Altruism and Reciprocity in the Long-Run*

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ABSTRACT

The aim of this paper is to provide a model where altruism is determined endogenously. Altruism is an attitude that influences our actions toward other people. The model presented in this paper assumes that this influence also operates in the opposite direction via reciprocity: that is, people modify their altruism based on the actions of other agents. The paper uses a dynamic setup with two agents whose incomes are random. Depending on the incomes realized, transfers are made. These transfers convey information about the level of altruism of the donors. The agents use this information to adjust their own level of altruism. If the transfer received by an agent implies that the altruism level of the other agent is higher (lower) than that of the receiver, then the latter will increase (decrease) his level of altruism. This behavior induces a stochastic process for the levels of altruism. The long level of altruism is studied using both analytic and computer simulations tools.

Keywords: Altruism, reciprocity, inter vivos transfers.

JEL Classification: D1, D64

Altruismo y reciprocidad en el largo plazo

RESUMEN

El objetivo de este trabajo es construir un modelo de altruismo endógeno. El altruismo es una actitud que influencia nuestras acciones hacia los otros. El modelo presentado asume que esta influencia también opera en la dirección opuesta vía reciprocidad: es decir, los agentes modifican su nivel de altruismo basados en las acciones de los otros agentes. Este trabajo se basa en un modelo con dos agentes cuyos ingresos son aleatorios. De acuerdo a la realización de los ingresos se realizan trasferencias entre los agentes. Estas transferencias transmiten información sobre el nivel de altruismo de los donantes. Los receptores usan esta información para ajustar su propio nivel de altruismo. Si la transferencia recibida implica que el nivel de altruismo del otro es mayor (menor) que el nivel del receptor, este último aumentará (disminuirá) su nivel de altruismo. Este comportamiento induce un proceso estocástico para los niveles de altruismo. Lo valores de largo plazo son estudiados por medio de herramientas analíticas y simulaciones numéricas.

Palabras clave: altruismo, reciprocidad, transferencias entre pares.

Clasificación JEL: D1, D64

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1 Introduction

Altruism is an attitude toward other people that influences our actions. Consequently, it is not unnatural to think that other people's actions toward us can influence our level of altruism. In this way, introspection indicates that if a person feels that another person cares about him or her, then he is more motivated to care about this person in return. This idea leads to reciprocity. Altruism and reciprocity are behaviors that are very connected.

This papers tries to elucidate the relation between altruism and reciprocity in the long run. It takes reciprocity as an innate behavior (exogenous) and assume that altruism is a dynamic behavior (endogenous). To do that I propose a particular model that makes explicit how reciprocity works and affects the levels of altruism. The model shows how reciprocity can sustain altruism or lead to its termination, depending on the parameters involved. The interesting point is that, when altruism is sustained it is not to its maximum level and that altruism can disappear despite positive initial values of it.

In the context of the present work altruism means that individual preference depends not only on own consumption but on the profile of "social consumption". The possible actions induced by altruism are private transfers. These transfers convey information about how much the donor cares about the recipient. I assume that the recipient adjusts his own level of altruism according to this information, trying to reciprocate the behavior perceived. The adjustment that I propose involves the idea of using one's own behavior as a rule to judge others' behavior. In some sense agents try to be as altruistic as the other: "if the other person cares about me more (less) than I care about her I will raise (diminish) my level of altruism toward her". This does not imply that the action of the agent is a direct consequence of the action of other people. In fact, as it will become clear later, it is not the action of the other agent but the "motivation" behind this action that is taken into account. The available information on this motivation is used to modify the "altruistic preference". It is in this way that future behavior is affected.

The paper uses a dynamic model with two individuals whose incomes are random. In each period agents obtain their incomes and determine the amount that they want to transfer to the other agent. These transfers convey information about the degree of altruism of the donor. The recipient of the transfer uses this information to revise their level of altruism. If the transfer the agent receives corresponds to a level of altruism higher (lower) than his own, then the agent will raise (diminish) his level of altruism. This behavior induces a stochastic process for the levels of altruism. We will see how the long-term level of altruism is related with the parameters of the process.

Due to its central motivation the present paper is a contribution to the explanation of persistence or extinguish of altruism.

Altruism, reciprocity, fairness and spitefulness are concepts of studies by Rabin (1993) and Levine (1998). The former uses the artifact of "psychological games" while my setup is more similar to the latter. Levine has players who "are willing to be more altruistic to an opponent who is more altruistic toward them". He uses this idea to explain data from experimental games. In this line is also Charness and Rabin (2002). My paper studies the dynamics and limit properties of the level of altruism induced by this behavior. Another paper in this line is Cervellati et al. (2010), who formulated a model in which sentiments (as they referred to altruism) toward other people are determined by the actions of these people.

The present paper has in common with Bejarano et al. (2018) that individuals are exposed to negative (and positive) shocks. While they analyze the effects of these shocks in a one shot Trust Game, I present the analysis in Bilateral Dictator Game[1]

The specific model presented here is closely related with the model presented by Stark and Falk (1998). They have a two-period game where a first-period transfer instills gratitude in the recipient. This gratitude (and not directly the transfer) will influence a second-period probable reverse transfer. My model is similar in the game played each period, but I consider a long horizon of interaction instead of only two periods. This is because I am interested in the long run value of altruism.

In Falk and Fischbacher (2006) a general theory of reciprocity in extensive games is proposed and applied to several games, the ultimatum and dictator

[1] In a Dictator Game one player transfers income to other that has no alternative to accept the transfer. In the game in this paper both agents have the possibility of make a transfer and none has the possibility to no accept the transfer received, a Bilateral Dictator Game.

games among them, but the approach is static in the sense that interaction is only a short term game. Our model is more specific, not general, but allows to analyze the long run value of altruism due to the reciprocity of agents. In some sense the model presented here can be viewed as an application of an alternative to the theory presented by Falk and Fischbacher (2006).

The structure of the paper is as follows. The basic model is outlined in Section 2. The results are presented in Section 3 and Section 4 concludes the paper.

2 The basic model

2.1 Preliminaries

Two agents, 1 and 2, live in a discrete time world, $t \in \{1, \dots, \infty\}$. Agent i at time t receives an income I_i^t that is the realization of a random variable. These random variables are independently and identically distributed respect to i and t. The distribution is common knowledge and both realizations are observed by both agents. In each period, once agent i knows the income realizations, he can make a private transfer to agent j. Let us call this transfer T_{ij}. After these transfers the wealth of agent i is $W_i = I_i - T_{ij} + T_{ji}$ and his utility[2] is:

$$(1 - \alpha_i)u(W_i) + \alpha_i u(W_j) \tag{1}$$

Here $u(\cdot)$ is a "direct" utility[3] and α_i is the level of altruism of agent i. As a natural restriction, let us impose $\alpha_i \leq \frac{1}{2}$, that is, no agent "cares" more about the other agent than about him or herself.

The agents are completely myopic when deciding how much to transfer to each other. As we will see, these transfers can cause a change in the level of altruism for the following periods, which in turn will affect the transfers the agents can receive in the future. The agents do not take into account all these effects when making the transfers; in every period they play a one shot simultaneous game that determines the transfers. The strategies of agent i are $T_{ij} \in [0, I_i]$ and expression (1) shows his payoff.

In order to have a closed-form equilibrium characterization, I assume that utility takes the well-known CRRA form:

$$u(I) \equiv -\frac{1}{\rho} I^{-\rho}$$

2 All variables must have the time index but as agents will be completely myopic, it is not necessary for the moment to include it.

3 A non-decreasing and concave function satisfying the Inada conditions

with[4] $\rho > -1$. With this utility we have the following proposition characterizing the Nash Equilibrium of the game:

Proposition 1 *The transfers game, with* $(\alpha_1, \alpha_2) \in [0, 1/2] \times [0, 1/2] \backslash \{(1/2, 1/2)\}$*, has a unique Nash Equilibrium in pure strategies. Player i's equilibrium strategy is:*

$$T_{ij} = \max\{0, \gamma_i I_i - (1 - \gamma_i) I_j\} \tag{2}$$

where $\gamma_i = \frac{(\alpha_i)^{\frac{1}{1+\rho}}}{(\alpha_i)^{\frac{1}{1+\rho}} + (1-\alpha_i)^{\frac{1}{1+\rho}}} \in [0, 1/2]$.

Appendix A contains the proof of the proposition and also discusses the case where $\alpha_1 = \alpha_2 = \frac{1}{2}$. This case has multiple equilibria, and Appendix A characterizes all of them. Only one of these equilibria has the agent with the lowest income realization making a zero transfer. I consider that this equilibrium is focal. In this focal equilibrium the strategies are also expressed by (2).

Then we can consider that in all the cases equation (2) determines the transfers. Note that to implement the Nash Equilibrium strategies, agents only have to know the income realizations and their owns alphas.

In equation (2) we can observe a well-known characteristic of this kind of model:

$$T_{ij} > 0 \Leftrightarrow \frac{I_i}{I_j} > \frac{1 - \gamma_i}{\gamma_i} = \left(\frac{1 - \alpha_i}{\alpha_i}\right)^{\frac{1}{1+\rho}} := B_i > 1$$

which is that a positive transfer is made if and only if the donor is sufficiently richer (in relative terms) than the recipient. In particular the agent with the lowest income realization will not make a positive transfer and when incomes are similar no transfers at all will be observed.

Another important aspect that will be used later is that (ceteris paribus) the transfer T_{ij} depends monotonically on α_i : $\frac{\Delta T_{ij}}{\Delta \alpha_i} \geq 0$, as we can see in Figure 1.

The timing of the model is as follows. At the beginning of each period, personal income is realized and observed by both agents; transfers, if any, are made; and finally the level of altruism is revised. The publicly observed variables are: I_1 , I_2 , T_{12} and T_{21} while α_i is private information of agent i.

Once transfers are made, the altruism levels of the individuals are updated using the rule proposed in the Introduction. In the following subsection I will describe this rule in detail and the stochastic process induced on the $\alpha's$. In section 3 the limit properties of the process are studied.

[4] For $\rho = 0$ we have $u(I) = \ln(I)$.

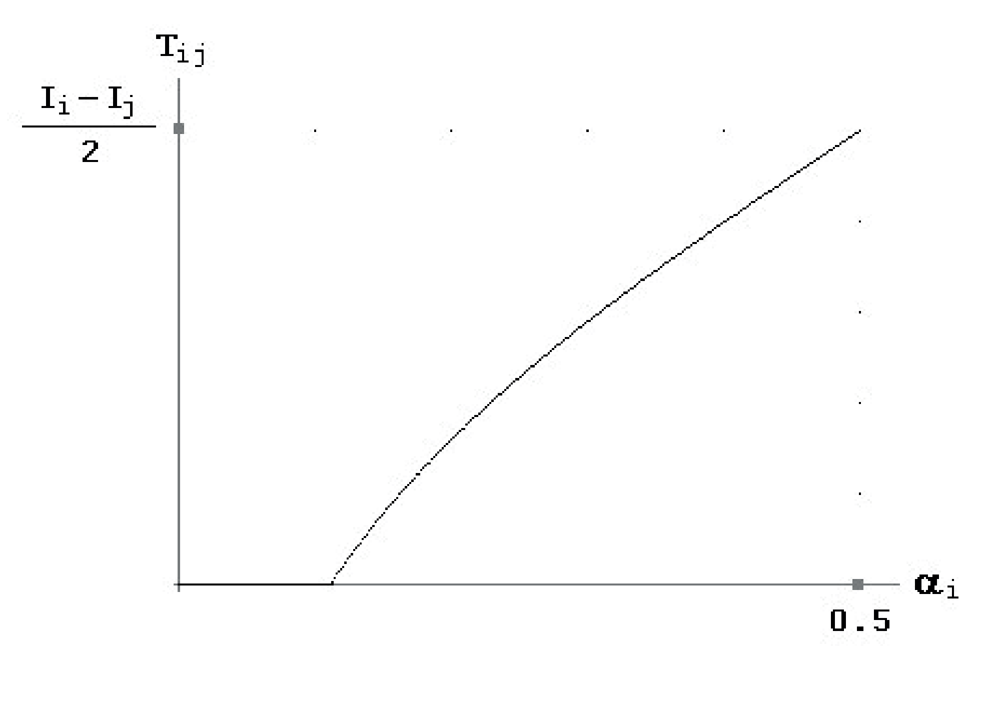

Figure 1: Monotonicity of the transfers

2.2 Dynamics

The dynamic of the model is generated by the revision of the level of altruism in every period. In this revision, agents also behave in a myopic way as they do not take into account the future implications of the altruism adjustment.

This adjustment has its basis in the intention of individuals to reciprocate altruism. If one agent has information about the level of altruism of the other, he uses this information to adjust his own level of altruism. In the next paragraph I will describe the exact way this adjustment is done.

Once incomes are realized, the agent with the lowest income is the potential recipient of a transfer. The agent with the highest realization is the potential donor. He does not expect any positive transfer from the other and consequently has no motive to revise his level of altruism. On the other hand, the agent with the lowest income realization can expect a transfer from the other and uses the information on the amount of the transfer (perhaps zero) to deduce if the other has a higher or lower level of altruism than him or herself. Based on this deduction, he revises his own level of altruism. If the high-income agent has a higher (lower) level of altruism, the low-income agent will raise (diminish) his own level of altruism.

Expressed informally, the agent's actions are based on the following rule: "If you give me less (more) than I would have given to you, then I will care less (more) about you in the future".

Note that the receiver does not in fact calculate the others level of altruism. Instead he compares the transfer received and the transfer that he would make if the incomes were interchanged. The equation that gives the exact value of the altruism coefficient of the other agent involves a lot of knowledge and some sophisticated reasoning. Assuming this level of sophistication in agents who are myopic in their decision making seems to be inconsistent. On the other hand, obtaining the hypothetical transfer involves the same type of calculations as obtaining real transfers. That is, the recipient of the transfer uses equation (2) interchanging I_i and I_j to calculate how much he would have given to the other. If we denote the hypothetical transfer from i to j if the incomes are interchanged, by T'_{ij} we have:

$$T'_{ij} = \max\{0, \gamma_i I_j - (1-\gamma_i) I_i\} \tag{3}$$

As we note before transfer T_{ij} depends monotonically on α_i so the order relation between the transfer received and the hypothetical one is related to the order relation between the alphas[5]:

$$\begin{aligned} T'_{ij} > T_{ji} &\Rightarrow \alpha_i > \alpha_j \\ T'_{ij} < T_{ji} &\Rightarrow \alpha_i < \alpha_j \\ T'_{ij} = T_{ji} > 0 &\Rightarrow \alpha_i = \alpha_j \\ T'_{ij} = T_{ji} = 0 &\Rightarrow \text{No information} \end{aligned}$$

Once the recipient agent (say agent i) compares these two quantities, he adjusts his level of altruism according to the ordering between them. If $\alpha_i < \alpha_j$ then he increases his α_i. If $\alpha_i > \alpha_j$ then he decreases his α_i. As the alphas have lower and upper bounds these adjustments have to consider these boundaries. I use the notation $\alpha_i^t \downarrow_0$ for the downward adjustment (toward zero) and $\alpha_i^t \uparrow^{0.5}$ for the upward adjustment (toward 0.5). If the potential recipient agent concludes that the alphas are equal, then no adjustment is made.

Finally we have to carefully consider the fourth case, where $T'_{ij} = T_{ji} = 0$. In this case no information about the relation between the alphas can be extracted. Here we have two options. The first one is consider that no alpha

[5] For example: $T'_{ij} > T_{ji} \Rightarrow \gamma_i I_j - (1-\gamma_i) I_i > \gamma_j I_j - (1-\gamma_j) I_i \Rightarrow (\gamma_i - \gamma_j)(I_i + I_j) > 0 \Rightarrow \gamma_i - \gamma_j > 0 \Rightarrow \alpha_i > \alpha_j$

is adjusted. The second one is consider that the agent with the low-income realization observes the difference between incomes but does not receive any transfer. I assume that this situation induces the agent to reduce his level of altruism, but in a weaker way. This adjustment is denoted by $\alpha_i^t \downharpoonleft_0$. Note that this approach include the first one only by consider $\alpha_i^t \downharpoonleft_0 = \alpha_i^t$. Putting all this information together and recovering the time superscript when i and j are such that $I_i^t < I_j^t$ we have:

$$
\begin{aligned}
{T'_{ij}}^t > {T_{ji}}^t &\rightarrow \alpha_i^{t+1} = \alpha_i^t \downarrow_0 \\
{T'_{ij}}^t < {T_{ji}}^t &\rightarrow \alpha_i^{t+1} = \alpha_i^t \uparrow^{0.5} \\
{T'_{ij}}^t = {T_{ji}}^t > 0 &\rightarrow \alpha_i^{t+1} = \alpha_i^t \\
{T'_{ij}}^t = {T_{ji}}^t = 0 &\rightarrow \alpha_i^{t+1} = \alpha_i^t \downharpoonleft_0
\end{aligned}
$$

The notation should clearly indicate whether the adjustments are toward 0 or 0.5, depending on the specific case, and should also indicate that in the last case the adjustment is weaker compared to the other cases. The exact specification of these adjustments is postponed to the next section.

Thus the change of the alphas depends on their present value (the state) and the realization of the incomes (which will determine the transfers). To analyze the evolution of altruism over time we have to derive the stochastic process that random incomes impose on the alpha values.

As we have $I_i \sim [0, +\infty[$, we can construct a distribution for $\frac{I_1}{I_2} \sim [0, +\infty[$ with CDF $F(\cdot)$, it can be show that this distribution satisfies: $F(x) = 1 - F(1/x)$ for all $x \geq 0$. In particular this implies: $F(1) = 0.5$.

In the Appendix B I make the study that gives us the following description of the stochastic process that determines the time path of the alphas:

$$
(\alpha_1^{t+1}, \alpha_2^{t+1}) = \begin{cases}
\left.\begin{array}{lll}
(\alpha_1^t \downarrow_0, \alpha_2^t) & \text{wp} & 1 - F(B_1^t) \\
(\alpha_1^t, \alpha_2^t \downharpoonleft_0) & \text{wp} & F(B_1^t) - 0.5 \\
(\alpha_1^t \downharpoonleft_0, \alpha_2^t) & \text{wp} & F(B_1^t) - 0.5 \\
(\alpha_1^t, \alpha_2^t \uparrow^{0.5}) & \text{wp} & 1 - F(B_1^t)
\end{array}\right\} \text{if } \alpha_1^t > \alpha_2^t \\
\left.\begin{array}{lll}
(\alpha_1^t, \alpha_2^t \downarrow_0) & \text{wp} & 1 - F(B_2^t) \\
(\alpha_1^t, \alpha_2^t \downharpoonleft_0) & \text{wp} & F(B_2^t) - 0.5 \\
(\alpha_1^t \downharpoonleft_0, \alpha_2^t) & \text{wp} & F(B_2^t) - 0.5 \\
(\alpha_1^t \uparrow^{0.5}, \alpha_2^t) & \text{wp} & 1 - F(B_2^t)
\end{array}\right\} \text{if } \alpha_1^t < \alpha_2^t \\
\left.\begin{array}{lll}
(\alpha_1^t, \alpha_2^t) & \text{wp} & 2(1 - F(B_1^t)) \\
(\alpha_1^t, \alpha_2^t \downharpoonleft_0) & \text{wp} & F(B_1^t) - 0.5 \\
(\alpha_1^t \downharpoonleft_0, \alpha_2^t) & \text{wp} & F(B_1^t) - 0.5
\end{array}\right\} \text{if } \alpha_1^t = \alpha_2^t
\end{cases} \tag{4}
$$

Where[6] $B_i^t = \left(\frac{1-\alpha_i^t}{\alpha_i^t}\right)^{\frac{1}{1+\rho}}$.

This corresponds to a bidimensional Markov Process (MP) in discrete time. As a first approach to the analysis of this MP let investigate its absorbing states.

First suppose $\alpha \downharpoonleft_0 = \alpha$, that is, when there is no information no alpha is adjusted. In this case $(\alpha_1^{t+1}, \alpha_2^{t+1}) = (\alpha_1^t, \alpha_2^t)$ with probability 1 if $\alpha_1^t = \alpha_2^t$ or if $F(B_i^t) = 1$ when $\alpha_i^t > \alpha_j^t$. For the last case, this is possible only if I_i/I_j is bounded above, let say by $\mathfrak{B}$. Then $F(B_i^t) = 1$ if and only if $B_i^t \geq \mathfrak{B}$, that is if and only if $\alpha_i^t \leq \frac{1}{1+\mathfrak{B}^{1+\rho}}$. Concluding we have a big set of absorbing states, those with $\alpha_1 = \alpha_2$ or with $\max\{\alpha_1, \alpha_2\} \leq \frac{1}{1+\mathfrak{B}^{1+\rho}}$.

On the other side if $\alpha \downharpoonleft_0 < \alpha$ it is easy to see that no state with $\alpha_1 \neq \alpha_2$ can be absorbing. To $\alpha_1 = \alpha_2$ be an absorbing state we need $\alpha_i^t \downharpoonleft_0 = \alpha_i^t$ or $F(B_1^t) = 0.5$. The first possibility corresponds to $\alpha_1 = \alpha_2 = 0$ and the second one to $B_1^t = 1$, that is to $\alpha_1 = \alpha_2 = 0.5$. Then in this case there are only two absorbing states: $(0, 0)$ and $(1/2, 1/2)$.

In the following section I will present the properties of this Markov Process for a particular specification of the adjustment process.

3 The Long-Run Behavior of the Altruism Level

In this section I will present the analysis of the Markov Process in (4). This analysis will be done in two complementary parts. First I present some theoretical results. As these results are not clear enough about the long behavior of (4) I close the analysis presenting some simulations.

I consider that the α's are real variables on the interval $[0, 0.5]$ and that the adjustment laws, for $k_r \in]0, 1[$ and $k_f \in [0, 1[$, take the form:

$$\begin{aligned} \alpha \uparrow^{0.5} &= (1-k_r)\alpha + (k_r)0.5 = \alpha + (k_r)(0.5-\alpha) \\ \alpha \downarrow_0 &= (1-k_r)\alpha + (k_r)0 = \alpha - k_r\,\alpha \\ \alpha \downharpoonleft_0 &= (1-k_f)\alpha + (k_f)0 = \alpha - k_f\,\alpha \end{aligned} \tag{5}$$

Under this specification, the changes in the alphas are proportional adjustments toward the corresponding boundaries. These changes depend inversely on the distance between α and the boundary. To incorporate the concept that $\downharpoonleft_0$ is a weaker adjustment than $\downarrow_0$, we have to consider that $k_f < k_r$. The case with $\alpha \downharpoonleft_0 = \alpha$ is captured with $k_f = 0$.

[6] If $\alpha_i^t = 0$ we consider $B_i^t = +\infty$ and so $F(B_1^t) = 1$.

Once we insert equations (5) into (4) we obtain the precise Markov Process for the alphas. The parameters of this MP are $k_r,\ k_f,\ (\alpha_1^0, \alpha_2^0)$, ρ and those of the specific CDF of the incomes. The complete analysis of this MP is very complex. I will present some analytic results and some computer simulations, which together provide rich evidence about the limit behavior of the process.

3.1 Analytic results

In this subsection I will decompose the Markov Process followed by the alphas in two pieces. Each of these pieces has clear properties on its limits. What will be not so clear is the way the pieces interact to generate the actual Markov Process. About this point we will have some intuitions. In the next subsection these intuitions will be confirmed by simulations.

Using (5) on (4) we obtain the MP defined by:

$$(\alpha_1^{t+1}, \alpha_2^{t+1}) = \left\{ \begin{array}{lll} (\alpha_1^t - k_r\,\alpha_1^t, \alpha_2^t) & \text{wp} & 1 - F(B_1^t) \\ (\alpha_1^t, \alpha_2^t - k_f\,\alpha_2^t) & \text{wp} & F(B_1^t) - 0.5 \\ (\alpha_1^t - k_f\,\alpha_1^t, \alpha_2^t) & \text{wp} & F(B_1^t) - 0.5 \\ (\alpha_1^t, \alpha_2^t + (k_r)(0.5 - \alpha_2^t)) & \text{wp} & 1 - F(B_1^t) \\ (\alpha_1^t, \alpha_2^t - k_r\,\alpha_2^t) & \text{wp} & 1 - F(B_2^t) \\ (\alpha_1^t, \alpha_2^t - k_f\,\alpha_2^t) & \text{wp} & F(B_2^t) - 0.5 \\ (\alpha_1^t - k_f\,\alpha_1^t, \alpha_2^t) & \text{wp} & F(B_2^t) - 0.5 \\ (\alpha_1^t + (k_r)(0.5 - \alpha_1^t), \alpha_2^t) & \text{wp} & 1 - F(B_2^t) \\ (\alpha_1^t, \alpha_2^t) & \text{wp} & 2(1 - F(B_1^t)) \\ (\alpha_1^t, \alpha_2^t - k_f\,\alpha_2^t) & \text{wp} & F(B_1^t) - 0.5 \\ (\alpha_1^t - k_f\,\alpha_1^t, \alpha_2^t) & \text{wp} & F(B_1^t) - 0.5 \end{array} \right. \begin{array}{l} \left.\vphantom{\begin{array}{l}a\\a\\a\\a\end{array}}\right\} \text{if } \alpha_1^t > \alpha_2^t \\ \left.\vphantom{\begin{array}{l}a\\a\\a\\a\end{array}}\right\} \text{if } \alpha_1^t < \alpha_2^t \\ \left.\vphantom{\begin{array}{l}a\\a\\a\end{array}}\right\} \text{if } \alpha_1^t = \alpha_2^t \end{array} \tag{6}$$

Taking a close look to this process and exploiting its symmetries we can decompose it in the following way:

$$(\alpha_1^{t+1}, \alpha_2^{t+1}) = \left\{ \begin{array}{lll} \Phi_1(\alpha_1^t, \alpha_2^t) & \text{wp} & 2F(\bar{B}^t) - 1 \\ \Phi_2(\alpha_1^t, \alpha_2^t) & \text{wp} & 2 - 2F(\bar{B}^t) \end{array} \right. \tag{7}$$

where $\bar{B}^t = \left(\frac{1 - \max\left\{\alpha_1^t, \alpha_2^t\right\}}{\max\left\{\alpha_1^t, \alpha_2^t\right\}} \right)^{\frac{1}{1+\rho}}$ and each of the components are:

$$\Phi_1(\alpha_1^t, \alpha_2^t) = \left\{ \begin{array}{lll} (\alpha_1^t, \alpha_2^t - k_f\,\alpha_2^t) & \text{wp} & 0.5 \\ (\alpha_1^t - k_f\,\alpha_1^t, \alpha_2^t) & \text{wp} & 0.5 \end{array} \right. \tag{8}$$

and

$$\Phi_2(\alpha_1^t, \alpha_2^t) = \begin{cases} \left.\begin{array}{lll} (\alpha_1^t - k_r\, \alpha_1^t, \alpha_2^t) & \text{wp} & 0.5 \\ (\alpha_1^t, \alpha_2^t + (k_r)(0.5 - \alpha_2^t)) & \text{wp} & 0.5 \end{array}\right\} \text{if } \alpha_1^t > \alpha_2^t \\ \left.\begin{array}{lll} (\alpha_1^t, \alpha_2^t - k_r\, \alpha_2^t) & \text{wp} & 0.5 \\ (\alpha_1^t + (k_r)(0.5 - \alpha_1^t), \alpha_2^t) & \text{wp} & 0.5 \end{array}\right\} \text{if } \alpha_1^t < \alpha_2^t \\ \left.\begin{array}{lll} (\alpha_1^t, \alpha_2^t) & \text{wp} & 1.0 \end{array}\right\} \text{if } \alpha_1^t = \alpha_2^t \end{cases} \tag{9}$$

Now I will study each of the process generated by Φ_1 and Φ_2. It is easy to see that the process defined by:

$$(\alpha_1^{t+1}, \alpha_2^{t+1}) = \Phi_1(\alpha_1^t, \alpha_2^t)$$

converges to zero:

Proposition 2 *If* $0 < k_f < 1$ *then the Markov Process:* $(\alpha_1^{t+1}, \alpha_2^{t+1}) = \Phi_1(\alpha_1^t, \alpha_2^t)$ *converges in mean square to* 0.

Proof. $E_t[\alpha_i^{t+1}] = 0.5\alpha_1^t + 0.5(\alpha_i^t - k_f\alpha_i^t) = (1 - 0.5k_f)\alpha_1^t$. Then: $E[\alpha_i^{t+1}] = (1 - 0.5k_f)E[\alpha_1^t]$. Solving the difference equation we obtain:

$$E[\alpha_i^t] = (1 - k_f/2)^t E[\alpha_1^0]$$

so

$$\lim_{t\to+\infty} E[\alpha_i^t] = 0$$

For the variance we have[7]:

$$V[\alpha_i^{t+1}] = \frac{k_f^2 - 2k_f + 2}{2} V[\alpha_i^t] + \frac{k_f^2}{2}(\frac{2-k}{2})^{2t} E[\alpha_i^0]^2 \tag{10}$$

that solves on:

$$V[\alpha_i^t] = (E[\alpha_i^0]^2 + V[\alpha_i^0])(\frac{k_f^2 - 2k_f + 2}{2})^t - (\frac{2-k}{2})^{2t} E[\alpha_i^0]^2$$

and in the limit:

$$\lim_{t\to+\infty} V[\alpha_i^t] = 0$$

■

The process governed by Φ_2 is harder to analysis. I will characterize the limit behavior of the expected value and variance of the alphas using the following:

---

7 Please see Appendix D for details

Lemma 3 *Consider the MP:*

$$(\alpha_1^{t+1}, \alpha_2^{t+1}) = \Phi_2(\alpha_1^t, \alpha_2^t)$$

If (α_1^0, α_2^0) *is uniformly distributed on the square* $[0, 1/2] \times [0, 1/2]$ *then for* $i = 1, 2$ *and at any* $t \geq 0$*:*

$$\Pr[\alpha_1^t > \alpha_2^t] = \Pr[\alpha_1^t < \alpha_2^t] = 0.5$$

and so $\Pr[\alpha_1^t = \alpha_2^t] = 0$.

Appendix C presents the proof of this lemma. Now we can state and prove

Proposition 4 *For the MP:*

$$(\alpha_1^{t+1}, \alpha_2^{t+1}) = \Phi_2(\alpha_1^t, \alpha_2^t)$$

when $t \to \infty$ *we have that:*

$$\begin{aligned} E\left[\alpha_i^t\right] &\to 1/4 \\ V\left[\alpha_i^t\right] &\to \frac{k_r}{16(2-k_r)} \end{aligned}$$

Proof. Direct calculation gives:

$$E_t\left[\alpha_i^{t+1}\right] = \begin{cases} \alpha_i^t - \frac{k_r}{2}\alpha_i^t & \text{if} \quad \alpha_i^t > \alpha_j^t \\ \alpha_i^t + \frac{k_r}{2}(0.5 - \alpha_i^t) & \text{if} \quad \alpha_i^t < \alpha_j^t \\ \alpha_i^t & \text{if} \quad \alpha_i^t = \alpha_j^t \end{cases}$$

Then

$$E[\alpha_i^{t+1}] = (E[\alpha_i^t] - \frac{k_r}{2}E[\alpha_i^t]) \Pr[\alpha_i^t > \alpha_j^t] + (E[\alpha_i^t] + \frac{k_r}{2}(0.5 - E[\alpha_i^t])) \Pr[\alpha_i^t < \alpha_j^t] + (E[\alpha_i^t]) Pr[\alpha_i^t = \alpha_j^t]$$

and by Lemma 3:

$$\begin{aligned} E[\alpha_i^{t+1}] &= (E[\alpha_i^t] - \frac{k_r}{2}E[\alpha_i^t])0.5 + (E[\alpha_i^t] + \frac{k_r}{2}(0.5 - E[\alpha_i^t]))0.5 \\ &= (1 - \frac{k_r}{2})E[\alpha_i^t] + k/8 \end{aligned}$$

Solving the difference equations:

$$E[\alpha_i^t] = (1 - \frac{k_r}{2})^t (E[\alpha_i^0] - \frac{1}{4}) + \frac{1}{4}$$

Finally

$$\lim_{t \to +\infty} E\left[\alpha_i^t\right] = \frac{1}{4}$$

For the variance $V[\alpha_i^t]$ we have[8]:

$$V[\alpha_i^{t+1}] = \frac{k_r^2 - 2k_r + 2}{2} V[\alpha_i^t] + \frac{k_r^2}{64}(\frac{2-k_r}{2})^{2t}(4E[\alpha_i^0] - 1)^2 + \frac{k_r^2}{32} \quad (11)$$

solving the difference equation we obtain:

$$\begin{aligned} V[\alpha_i^t] &= (\frac{k_r^2 - 2k_r + 2}{2})^t V[\alpha_i^0] + \\ &+((\frac{2-k_r}{2})^{2t} - (\frac{k_r^2 - 2k_r + 2}{2})^t)\frac{E[\alpha_i^0](1 - 2E[\alpha_i^0])}{2} + \\ &+\frac{k_r - 1}{8(k_r - 2)}(\frac{k_r^2 - 2k_r + 2}{2})^t - \frac{1}{16}(\frac{2-k_r}{2})^{2t} + \frac{k_r}{16(2-k_r)} \end{aligned}$$

taking limits we finally obtain:

$$\lim_{t \to +\infty} V[\alpha_i^t] = \frac{k_r}{16(2-k_r)}$$

■

This proposition tells us that, for the process generated by Φ_2 (9), we can expect the alphas to oscillate over time around the 0.25 value with an amplitude that depends positively on the magnitude of k_r.

The analysis of how these two processes compound the actual MP we are interested in, equation (6), is not easy. I will provide some insights of this analysis. Roughly speaking the long run behavior of the complete MP (7) depends on the behavior of $F(\bar{B}^t)$ and on the relative speed of convergence of each of the two components. As an example suppose that the adjustments in Φ_1 are very strong and then the convergence to zero is very fast while in Φ_2 we have small changes with a correspondent slow convergence. In this case we have to expect that the whole Markov Process converge to zero. In the opposite case, with Φ_2 "strong" and Φ_1 "soft", the convergence to a positive value (or support) will be not surprising. The other determinant of the long run behavior of the levels of altruism is the behavior of $F(\bar{B}^t)$. If this value is persistently low (near 0.5, in correspondence with α's also near 0.5) then Φ_2 will dominate the MP and a positive long run level of altruism can be expected. If $F(\bar{B}^t)$ is persistently high (near 1, in correspondence with α's near 0) then Φ_1 will dominate the MP and the level of altruism will go to zero without surprise.

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[8] Please see Appendix D for details

In terms of the basic parameters of the MP we have that while k_r and k_r affect the speed of convergence, ρ and the specific parameters of the income random variable distribution affect $F(\bar{B}^t)$. The effect of initial values α_1^0 and α_2^0 deserves a carefully discussion. The next paragraphs present this discussion and also those for the other parameters.

α_1^0 and α_2^0: The results stated in the propositions do not depend on α_1^0 and α_2^0, but if these values are very close to zero then $F(\bar{B}^0)$ will be very close to 1 and it is very probable that Φ_1 will determine the entire MP. So if in this case $k_f \neq 0$ the process can converge to zero and if $k_f = 0$ the process can stay permanently in (α_1^0, α_2^0).

k_r and k_f: As we said, these parameters affect the speed of convergence of the two basic processes. In general if k_f is low enough respect to k_r we can expect that Φ_2 dominates Φ_1, because of its greater speed of convergence. If this is the case the MP will oscillate around $1/4$ if $k_f = 0$ and around a value near but below $1/4$ if $k_f > 0$.

But if k_f is too high with respect to k_r, it is possible that the dominance of over (9) leads to the convergence of the whole MP to $(0, 0)$.

ρ: The value of $F(\bar{B}^t)$ depends negatively on ρ. Then, with a high ρ it is easier to have a low value of $F(\bar{B}^t)$ than with a low ρ. This means that Φ_2 will be more frequently the law that the MP follows, and that we can expect the convergence to a positive value of the altruism levels.

CDF: The particular distribution of the income random variable affects directly $F(\bar{B}^t)$. As $\bar{B}^t \geq 1$ and $F(1) = 0.5$, it will be in favor of a positive convergence that the cumulative $F(x)$ stay close to 0.5 as far as possible for $x > 1$. This mean that the probability of I_1 and I_2 being similar has to be low. One case where we have this is when the random variables I_i are draw from a distribution with a high (relative to the mean) variance.

In resume we can state:

Fact 1 *α_1^0 and α_2^0 not too low, k_r not too high and k_f relatively low, ρ high and an income random process with high variability are in favor of the persistence of a positive level of altruism in he long run.*

In plain language, in order to have a long run persistence of altruism we need some minimal level of initial altruism, a smooth adjustment process, an agents utility concave enough and a high income variability.

In the following subsection, the computer simulations illustrate all these points.

3.2 Computer Simulations

In this subsection I will present some simulations that sustain Fact 1.

The parameters of the MP defined by equations (4) and (5) are $\{\alpha_1^0, \alpha_2^0, k_r, k_f, \rho\}$ plus those of the specific CDF of the incomes. I fix this CDF on the family of Lognormal Distributions with mean μ and variance σ.

Before the simulations let note that if I_1 and I_2 are draw from a Lognormal Distribution with mean μ and variance σ^2, then the distribution of the ratio I_i/I_j is also Lognormal, with mean $1 + (\sigma/\mu)^2$ and variance $(\sigma/\mu)^2(2 + (\sigma/\mu)^2)(1 + (\sigma/\mu)^2)^2$. Then the distribution of the ratio I_i/I_2 depends on the Coefficient of Variation, $CV = \sigma/\mu$, of the original random distribution of incomes.

Consequently, the set of parameters is now $\alpha_1^0, \alpha_2^0, \{k_r, k_f, \rho, CV = \frac{\sigma}{\mu}\}$.

For the simulations I will present I take the following values of the parameters as a base:

$\alpha_1^0 = 0.49 \quad \alpha_2^0 = 0.01 \quad k_r = 0.01 \quad k_f = 0.0009 \quad \rho = 0.9 \quad CV = 0.5$

From this base I study each parameter one by one in order to illustrate its effect on the long run behavior of altruism. The study of this effect is do it by carrying 100 long run simulations of the MP for each value of the parameter of interest. From each of these simulations I take the "late mean" of the alphas values, that is the mean of the last 1% of each alpha time series. So for each set of parameters I have 100 mean values. I calculate the maximum, the average and the minimum of this set and plot these values vas the parameter of study in each case.

In addition a particular time series for some relevant parameters values are showed.

Effect of $\alpha_{\mathbf{i}}^{\mathbf{0}}$:

As a first exercise I take $\alpha_2^0 = 0.01$, $k_r = 0.01$, $k_f = 0.0009$, $\rho = 0.9$, $CV = 0.5$ and make α_1^0 going from 0 to 0.5 (in steps of 0.005). Fact 1 suggest that the long run behavior of altruism has to depend positively on the α_1^0 value.

For each set of parameters 100 simulations were done. From each of these simulations the mean of the last 1% of each alpha time series was calculate. The maximum, the average and the minimum of these 100 values are showed in the Figure 2. The picture in this figure plots these three values in the vertical axis for each value of α_1^0 in the horizontal axis.

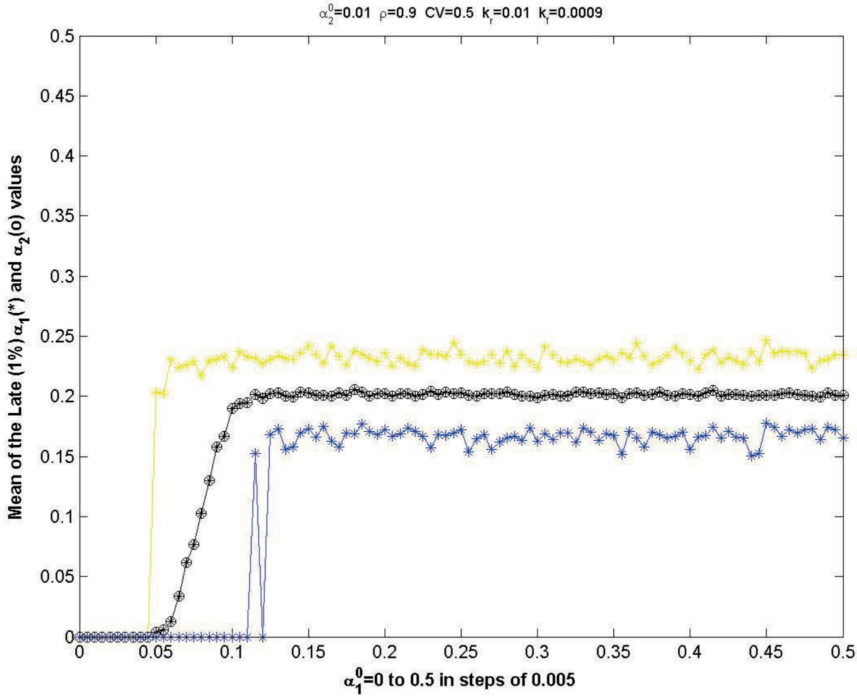

Figure 2: Effect of $\alpha_{\mathbf{i}}^{\mathbf{0}}$ from 0 to 0.5

As we can see if α_1^0 is low enough altruism disappears in the long run, while for α_1^0 bigger than some lower bound the long run level of altruism is positive. Note that it seems that this level does not depend on the particular value of the initial alphas, once the maximum value is greater than the lower bound.

For more illustration Figure 3 shows the time series of then alphas for three specific values of α_1^0.

In the first window we have the case of $\alpha_1^0 = 0.05$, in this case altruism begins so low that not survive the process of adjustment. In the other two cases altruism stabilizes, in some way, at a value closely below 0.25.

<u>Effect of $\mathbf{k_r}$:</u>

Now I fix $\alpha_1^0 = 0.49, \alpha_2^0 = 0.01, k_f = 0.0009, \rho = 0.9, CV = 0.5$ and let k_r take values between 0.0009 (the k_f value) and 0.5009 in steps of 0.005. We expect to observe that for low values of k_r, that is near to the value of

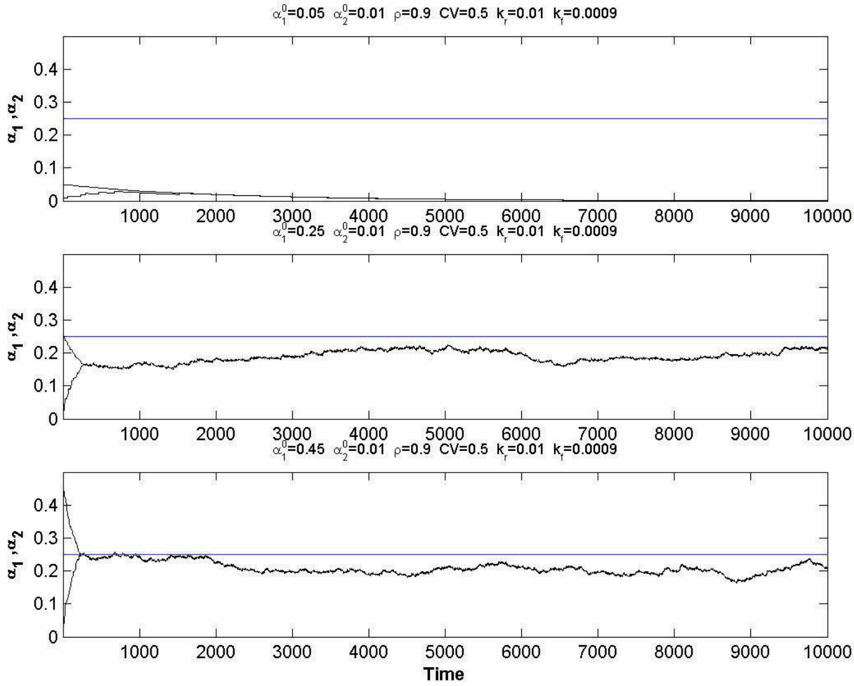

Figure 3: Effect of $\alpha_{\mathbf{i}}^{\mathbf{0}}$

$k_f = 0.009$, the level of altruism tends to zero or low values. In Figure 4 we can see that for values of k_r near 0.009 the long run level of altruism is low but not zero. With k_r in a central range the level of altruism takes positive values in the long run, and again this value seems not to depend on the exact value of k_r. One additional feature we can observe is that for large value of k_r altruism can eventually disappear and for very large values of k_r altruism disappears for sure. This is because the adjustment at each period is so strong that, with some positive probability, the adjusted level of altruism can be very low. From this level as initial value the altruism disappears as in the previous simulations.

Let now take a look to the time path of altruism for three specific values of k_r. Figure 5 shows its.

In this figure we can clearly observe the effect of k_r on the alpha's Markov Process. For $k_r = 0.05$ the adjustment is smooth and the level of altruism stabilizes near 0.25. For a larger k_r, 0.25 in the middle graph, altruism does not disappear but is very volatile. With $k_r = 0.5$ this volatility is so strong

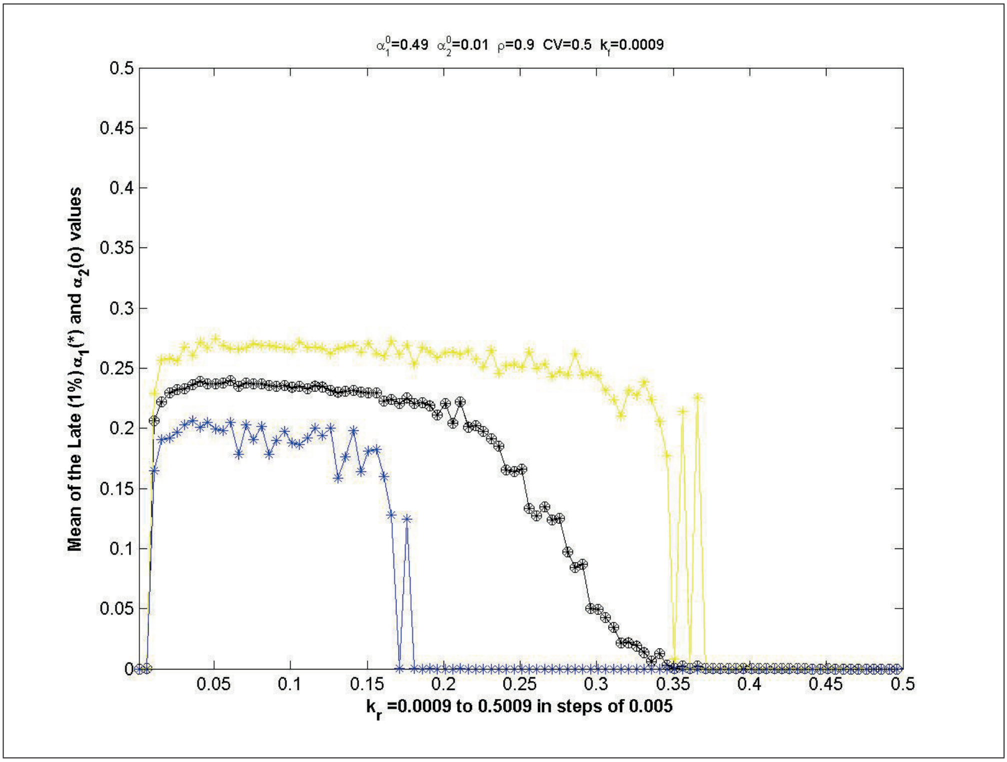

Figure 4: Effect of $\mathbf{k_r}$

that altruism disappears. I have to mention that this last case is only one possibility, in some simulations with the same k_r value altruism survives for as long as for 10,000 periods .

Effect of $\mathbf{k_f}$:

To see how k_f affects the long run behavior of altruism I fix all other parameters: $\alpha_1^0 = 0.49, \alpha_2^0 = 0.01, k_r = 0.01, \rho = 0.9, CV = 0.5$ and let k_f go from 0 to 0.01 in steps of 0.0001. Figure 6 shows the relation between the limit of the alphas and the value of k_f.

This relation is pretty clear. With $k_f = 0$ the long run value of the level of altruism is very close to 0.25. When k_f grows toward 0.01 this level decreases toward zero. For k_f greater than some value between 0.001 and 0.002 altruism tends to zero in the long run. Here, in opposition with the previous cases, the level of altruism depends monotonically on the value k_f.

For this case I also present the time series of the alphas for three values of k_f. We can see them in Figure 7

These graphs only confirm what we say before. The only particularity is

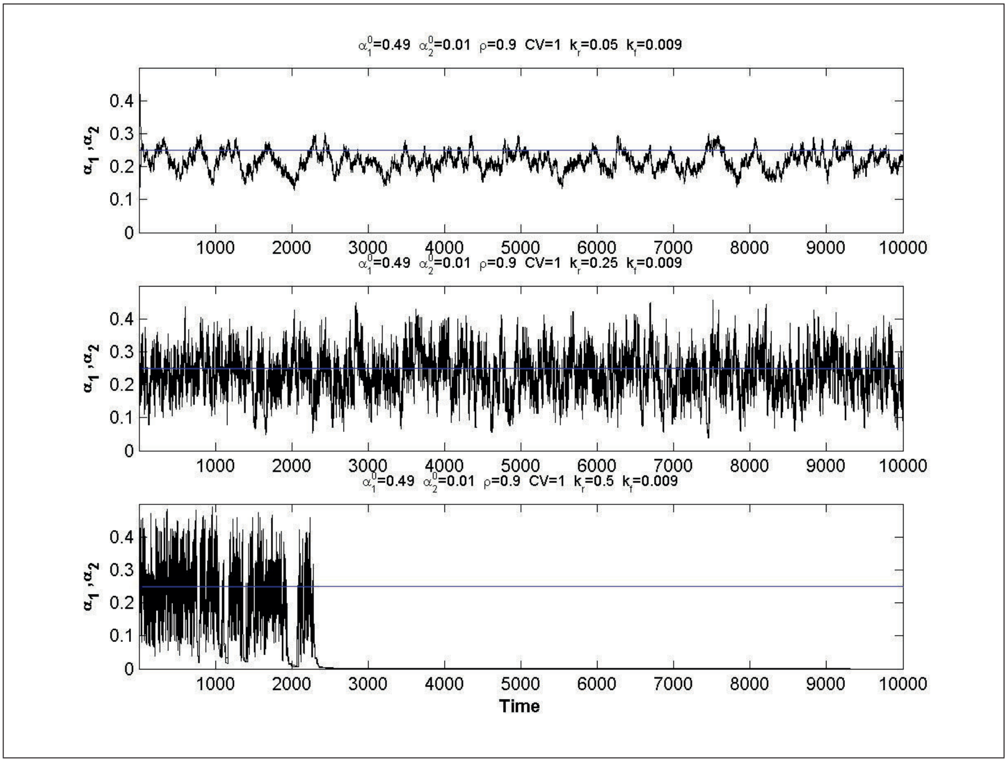

Figure 5: Effect of $\mathbf{k_r}$

that in the last case, $k_f = 0.01$, the level of altruism does not disappear in the first 10,000 periods. This constitutes a particular example, most of the times the simulation with this value present the convergence to zero of the level of altruism.

Effect of ρ:

Fact 1 indicates that a low ρ can cause the level of altruism to converge to zero while a high ρ permits the convergence to a positive value. We can see this in Figure 8 below.

In this graph $\alpha_1^0 = 0.49, \alpha_2^0 = 0.01, k_r = 0.01, k_f = 0.0009, CV = 0.5$ are fixed and ρ goes from 0 to 4 in steps of 0.04. We can observe that for ρ below some value less than one the level of altruism goes to zero. For values of ρ greater than one the level of altruism converges to a positive value. We can see also that this value softly depends monotonically on ρ.

Figure 9 shows three time path of the alphas values for three representative ρ. With $\rho = 0.5$ the level of altruism goes monotonically to zero while with $\rho = 2$ or 4 the two alpha values are quickly stabilized below 0.25.

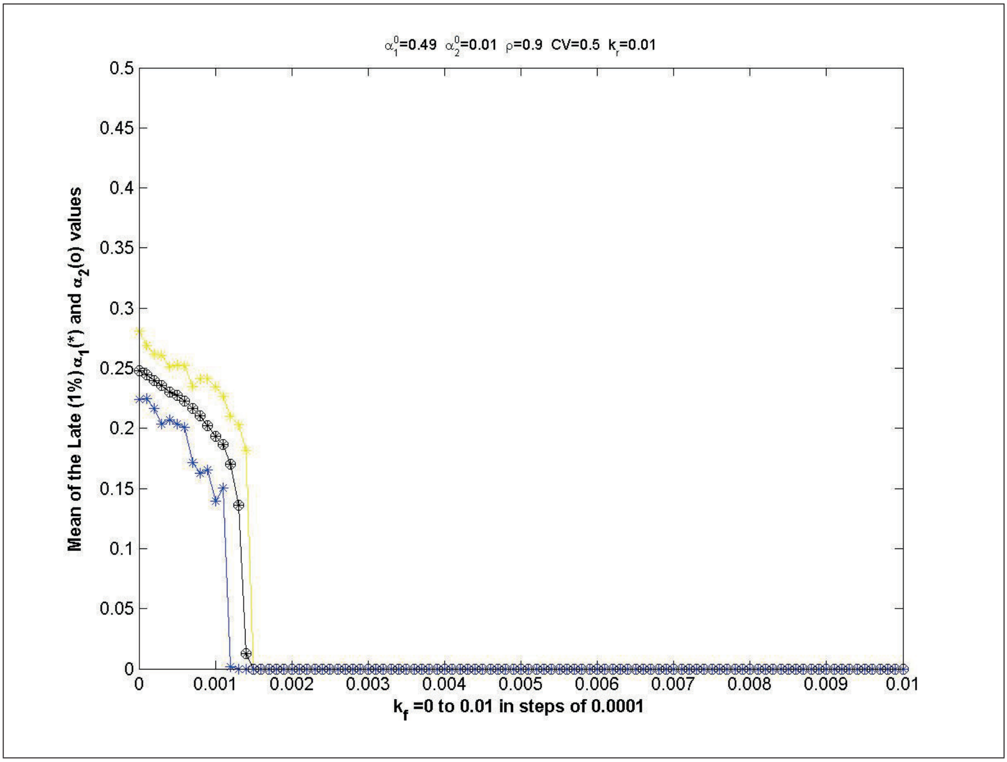

Figure 6: Effect of $\mathbf{k_f}$

Effect of **CV**:

The last simulation exercise investigates the effect of the income variability on the level of altruism. The income variability corresponds to the coefficient of variation (CV) of the random variable that produces income realizations. We expect that a sufficiently high CV causes altruism to be sustained along time. We can see this in Figure 10. For low values, say less than 0.5, of CV altruism goes to zero in the long run. When the CV goes from 0.5 to 1 the long run level of altruism grows from zero to some value below 0.25. Beyond CV=1 the alphas values still grow but slowly than before.

At last I present the time path of the alphas for three CV values. We can observe how for CV=0.5 the level of altruism goes to zero very quickly. Also we can observe that there is not too much difference between the time path when CV=2 with the time path when CV=4. In both cases altruism is quickly stabilized around its mean.

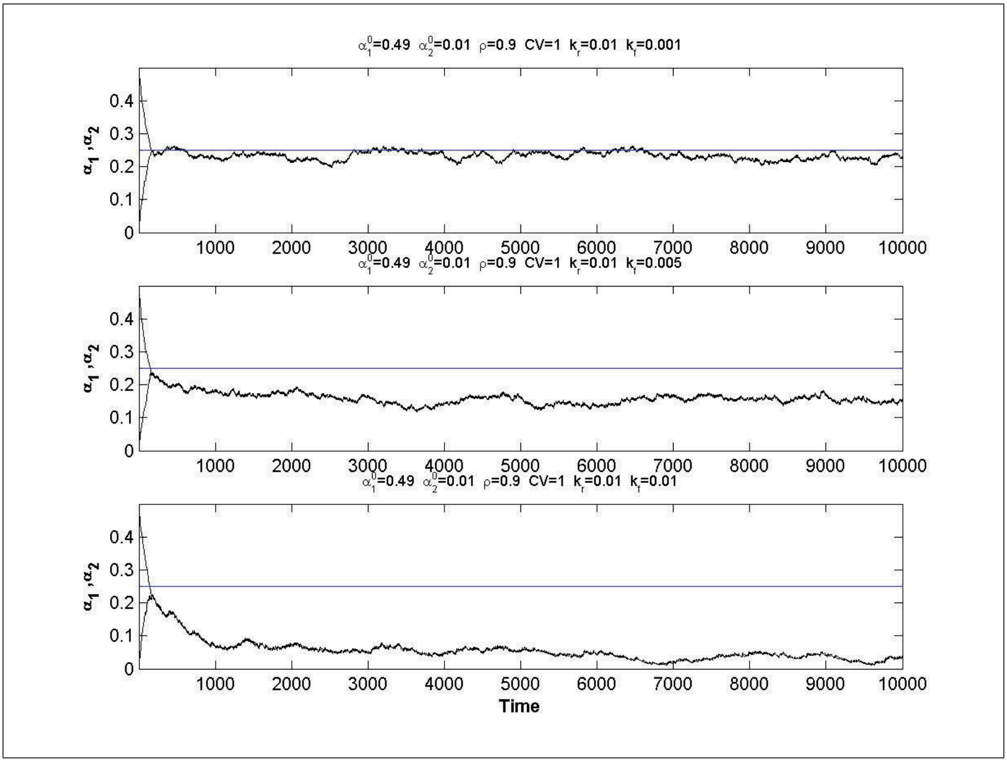

Figure 7: Effect of $\mathbf{k_f}$

4 Conclusion

The model presented in this paper study endogenous altruism in a dynamic way. The focus was on the conditions that allow altruism ton survive in the long run. The model implement the idea that altruistic motivated actions modify he altruism concern of the receptors of these actions. Then the levels of altruism are under a feedback process. In this paper I assumed that this feedback is product of the adjustment by the agents of its own level of altruism after observe the action of other agents. This process can cause the altruism to disappear after some time or to persist in the long run at a positive level.

With a particular adjustment process I had study this two possibilities. I obtain that the persistence or not of altruism depends on the the concavity of the utility function, the variability of the income process and the specific parameters of adjustment. Those parameters can be differentiated as personal and environmental. Personals parameters are the concavity of the utility

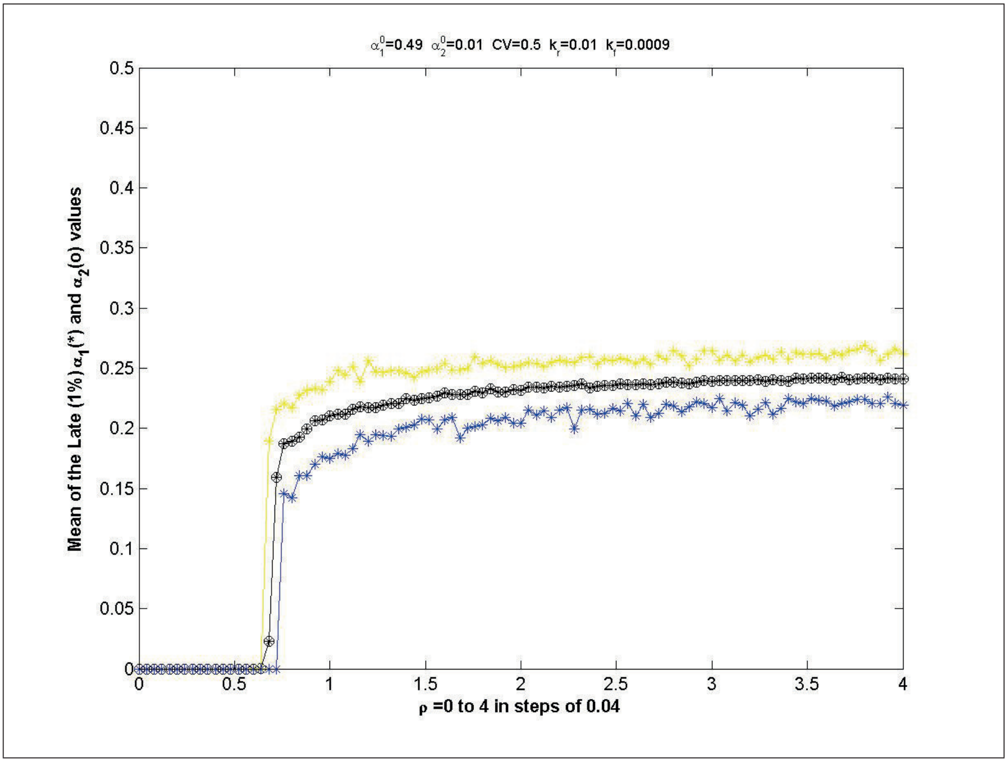

Figure 8: Effect of ρ

function and the specific parameters of adjustment while the variability of the income process is an environmental parameter.

In terms of the concavity of the utility function the persistence of altruism needs utilities sufficiently concave. If we relate concavity with the smoothness of consumption we can say that altruism is more likely to survive if consumers love smooth consumption paths. Another interpretation can be done in terms of risk aversion. In this line we have that consumers have to be enough risk averse to sustain positive levels of altruism in the long run.

The others personal parameters correspond to the adjustment process. The computer simulations indicate that its effect is secondary in terms of the persistence of altruism. A very strong adjustment process, in terms of a large change of the level of altruism if adjustment takes place, can cause the altruism to disappear. But this phenomenon is marginal and it is more a consequence of the main effect the kind of adjustment. With a strong adjustment process we observe that the levels of altruism are very variable having a large range of values taken along time. What can happen

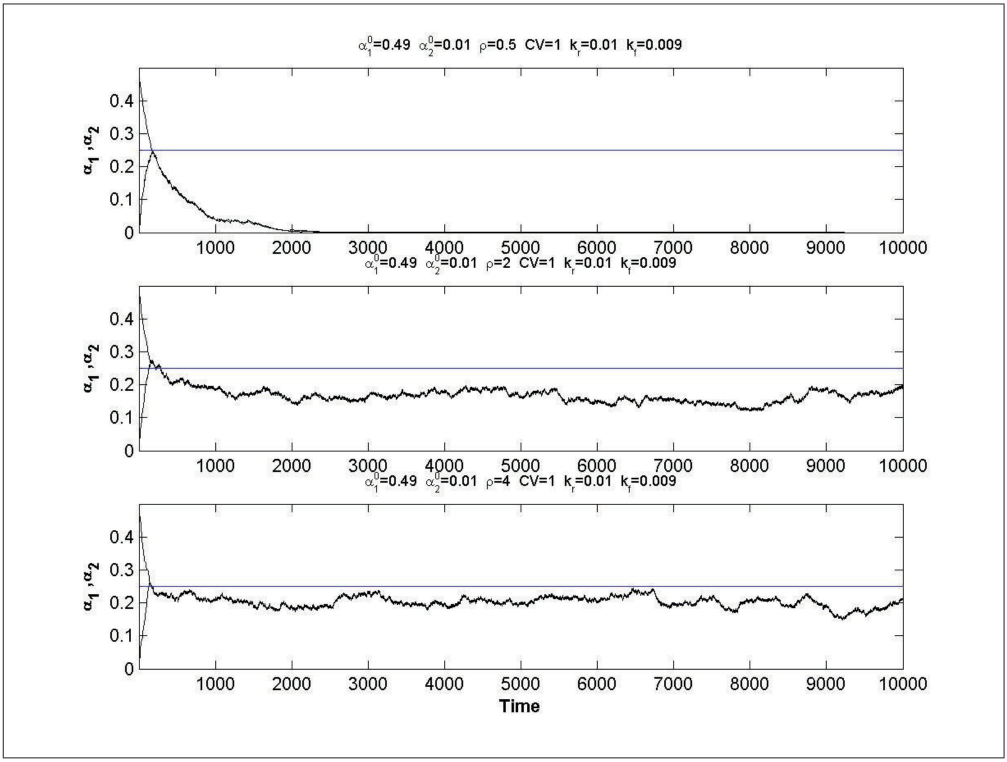

Figure 9: Effect of ρ

is that eventually the levels of altruism are adjusted to a very low value and thereafter a diminishing process begins.

The variability of the income process seems to be crucial to have an operative altruism in the long run. In a stable scenario, when income variance is small relative to its mean, altruism levels tend to zero in the short run. In the other hand, if income has a high enough coefficient of variation (the ratio between variance and mean) the level of altruism tends to follow a random process with mean near its middle value (0.25) and a variance that depend on other parameters, as those of the adjustment process. This last affirmation has been showed to be very robust.

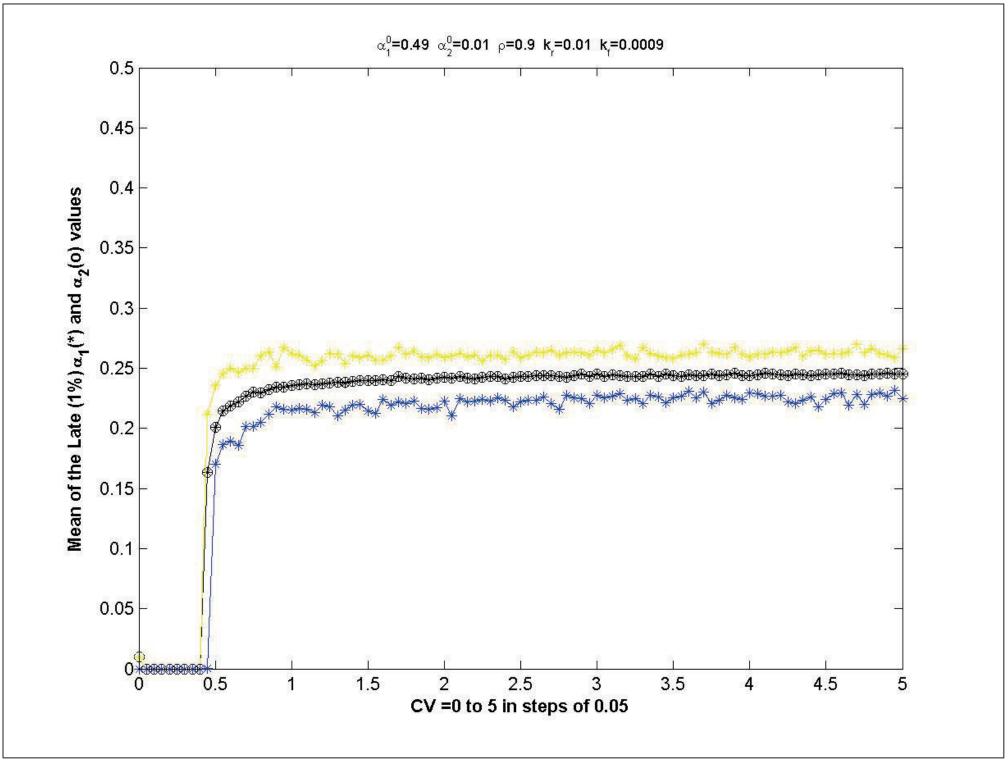

Figure 10: Effect of CV

APPENDIX

A Proof of Proposition 1

Proof. The best response of agent i to agent j transfer T_{ji} is the solution to:

$$\max_{0\le T_{ij}\le I_i} -\frac{1-\alpha_i}{\rho}\left(I_i - T_{ij} + T_{ji}\right)^{-\rho} - \frac{\alpha_i}{\rho}\left(I_j - T_{ji} + T_{ij}\right)^{-\rho} \tag{12}$$

that is:

$$BR_i(T_{ji}) = \min\left\{\max\left\{0, \gamma_i I_i - (1-\gamma_i)I_j + T_{ji}\right\}, I_i\right\}$$

Let us examine first the case where $0 < I_i$ for $i = 1, 2$ and $0 < \alpha_i \le 1/2$ for $i = 1, 2$ with at least one alpha strictly less than $1/2$. In this case all the possible equilibria are:

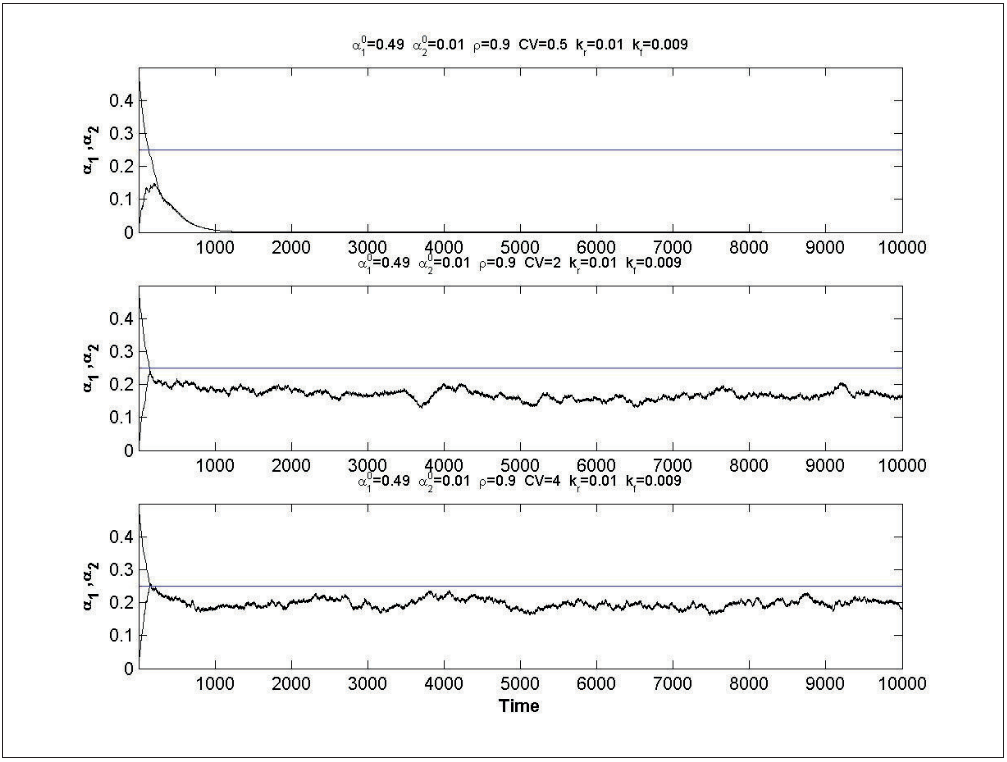

Figure 11: Effect of CV

1. Both transfers equal zero: $T_{12} = T_{21} = 0$

   For agent 1, $BR_1(0) = 0$ if and only if $0 \geq \gamma_1 I_1 - (1 - \gamma_1) I_2$ and this is equivalent to $I_1/I_2 \leq \frac{1-\gamma_1}{\gamma_1} = B_1$. This is true also for agent 2. As a result, the necessary and sufficient condition for $(0, 0)$ to be an equilibrium is: $1/B_2 \leq I_1/I_2 \leq B_1$.

2. One transfer equals zero and one transfer is positive, say: $T_{12} = 0$, $T_{21} > 0$

   In this case, $T_{21} = BR_2(0) = \min\{\max\{0, \gamma_2 I_2 - (1-\gamma_2) I_1\}, I_2\}$, as $\gamma_2 I_2 - (1-\gamma_2) I_1 < I_2$ the only possibility to the best reply be positive is $T_{21} = \gamma_2 I_2 - (1-\gamma_2) I_1 > 0$. Then $BR_2(0) > 0$ if and only if $0 < \gamma_2 I_2 - (1-\gamma_2) I_1$, that is if and only if $I_2/I_1 > \frac{1-\gamma_2}{\gamma 2} = B_2$.

   For agent 1 : $BR_1(\gamma_2 I_2 - (1-\gamma_2) I_1) = 0$ if and only if $0 \geq \gamma_1 I_1 - (1 - \gamma_1) I_2 + \gamma_2 I_2 - (1-\gamma_2) I_1 = (\gamma_1 + \gamma_2 - 1)(I_1 + I_2)$. As $I_1 + I_2$ is positive, the necessary and sufficient condition is $1 \geq \gamma_1 + \gamma_2$. This condition is always satisfied under $(\alpha_1, \alpha_2) \neq (1/2, 1/2)$.

Then $(0, \gamma_2 I_2 - (1-\gamma_2)I_1)$ is a Nash Equilibrium if and only if $I_2/I_1 > B_2$.

By symmetry $(\gamma_1 I_1 - (1-\gamma_1)I_2, 0)$ is a Nash Equilibrium if and only if $I_1/I_2 > B_1$.

3. Both transfers are positive: $T_{12} > 0$, $T_{21} > 0$

   We have the following subcases:

   (a) One maximal transfer, say: $T_{21} = I_2$
   Then $T_{12} = BR_1(I_2) = \min\{\gamma_1 (I_1 + I_2), I_1\}$

   i. If $T_{12} = \gamma_1 (I_1 + I_2)$

$$\begin{aligned} BR_2(\gamma_1 (I_1 + I_2)) &= \min\{\max\{0, \gamma_2 I_2 - (1-\gamma_2)I_1 + \gamma_1 (I_1 + I_2)\}, I_2\} \\ &= \min\{\max\{0, (\gamma_1 + \gamma_2)I_2 - (1-\gamma_1-\gamma_2)I_2\}, I_2\} \end{aligned}$$

   then as $(\gamma_1+\gamma_2)I_2-(1-\gamma_1-\gamma_2)I_1 < I_2$ we have $BR_2(\gamma_1 (I_1 + I_2)) < I_2$, a contradiction.

   ii. If $T_{12} = I_1$. Then $\gamma_1 (I_1 + I_2) \geq I_1$ and $\gamma_2 (I_1 + I_2) \geq I_2$. Adding both inequalities and simplifying we obtain $\gamma_1 + \gamma_2 \geq 1$. This condition is satisfied only when $\alpha_1 = \alpha_2 = 1/2$.

   Then there is no Nash Equilibrium with one agent giving all his income.

   (b) Both transfers are interior solutions: $0 < T_{12} < I_1$ and $0 < T_{21} < I_2$
   If $0 < T_{21} < I_2$ then $T_{21} = \gamma_2 I_2 - (1-\gamma_2)I_1 + T_{12}$. Now, as $0 < T_{12} < I_1$:

$$\begin{aligned} T_{12} &= BR_1(\gamma_2 I_2 - (1-\gamma_2)I_1 + T_{21}) \\ &= \gamma_1 I_1 - (1-\gamma_1)I_2 + \gamma_2 I_2 - (1-\gamma_2)I_1 + T_{12} \\ &= T_{12} - (1-\gamma_1-\gamma_2)(I_1 + I_2) \end{aligned}$$

   This is impossible unless $\gamma_i + \gamma_j = 1$ or $I_j + I_i = 0$, both excluded in this part of the proof.

Then in this case there is no Nash Equilibrium with both transfers being positive.

Summarizing, with $0 < \alpha_i < 1/2$ and $0 < I_i$ for $i = 1, 2$, there are only three possible Nash Equilibrium cases:

$$\begin{aligned}
(0,0) &\text{ iff } 1/B_2 \leq I_1/I_2 \leq B_1 \\
(\gamma_1 I_1 - (1-\gamma_1)I_2, 0) &\text{ iff } I_1/I_2 > B_1 \\
(0, \gamma_2 I_2 - (1-\gamma_2)I_1) &\text{ iff } I_2/I_1 > B_2
\end{aligned}$$

This corresponds to expression (2).

Now I examine the remaining cases:

1. $I_i = 0$: In this case agent i has only one strategy: $T_{ij} = 0$. Equation $T_{ij} = \max\{0, \gamma_i I_i - (1-\gamma_i)I_j\}$ still holds.
2. $\alpha_i = 0$: In this case the trivial solution to (12) is $T_{ij} = 0$. As $\gamma_i = 0$ in this case, once again equation $T_{ij} = \max\{0, \gamma_i I_i - (1-\gamma_i)I_j\}$ still holds.

This concludes the proof. ■

To finish the equilibrium analysis, I examine the only case that is not covered by the proposition: $\alpha_1 = \alpha_2 = 1/2$.

In this case $\gamma_1 = \gamma_2 = 1/2$, $B_1 = B_2 = 1$ and:

$$BR_i(T_{ji}) = \min\left\{\max\left\{0, \frac{I_i - I_j}{2} + T_{ji}\right\}, I_i\right\}$$

Suppose $I_1 = I_2$ then $BR_i(T_{ji}) = \min\{T_{ji}, I_i\} = T_{ji}$. Then there is a continuum of Nash Equilibria: (T, T) with $T \in [0, I_i]$.

If $I_1 > I_2$ then $BR_1(T_{21}) = \min\left\{\frac{I_1-I_2}{2} + T_{21}, I_1\right\}$. As $T_{21} \leq I_2$, we have $\frac{I_1-I_2}{2} + T_{21} \leq \frac{I_1+I_2}{2} < I_1$, then $BR_1(T_{21}) = \frac{I_1-I_2}{2} + T_{21}$. Now $BR_2(\frac{I_1-I_2}{2} + T_{21}) = \min\{T_{21}, I_2\}$. Then again we have a continuum of equilibria: $(\frac{I_1-I_2}{2} + T, T)$ with $T \in [0, I_2]$.

If $I_1 < I_2$ by symmetry the set of Nash Equilibria is composed of $(T, \frac{I_2-I_1}{2} + T)$ with $T \in [0, I_1]$.

In all cases the level of wealth after transfers is $(\frac{I_1+I_2}{2}, \frac{I_1+I_2}{2})$.

I consider that the focal equilibrium in this case corresponds to the agent with the lowest income realization making a zero transfer ($T = 0$). This equilibrium is included in (2).

B Derivation of the basic dynamic process (Equation 4)

In order to develop the adjustment dynamics we need the hypothetical transfers:

$$\left.\begin{array}{ll} T'_{12}=0 & T'_{21}=\gamma_2 I_1-(1-\gamma_2)I_2 \\ T'_{12}=0 & T'_{21}=0 \\ T'_{12}=\gamma_1 I_2-(1-\gamma_1)I_1 & T'_{21}=0 \end{array}\right\} \text{if} \left\{\begin{array}{c} I_1/I_2\leq 1/B_1 \\ 1/B_1\leq I_1/I_2\leq B_2 \\ B_2\leq I_1/I_2 \end{array}\right. .$$

Now we can compare T with T'. We have to distinguish three possibilities $\alpha_2<\alpha_1$, $\alpha_1<\alpha_2$ and $\alpha_1=\alpha_2$.

When $\alpha_2<\alpha_1$, the order $\frac{1}{B_2}<\frac{1}{B_1}<1<B_1<B_2$ is satisfied and we have five cases:

$\frac{I_1}{I_2}<\frac{1}{B_2}$	$\frac{1}{B_2}<\frac{I_1}{I_2}<\frac{1}{B_1}$	$\frac{1}{B_1}<\frac{I_1}{I_2}<B_1$	$B_1<\frac{I_1}{I_2}<B_2$	$B_2<\frac{I_1}{I_2}$
$T_{12}=0$	$T_{12}=0$	$T_{12}=0$	$T_{12}>0$	$T_{12}>0$
$T'_{21}=0$	$T'_{21}=0$	$T'_{21}=0$	$T'_{21}=0$	$T'_{21}>0$
$T_{21}>0$	$T_{21}=0$	$T_{21}=0$	$T_{21}=0$	$T_{21}=0$
$T'_{12}>0$	$T'_{12}>0$	$T'_{12}=0$	$T'_{12}=0$	$T'_{12}=0$

From the table we can conclude that:

1. $T_{21}<T'_{12}$ iff $\frac{I_1}{I_2}<\frac{1}{B_1}$ (then $\alpha_1\downarrow_0$).
2. $T_{12}>T'_{21}$ iff $B_1<\frac{I_1}{I_2}$ (then $\alpha_2\uparrow^{0.5}$).
3. $T_{12}=T'_{21}=0$ and $I_2<I_1$ iff $1<\frac{I_1}{I_2}<B_1$ (then $\alpha_2\downharpoonleft_0$).
4. $T_{21}=T'_{12}=0$ and $I_1<I_2$ iff $\frac{1}{B_1}<\frac{I_1}{I_2}<1$ (then $\alpha_1\downharpoonleft_0$).
5. $T_{12}<T'_{21}$ never occurs.
6. $T_{21}>T'_{12}$ never occurs.

In a similar way when $\alpha_2>\alpha_1$ we have:

1. $T_{12}<T'_{21}$ iff $B_2<\frac{I_1}{I_2}$ (then $\alpha_2\downarrow_0$).
2. $T_{21}>T'_{12}$ iff $\frac{I_1}{I_2}<\frac{1}{B_2}$ (then $\alpha_1\uparrow^{0.5}$).
3. $T_{12}=T'_{21}=0$ and $I_2<I_1$ iff $1<\frac{I_1}{I_2}<B_2$ (then $\alpha_2\downharpoonleft_0$).

4. $T_{21} = T'_{12} = 0$ and $I_1 < I_2$ iff $\frac{1}{B_2} < \frac{I_1}{I_2} < 1$ (then $\alpha_1 \lfloor_0$).

5. $T_{12} > T'_{21}$ never occurs.

6. $T_{21} < T'_{12}$ never occurs.

Finally if $\alpha_2 = \alpha_1$ the only cases where there is some adjustment are:

1. $T_{12} = T'_{21} = 0$ and $I_2 < I_1$ iff $1 < \frac{I_1}{I_2} < B_1$ (then $\alpha_2 \lfloor_0$).

2. $T_{21} = T'_{12} = 0$ and $I_1 < I_2$ iff $\frac{1}{B_1} < \frac{I_1}{I_2} < 1$ (then $\alpha_1 \lfloor_0$).

As we have $I_i \sim [a, b] \subset [0, +\infty[$, we can derive a distribution for $\frac{I_1}{I_2} \sim [\frac{a}{b}, \frac{b}{a}]$ with CDF $F(\cdot)$. This distribution has the following property: $F(x) = 1 - F(1/x)$ for all $x > 0$ [9]

In particular this implies: $F(1) = 0.5$.

Then we have:

$$(\alpha_1^{t+1}, \alpha_2^{t+1}) = \left\{ \begin{array}{lll} \left.\begin{array}{lll} (\alpha_1^t \downarrow_0, \alpha_2^t) & \text{wp} & 1 - F(B_1) \\ (\alpha_1^t, \alpha_2^t \lfloor_0) & \text{wp} & F(B_1) - 0.5 \\ (\alpha_1^t \lfloor_0, \alpha_2^t) & \text{wp} & F(B_1) - 0.5 \\ (\alpha_1^t, \alpha_2^t \uparrow^{0.5}) & \text{wp} & 1 - F(B_1) \end{array} \right\} & \text{if } \alpha_1 > \alpha_2 \\ \left.\begin{array}{lll} (\alpha_1^t, \alpha_2^t \downarrow_0) & \text{wp} & 1 - F(B_2) \\ (\alpha_1^t, \alpha_2^t \lfloor_0) & \text{wp} & F(B_2) - 0.5 \\ (\alpha_1^t \lfloor_0, \alpha_2^t) & \text{wp} & F(B_2) - 0.5 \\ (\alpha_1^t \uparrow^{0.5}, \alpha_2^t) & \text{wp} & 1 - F(B_2) \end{array} \right\} & \text{if } \alpha_1 < \alpha_2 \\ \left.\begin{array}{lll} (\alpha_1^t, \alpha_2^t) & \text{wp} & 2(1 - F(B_1)) \\ (\alpha_1^t, \alpha_2^t \lfloor_0) & \text{wp} & F(B_1) - 0.5 \\ (\alpha_1^t \lfloor_0, \alpha_2^t) & \text{wp} & F(B_1) - 0.5 \end{array} \right\} & \text{if } \alpha_1 = \alpha_2 \end{array} \right.$$

C Proof of Lemma 3

In fact I will prove that if α_1^0 and α_2^0 are draw from the same distribution then α_1^t and α_2^t have also the same distribution at all $t > 0$.

Lets proceed by induction; for $t = 0$ as α_1^0 and α_2^0 are draw from the same distribution it is direct that $\Pr[\alpha_1^0 \leq a] = \Pr[\alpha_2^0 \leq a]$ and that $\Pr[\alpha_1^0 > \alpha_2^0] = \Pr[\alpha_1^0 < \alpha_2^0] = 0.5$ and $\Pr[\alpha_1^0 = \alpha_2^0] = 0$.

[9] As I_i and I_j are independently and identically distributed, $\frac{I_i}{I_j}$ and $\frac{I_j}{I_i}$ have the same CDF $F(\cdot)$. Then $F(x) = \Pr[\frac{I_i}{I_j} \leq x] = \Pr[\frac{1}{x} \leq \frac{I_j}{I_i}] = 1 - F(1/x)$

Now assume that α_1^t and α_2^t have the same distribution. From (9) we have:

$$\begin{aligned}\Pr_t[\alpha_1^{t+1} < a] &= 0.5(1_{\alpha_1^t - k_r\alpha_1^t < a} + 1_{\alpha_1^t < a})1_{\alpha_1^t > \alpha_2^t} \\ &\quad +0.5(1_{\alpha_1^t < a} + 1_{\alpha_1^t + (k_r)(0.5-\alpha_1^t) < a})1_{\alpha_1^t < \alpha_2^t} \\ &\quad +1_{\alpha_1^t < a}1_{\alpha_1^t = \alpha_2^t}\end{aligned}$$

and

$$\begin{aligned}\Pr_t[\alpha_2^{t+1} < a] &= 0.5(1_{\alpha_2^t < a} + 1_{\alpha_2^t + (k_r)(0.5-\alpha_2^t) < a})1_{\alpha_1^t > \alpha_2^t} \\ &\quad +0.5(1_{\alpha_2^t - k_r\,\alpha_2^t < a} + 1_{\alpha_2^t < a})1_{\alpha_1^t < \alpha_2^t} \\ &\quad +1_{\alpha_2^t < a}1_{\alpha_1^t = \alpha_2^t}\end{aligned}$$

and then:

$$\begin{aligned}\Pr[\alpha_1^{t+1} < a] &= 0.5(\Pr[\alpha_1^t - k_r\,\alpha_1^t < a] + \Pr[\alpha_1^t < a])\Pr[\alpha_1^t > \alpha_2^t] \\ &\quad +0.5(\Pr[\alpha_1^t < a] + \Pr[\alpha_1^t + (k_r)(0.5-\alpha_1^t) < a])\Pr[\alpha_1^t < \alpha_2^t] \\ &\quad +\Pr[\alpha_1^t < a]\Pr[\alpha_1^t = \alpha_2^t]\end{aligned}$$

$$\begin{aligned}\Pr[\alpha_2^{t+1} < a] &= 0.5(\Pr[\alpha_2^t < a] + \Pr[\alpha_2^t + (k_r)(0.5-\alpha_2^t) < a])\Pr[\alpha_1^t > \alpha_2^t] \\ &\quad +0.5(\Pr[\alpha_2^t - k_r\,\alpha_2^t < a] + \Pr[\alpha_2^t < a])\Pr[\alpha_1^t < \alpha_2^t] \\ &\quad +\Pr[\alpha_2^t < a]\Pr[\alpha_1^t = \alpha_2^t]\end{aligned}$$

It is clear that if α_1^t and α_2^t have the same distribution then $\Pr[\alpha_1^{t+1} < a] = \Pr[\alpha_2^{t+1} < a]$ and so α_1^{t+1} and α_2^{t+1} has also the same distribution. This implies that:

$$\Pr[\alpha_1^{t+1} > \alpha_2^{t+1}] = \Pr[\alpha_1^{t+1} < \alpha_2^{t+1}] = 0.5$$

D Calculation of equations (11) and (10)

D.1 Equation (11)

We know that

$$E[\alpha_i^{t+1}] = (1 - \frac{k_r}{2})(E[\alpha_i^t]) + \frac{k}{8} \tag{13}$$

and

$$E[\alpha_i^t] = (1 - \frac{k_r}{2})^t(E[\alpha_i^0] - \frac{1}{4}) + \frac{1}{4} \tag{14}$$

Direct calculation give us:

$$E_t[(\alpha_i^{t+1})^2] = \frac{k_r^2 - 2k_r + 2}{2}(\alpha_i^t)^2 + \frac{k(1-k_r)}{4}\alpha_i^t + \frac{k_r^2}{16}$$

then:

$$E[(\alpha_i^{t+1})^2] = \frac{k_r^2 - 2k_r + 2}{2}E[(\alpha_i^t)^2] + \frac{k_r(1-k_r)}{4}E[\alpha_i^t] + \frac{k_r^2}{16} \qquad (15)$$

Using (13) and (15):

$$\begin{aligned} V[(\alpha_i^{t+1})] &= E[(\alpha_i^{t+1})^2] - E[(\alpha_i^{t+1})]^2 \\ &= \frac{(k_r-2)^2}{4}E[(\alpha_i^t)]^2 + \frac{k_r^2}{8}E[\alpha_i^t] - \frac{k_r^2 - 2k_r + 2}{2}E[(\alpha_i^t)^2] - \frac{3k_r^2}{64} \end{aligned}$$

Replacing $E[(\alpha_i^t)^2] = V[\alpha_i^t] + E[\alpha_i^t]^2$:

$$V[(\alpha_i^{t+1})] = \frac{k_r^2}{4}E[(\alpha_i^t)]^2 - \frac{k_r^2}{8}E[\alpha_i^t] + \frac{k_r^2 - 2k_r + 2}{2}V[\alpha_i^t] + \frac{3k_r^2}{64}$$

Using now (14) we obtain equation (11):

$$V[(\alpha_i^{t+1})] = \frac{k_r^2 - 2k_r + 2}{2}V[\alpha_i^t] + \left(\frac{k_r^2}{8}\left(\frac{2-k}{2}\right)^{2t}(4E[a_i^0]-1)^2 + \frac{k_r^2}{32}\right)$$

D.2 Equation (10)

Following the same procedure:

$$E[\alpha_i^{t+1}] = (1 - \frac{k_f}{2})E[\alpha_i^t] \qquad (16)$$

and

$$E[\alpha_i^t] = (1 - \frac{k_f}{2})^t E[\alpha_i^0] \qquad (17)$$

Direct calculation give us:

$$E[(\alpha_i^{t+1})^2] = \frac{k_f^2 - 2k_f + 2}{2}E[(\alpha_i^t)^2] \qquad (18)$$

Using (16) and (18):

$$\begin{aligned} V[(\alpha_i^{t+1})] &= E[(\alpha_i^{t+1})^2] - E[(\alpha_i^{t+1})]^2 \\ &= \frac{k_f^2 - 2k_f + 2}{2}E[(\alpha_i^t)^2] - \frac{(k_f-2)^2}{4}E[(\alpha_i^t)]^2 \end{aligned}$$

Replacing $E[(\alpha_i^t)^2] = V[\alpha_i^t] + E[\alpha_i^t]^2$:

$$V[(\alpha_i^{t+1})] \quad = \quad \frac{k_f^2}{4}E[(\alpha_i^t)]^2 + \frac{k_f^2 - 2k_f + 2}{2}V[\alpha_i^t]$$

Using now (17) we obtain equation (10):

$$V[\alpha_i^{t+1}] = \frac{k_f^2 - 2k_f + 2}{2}V[\alpha_i^t] + \frac{k_f^2}{2}(\frac{2 - k_f}{2})^{2t}E[\alpha_i^0]^2$$

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