every element of B is matched with at least one element of A is called a mapping of set A *onto* set B.

Definition: F is a mapping for a set A into a set B if, for every element of A, there exists a single element b from B to match with a; this matching is expressed as $f(a) = b$, where set A is considered the domain of f, and set B as the codomain of f.

Definition: A mapping f of A and B is one to one if every element in the range of f is the exact image of an element of A., thus, if $f(a) = f(c)$, then $a = c$.

Definition of transformation: A transformation is a mapping of A onto B so that every element of B is the exact image of an element of A. In other words, a transformation is both a *one to one* and *onto* mapping.

Definition of a group of transformations: a group of transformations of a set A over itself is a set S that is not empty with an operation such that:

If f and g are in S, then fg and gf are in S. (Closure)

If f, g and h are in S, then $(fg)h = f(gh)$. (Associativity)

There is but a single element in S which satisfies $If = fI = f$ for all f in S. (Identity)

Given f in S, a single f -1 element exists to satisfy f -1 $f = ff^{-1} = I$. (Inversion)

The property of *closure* means that the product of any two transformations in the set is in itself also a transformation in the set.

[42] This thesis, which received early validation by Cassirer, has been called the “identity thesis” by Thomas Mormann, although Mormann himself did not consider the case of perception. See his “Idealización y matematización en el idealismo crítico de Cassirer”, in: Txapartegi, E. (comp.), *op.cit.*

[43] For a detailed survey of these concepts see Meserve, B., *Fundamental Concepts of Geometry*, Cambridge: Addison-Wesley Publishing Company, 1955 and Smart, J., *Modern Geometries*, Belmont: Brooks/Cole Pub. Company, 1994.

The second property is that of *associativity*. The notation of $f(gh)$ and $(fg)h$ indicates the product of the same three transformations in that very order. For example, let us consider the following three transformations, which have been defined as follows:

$f : x \to x + 3$

$g : x \to x - 2$

$h : x \to 2x$

$f(gh) : x \to ([2x] - 2) + 3 = 2x + 1$

$(fg)h : x \to 2x + (-2 + 3) = 2x + 1$

The third property established the existence of a unique *identity*. Since the product of a transformation and the identity is the transformation, the effect of identity is to leave each point as an invariant.

The fourth property determines the existence of a single inverse transformation for every transformation of the set. Since the product of a transformation and its inverse is identity, the effect of inversion is to undo the transformation, returning each point to its original position.

Just as had occurred with the concept of number before it, the concept of group is perceived as a fundamental principle of order, with the difference that what is brought into a unity are not elements, but operations. The creation of the series of natural numbers began by fixing a first element and giving a rule which always allowed for the generation of new elements by way of its repeated application. All of the elements were united into a unitary whole by virtue of which every connection performed between the elements of the series of numbers comes to "define" a new number. If we were to form the sum of two numbers a and b, or their difference, or their product, etc., the values $a + b$, $a - b$, $a.\ b$ are not extraneous to the basic series: they belong to it by occupying a determined position in it, or they can be indirectly referred to the elements of the basic series in agreement with fixed rules. Therefore, no matter how far we advance into the new syntheses, we shall always have the safety that the logical frame for our investigation will never be completely broken, disregarding how far it is broadened. The notion of the unitary realm of number means that the combination of arithmetic operations, no matter how numerous, will always result in arithmetic elements. Group theory elevates that same vantage to a level of the strictest, true universality, as this theory –so to say– eliminated the dualism implicit when speaking of "elements" and "operation": in group theory, the operation has, itself, become the element. Just as Kepler termed the number as "the object of the spirit" that allows us to pry into reality, we can also validly claim that group theory, considered by some to be the single most brilliant example of purely intellectual mathematics[44], has made the interpretation of certain spatial, physical and perceptual connections possible. Let us now consider some different applications for it.

When applied to geometry, a group is formed by all the geometric transformations that result from allowing the elements to move in ordinary three-dimensional space. In this concept of group, the general principle of classification is obtained by means of which the different types of geometries may be unified under a single, simple vantage. If we pose the question as to what we should consider as geometry, the answer is: those properties that remain invariant throughout certain spatial transformations; or, in other words, those structures that persist when we vary the absolute position of this structure in space, when we proportionally augment or diminish the absolute magnitude of its parts, or when we finally revert the order of its individual parts, as in when we substitute the original figure for another that relates to it as with its mirror image.

Felix Klein developed the idea into the center of what is known as the Erlangen Program; which we can best appreciate in the following passage: "Given any group of transformations in space that includes the main group as a subgroup, the invariant theory for this group casts a definite type of geometry, and every possible geometry can be obtained in this same way"[45]. The sense of this passage can be explained in the following manner: the differences between geometries are in fact the differences between the relationships that they explore. The relationships or properties that geometry explores are those that are invariant under a set or group of transformations; invariant properties and allowed transformations are mutually determined, so that geometries can be characterized by their invariant properties or groups of transformations.

The groups of transformations to which the geometric figures are subjected vary in their degree of radicality and, the more radical this measure is, the more general the invariant properties are. This makes it possible to classify geometries in relation to the degree of generality produced by the degree of radicality in the transformations, each of which contains the one before it. At the cusp of this classification we find topology, followed by projective geometry and Euclidean geometry, respectively.

Euclidean geometry is the study of the invariant properties under the group of so-called rigid movements: translation, rotation and reflection. The

[44] The expression is H. Weyl's. See his *Filosofía de las matemáticas y de la ciencia natural*, Mexico D.F.: UNAM, 1965.
[45] Klein, F., *Elementary Mathematics from an Advanced Standpoint*, New York: The Macmillan Company, 1939, p. 133.

essential property preserved by this group of movements is distance or, in other words, the feature of isometry.

Projective geometry is the study of the properties of sets of points that are invariant under the group of projective transformations. These transformations, which make so-called figures of perspective possible, constitute a one to one mapping onto points, that is, a one to one correspondence of corresponding elements exists, as well as a relation of incidence shared by them. The relations preserved under these conditions are those of incidence and existence: the copunctuality of points, the colineality of lines and the complanarity of planes. In such a way, the relations explored by projective geometry are relations of incidence, completely independent to relations of magnitude.

Last but not least, topology is the study of the properties of a set of invariant points under the group of bicontinuous transformations of space into itself. A transformation f is bicontinuous, or a *homeomorphism*, if and only if f and f-1 are both continuous. A transformation of points in the planar region S1 onto a point in planar region S2 is continuous if for every point on S2 and every positive number ε there is a positive number δ such that the image of any given point in S1 that is in the vicinity of a point A with a radius δ is in the vicinity of the image of A with a radius ε. The invariant property that casts the set of topological transformations is the *connectivity* of a set of points. A set is connected if and only if any two points in the set can be joined by a curve that is completely outside of the set.

Let us now move to perception. According to a thesis backed by psychologists as prominent as Katz, Buehler, Brunswik and Frischeisen-Koehler, and currently defended by researchers like Roger Shepard, amongst others, perception is not confined to a "here and now"; instead, it expands particular data and incorporates it into a total experience, and it is only by virtue of this integration that perception can exert its own function as an objective factor in knowledge. If perception were limited to the flow of impressions, it would disintegrate, because each of these impressions presents the scale, size and color of objects in a different way.

As a matter of fact, psychologists sustain perception does not adhere to this kaleidoscopic succession of images, but that it constructs the true perceptual from them. For example, depending on the circumstances in which an object is seen, solar lighting is subject to great variations in spectral composition. The light that strikes the object directly may be stronger at a medium wavelength (yellows), while the light spread on the object from a clear sky may be stronger in short wavelengths (blues), not to say that the light in a solar environment may be stronger at the higher wavelengths (reds). Depending on such conditions, the light spread to our eyes from any given object may be strongly directed towards lower, medium or higher wavelengths, towards yellows, blues or reds. Despite such great variations in the light that surface spreads over our eyes under such different conditions, though, the color we perceive in a surface persists as a fixed –an even an apparently inherent– property of the thing itself[46].

So it is that, from the vantage of the evidence afforded us by the psychology of perception, we could properly say that it is dominated by a constitutive factor that manifests itself in the possibility to conform invariants. The original instability of our flow of impressions is, by virtue of this function of constitution of invariants, substituted by a world of stable, permanent objects that are the essential feature of what we aptly designate as an objective world. Perception is possible through a process known as *idealization*, insofar as it introduces a structure that was in no way established beforehand into the flux of sense impressions.

As for physical science, it seems to operate analogously by introducing –albeit I would think that not into the flow of impressions as much as into the world of ordinary perception, -- more refined structures and determinations that attain some degree of perfection in form. Perhaps in idealizing scientific practice we shall find perfection in the search and constitution of invariants, something we cannot sustain is the case for ordinary experience since –as has also been observed by empirical psychology– there are limits to perceptual constancy. In physics, on the other hand, once the group of transformations becomes specified, all modifications possible with respect to that group can be determined through exact laws.

Physics has bestowed a great importance on Lorentz's transformations[47], which determine that something be considered more objective tan

[46] For further details, see Shepard, R., "The Perceptual Organization of Colors: An Adaptation to Regularities of the Terrestrial World?", in: Barkow, J., L. Cosmides and J. Tooby (eds.), *The Adapted Mind. Evolutionary Psychology and the Generation of Culture*, New York: Oxford University Press, 1992; Maloney, L.T. and B.A. Wandell, "Color Constancy: A Method for Recovering Surface Spectral Reflectance", in: *Journal of the Optical Society of America*, III (1986), pp. 29-33.

[47] Lorentz' transformations emerge in the context of Maxwell's equations for the electromagnetic field. Said equations describe the behaviour of this field in every point in space and every instant in time. Thus the question arose: can the position and time in Maxwell's equations be changed, without changing their form? In the case of the equations of Newtonian mechanics, the answer is affirmative, given Galileo's principle of relativity: one can move from one frame of reference to another without changing the form of the equations, provided that the time measured in each system is the same. In the case of electromagnetism, the problem is more involved because one cannot recur to Galilean relativity. Lorentz, however, did prove that a mathematical

something else if it is invariant under such transformations. Einstein taught us that the measurements for time and space are relative to an observer; their magnitudes will be measured differently by different inertial observers, and space and time intervals are not invariant under Lorentz's transformations. Inertial observers will, however, coincide on another interval of events that involves not only temporal and spatial separations, but a mixture of both, that is, the square root of the square of the temporal separation minus the square of the spatial separation. This, more complicated, interval is invariant under Lorentz's transformations. The principle underlying Einstein's theory of relativity sustains that all the laws of physics are the same for all inertial observers; they are the same in every inertial frame of reference and hence, they are invariant under Lorentz's transformations. It's not to wonder, then, that Einstein said that, rather than calling his theory one of "relativity", he had called it a theory of "invariants"[48].

Conclusions

To conclude, allow me to return to Kant for a moment. In his "Analytic of principles", in the chapter on "The schematism of the concepts of pure understanding", Kant confronts the problem of showing what it is that makes it possible for objects to be subsumed by concepts; that is, he worries about elucidating how the conceptual *a priori* determines what is given by experience. According to Kant, when we say that "an object is contained in a concept", what we mean is that the concept must include what is represented in the object it is to subsume. Now then, pure concepts of understanding contain the rules for the determination of phenomena *a priori*, and are hence free of experience, requiring a terminus that invests them of the empirical content to which they are addressed. This content, in their case, cannot be more than a formal condition of sensibility supplying the synthetic rule for phenomena. Such a formal and pure condition of sensibility is what Kant terms his "schema of concepts".

A schema, according to Kant, must be clearly distinguished from an image. The latter is the product of the reproductive imagination and as such, always particular. The schema, on the other hand –insofar as it must enable the application of a universal-- cannot be a merely particular representation: it has to be intellectual and sensible at one same time. If we

transformation exists which leaves the form of Maxwell's equations invariant, if and only if both the position *and* time of a point are changed.

[48] For a profound elucidation of these matters, refer to Friedman, M., *Foundations of Space-Time Theories*, Princeton: Princeton University Press, 1983.

think that a concept, according to Kant, is an *a priori* rule for the synthesis of representation, and that the schema, being a product of the imagination, is therefore the product of a sensible faculty, it seems that what Kant is suggesting here is that this schema is a sort of image determined by a concept, which makes it different from a mere image. He appears to think so when he says: "Thus, I write five successive points…, I have an image of the number 5. If, on the contrary, I simply thought of a number in general, be it five, be it one hundred, *such thinking is a method of representation, in agreement with a certain concept, a quantity in an image*, more than that image in itself which, in this last case, I could hardly grasp and compare to the concept"[49]. What I would like to suggest is that when Kant speaks of "schema", he takes it to be the image of those properties common to several particular objects, which is a product of the productive imagination *a priori*, ruled by a concept. In such a way, the schema represents the invariant or essential features of an object, and emerges from the determination of the particular images according to an *a priori* rule that is the concept. In his opinion, this function is not reserved to pure concepts, it also includes empiric concepts. Kant says: "The concept of dog means a rule according to which my imagination is capable of drawing the figure of a four-legged animal in general, without being limited to a particular figure offered by experience nor to any possible image that may concretely represent it"[50]. Thus, it seems that for Kant the application of a concept –and with it, the possibility of the object itself– depends on the pursuit and constitution of invariants in the flow of experience, which in itself proceeds by recurring to transcendental schemes.

In our previous section, we have shown how mathematics, physics and perception itself display a stability and permanence in their respective structures, albeit with differing degrees of precision. To my opinion, this invariance –comprising the most general features of what we call the objective world– is the result of applying a synthetic, formal and original principle based on human understanding which, together with the principles determining the very order of the invariant, sets us apart as rational creatures. Mathematical objects, like physical events and the objects of ordinary perception, are the result of an operation that, in spanning a variety of simple elements, unifies them to conform classes of objects sharing a same set of structural and permanent properties. The concept of group of transformations can be seen as that essential synthetic principle that is responsible

[49] *CPR*, A140-B179 (the emphasis is mine).
[50] *Ibid.*, A141.

for the syntheses performed in the different domains of cognition. The task remains, however, of theorizing not only on the nature of this constitutive-formal function, but also on the specifications –or, to say in the jargon of group theory– on the permissible transformations that determine the set of invariant properties running through the different realms of cognition. In other words, the more general categories –the rule of which obeys to the formal principle of synthesis– has yet to be specified. And, as regards this matter, we might identify two sets of conditions: an *a priori* one, universal and necessary, ahistorical and nestled in the structure of our rationality; and another, conventional, historic one. But the development of this point is the subject for another essay.

(Translated from Spanish by Monica Belevan)