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EXTRAPOLATION AND COMMUTATORS OF SINGULAR INTEGRALS

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1. Introduction

In these notes we shall present results concerning L^p inequalities with different but related weights for commutators of singular and strongly singular integrals. These commutators turn out to be controlled by commutator of fractional order of the Hardy-Littlewood maximal operator. The boundedness properties are obtained by extrapolation from infinity. These notes are based mainly on [G-H-S-T].

We denote by R^n the n-dimensional euclidean space. The Lebesgue measure of a Lebesgue measurable set $E \subset R^n$ is denoted by $|E|$. If Q is a cube in R^n and γ is a real number, then γQ shall stand for the cube with the same center as Q and side γ times that of Q. A weight $w(x)$ is a non-negative measurable function on R^n.The measure associated with w is the set function

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given by $w(E)=\int_E w(x)dx$. By $L^p(w), 0<p<\infty$, we denote the space of all Lebesgue measurable functions $f(x)$ such that

$$\|f\|_{L^p(w)} = (\int_{R^n} |f(x)|^p w(x)dx)^{1/p} < \infty.$$

The average of locally integrable function f over a cube Q is defined as $m_Q f = |Q|^{-1}\int_Q f(x)dx$. Given a cube $Q \subset R^n$ and $0<r<\infty$, the Hardy-Littlewood maximal function with respect to Q of a function $f(x), x\in Q$, is defined as

$$M_r^Q(f)(x) = \sup_{x\in J\subset Q} (|J|^{-1}\int_J |f(y)|^r dy)^{1/r},$$

where J is any cube satisfying the condition. If $Q=R^n$ we simply write $M_r(f)$ instead of $M_r^{R^n}(f)$ and if, in addition, $r=1$, just $M(f)$. Given $1<p<\infty$, a necessary and sufficient condition for a weigth $w(x)$ to satisfy

$$\int_Q (M_1^Q(f)(x))^p w(x)dx \le C\int_Q |f(x)|^p w(x)dx,$$

with a constant C independent of f, is that w belongs to the class A_p of Muckenhoupt, i.e. that w satisfies the condition:

$$\sup_{J\subset Q}\left(|J|^{-1}\int_J w(x)dx\right)\left(|J|^{-1}\int_J w(x)^{-\frac{1}{p-1}}\right)^{p-1} = C_p(w) < \infty,$$

where J is a cube. The condition A_1 is, by definition, the limit of A_p for $p\to 1$, that is to say $w\in A_1$ if and only if

$$\sup_{J\subset Q}\left(|J|^{-1}\int_J w(x)dx\right)\left(\operatorname{ess\,sup}_{x\in J} . w(x)^{-1}\right) = C_1(w) < \infty.$$

The class A_∞ is the union $\cup A_p$; $1<p<\infty$. Equivalently $w\in A_\infty$ if and only if there exist two constants $0<\delta\le 1$ and $C>0$ such that for every cube J and every measurable set $E\subset J$

$$\frac{w(E)}{w(J)} \le C\left(\frac{|E|}{|J|}\right)^{\delta}$$

holds. As references for the A_p classes we give [M] and [G-R].

2. Extrapolation

Let ν be a weight By $A_p(\nu)$, $1<p<\infty$, we denote the class of all pairs of weights α and β such that α^p and β^p belong to A_p and $\nu=\alpha\beta^{-1}$. For these classes we have the following so called extrapolation theorem:

Theorem 2.1 *Let T be a sublinear operator and $1<q<\infty$. If for every pair $(\alpha,\beta)\in A_q(\nu)$,*

$$\|Tf\|_{L^q(\beta^q)} \le C_{\alpha\beta}\ \|f\|_{L^q(\alpha^q)}$$

holds with a constant $C_{\alpha,\beta}$ depending only on $C_q(\alpha^q)$ and $C_q(\beta^q)$, then

$$\|Tf\|_{L^p(\beta^p)} \le C_{\alpha\beta}\ \|f\|_{L^p(\alpha^p)}$$

holds for every $1<p<\infty$, provided that $(\alpha,\beta)\in A_p(\nu)$.

For the proof of this result and more references, see [S-T1], [H-M-S] and [G-R].

The following theorem deals with extrapolation from infinity:

Theorem 2.2 *Let ν be a given weight, and T an operator such that*

$$\|\beta\ Tf\|_\infty \le c\|\beta\ f\|_\infty$$

holds whenever β^{-1} and $(\nu\beta)^{-1}$ belongs to A_1 with a constant c depending on 3 and ν only. Then, for $1<p<\infty$,

$$\|Tf\|_{L^p(\beta^p)} \le c_p\|f\|_{L^p(\beta^p)}$$

holds whenever the pair $(\alpha,\beta)\in A_p(\nu)$, with a constant c_p depending on p,α,β and ν only.

This theorem has its origin in [H-M-S] and was proved in [S-T 1]. In order to prove Theorem (2.2) we shall need a result due to J.L.Rubio de Francia and J.García Cuerva, see [G-R].

Lemma 2.3 *Let μ be a positive measure and $S(u)$ an operator defined for $u \in L^p(\mu)$. We assume that S satisfies*

$$S(u)(x) \geq 0,$$
$$S(\lambda u)(x) = |\lambda| S(u)(x),$$
$$S(u_1 + u_2)(x) \leq S(u_1)(x) + S(u_2)(x) \quad \textit{and}$$
$$\|S(u)\|_{L^p(\mu)} \leq \|S\| . \|u\|_{L^p(\mu)} .$$

Then, given $u \geq 0$ there exists U such that

$$u(x) \leq U(x),$$
$$\|U\|_{L^p(\mu)} \leq 2 \|u\|_{L^p(\mu)} \quad \textit{and}$$
$$S(U)(x) \leq 2 \|S\| U(x).$$

Proof. Let $S^0(u) = u$ and $S^{k+1}(u) = S(S^k(u))$. We define

$$U(x) = \sum_{k=0}^{\infty} (2 \|S\|)^{-k} S^k (u)(x).$$

Then, $u(x) \leq U(x)$ and

$$\|U\|_{L^p(\mu)} \leq \sum_{k=0}^{\infty} (2 \|S\|)^{-k} \|S^k (u)\|_{L^p(\mu)}$$
$$\leq \left(\sum_{k=0}^{\infty} 2^{-k} \right) \|u\|_{L^p(\mu)} = 2 \|u\|_{L^p(\mu)} .$$

Moreover,

$$S(U) \leq \sum_{k=0}^{\infty} (2 \|S\|)^{-k} S^{k+1} (u)$$
$$\leq 2 \|S\| \sum_{k=0}^{\infty} (2 \|S\|)^{-(k+1)} S^{k+1} (u) \leq 2 \|S\| U$$

❑

Proof of Theorem 2.2 Let $f \in L^p(\beta^p)$, $1 < p < \infty$, where β^p and $\alpha^p \in A^p$, $\alpha\beta^{-1} = v$. We can assume that $\beta(x) > 0$ for every x. We define $g(x)$ as

$$g(x) = |f(x)|\,\beta^{p'}(x)/\|f\|_{L^p(\beta^p)} \quad \text{if } f(x) \neq 0 \text{ and}$$

$$g(x) = e^{-\pi (x)^2/p}\,\beta(x)^{p'/p} \quad \text{otherwise.}$$

For this function $g(x)$ we have

a) $g(x) > 0$ for every x,

b) $\|f(x)\beta(x)^{p'} g^{-1}(x)\|_\infty = \|f\|_{L^p(\beta^p)}$ and

c) $\int g(x)^p \beta(x)^{-p'} dx \leq 2$.

Let us define

$$S(h)(x) = \beta(x)^{p'} M(\beta^{-p'}h)(x) + v(x)\beta(x)^{p'} M(v^{-1}\beta^{-p'}h)(x).$$

This operator S satisfies

$$\int S(h)(x)^p \beta^{-p'}(x)dx = \int \beta^p [M(\beta^{-p'}h)]^p dx + \int v^p \beta^p [M(v^{-1}\beta^{-p'}h)]^p dx$$

$$\leq [c_p(\beta^p)^p + c_p((v\beta)^p)^p] \int |h|^p \beta^{-p'} dx.$$

Thus, by Lemma (2.3) there is a function G such that

e) $g(x) \leq G(x)$ a.e.,

f) $\int G^p \beta^{-p'} dx \leq 4$ and

g) $S(G)(x) \leq 2[c_p(\beta^p)^p + c_p((v\beta)^p)^p]G(x)$ a.e.

The inequality in g) implies

h) $M(\beta^{-p'}G)(x) \leq c\beta^{-p'}(x)G(x)$ and

i) $M(v^{-1}\beta^{-p'}G)(x) \leq cv^{-1}(x)\beta(x)^{-p'}G(x)$.

Then, by a), b) and e) we have

$$\|f\|_{L^p(\beta^p)} = \|f\beta^{p'}g^{-1}\|_\infty \geq \|f\beta^{p'}G^{-1}\|_\infty.$$

On the other hand, since $G\beta^{-p'}$ and $v^{-1}\beta^{-p'}G$ belong to A_1, by hypothesis, we get

$$\| \beta^{p'} G^{-1} Tf \|_\infty \le c \| \beta^{p'} G^{-1} f \|_\infty .$$

Thus,

$$\|Tf\|_{L^p(\beta^p)} = \left(\int |Tf|^p \beta^p dx\right)^{1/p} \le \| \beta^{p'} G^{-1} Tf \|_\infty \left(\int G^p \beta^{-p'} dx\right)^{1/p}$$

$$\le 4^{1/p} c \| \beta^{p'} G^{-1} f \|_\infty \le 4^{1/p} c \| f \|_{L^p(\beta^p)} \qquad \square$$

3. The sharp maximal function

We shall need the following version of the sharp maximal function:

Definition 3.1 *Let $0<r<\infty$ and Q a cube properly contained in R^n. Given a function $f(x)$, $x \in Q$, we define:*

$$f_r^{\#,Q}(x) = \sup_{x \in J} \left\{ \inf_{c \in R} \left(|J|^{-1} \int_J |f(y) - c|^r dy \right)^{1/r} \right\},,$$

where J is any cube contained in Q. Moreover, $f_r^{\#}(x)$ stands for the function defined above where the supremum is taken for all cubes $J \subset R^n$.

Then we have the following theorem:

Theorem 3.2 *Let* c_Q be a constant satisfying:

$$\int_Q |f(y) - c_Q|^r dy \le 2 \inf_{c \in R} \int_Q |f(y) - c|^r dy .$$

Then, given $0<p<\infty$ and $w \in A_\infty$, there exists a constant C such that

$$\int_Q |f(x) - c_Q|^p w(x) dx \le \int_Q M_r^Q (f - c_Q)(x)^p w(x) dx \le C \int_Q [f_r^{\#,Q}(x)]^p w(x) dx$$

holds for every f. The constant C does not depend on Q.

Proof: It is enough to prove the theorem for the dyadic maximal function $\tilde{M}_r^Q(f - c_Q)$ instead of $M_r^Q(f - c_Q)$ since for $w \in A_\infty$ they have equivalent $L^p(w)$ norms, see [G-R], p.136. Let $g = f - c_Q$. We observe that $f_r^{\#,Q}(x) = g_r^{\#,Q}(x)$ holds for every point $x \in Q$. Let $\lambda_0^r = |Q|^{-1} \int_Q |g(x)|^r dx$

and $\lambda \geq \lambda_0$. We apply the Calderón-Zygmund decomposition to the function $|g|^r$ with parameter λ^r obtaining a disjoint sequence $\{I_k\}$ of dyadic subcubes of Q satisfying:

$$\lambda^r < |I_k|^{-1} \int_{I_k} |g(x)|^r \, dx \leq 2^n \lambda^r .$$

Applying again the Calderón-Zygmund decomposition to $|g|^r$, but this time with parameter $3^n\lambda^r$, we get a disjoint sequence $\{J_h\}$ of dyadic subcubes of Q satisfying:

$$3^n \lambda^r < \left|J_h\right|^{-1} \int_{J_h} |g(x)|^r dx \leq 2^n 3^n \lambda^r .$$

We observe that every cube J_h is contained in one and only one cube I_k. Besides, we have:

$$\{x \in Q: \tilde{M}_r^Q(g)(x) > \lambda\} = \cup I_k \quad \text{and} \quad \{x \in Q: \tilde{M}_r^Q(g)(x) > 3^{n/r}\lambda\} = \cup J_h .$$

Thus, if $J_h \subset I_k$, we have:

$$\lambda^r \leq (3^n - 2^n)\lambda^r \leq |J_h|^{-1} \int_{J_h} |g(x)|^r dx - |I_k|^{-1} \int_{I_k} |g(y)|^r dy$$

$$\leq |J_h|^{-1} |I_k|^{-1} \int_{J_h} \int_{I_k} \left| |g(x)|^r - |g(y)|^r \right| dx dy.$$

Therefore, since the cubes J_h are disjoint, we get:

$$\lambda^r |I_k|^{-1} \sum_{J_h \subset I_k} |J_h| \leq |I_k|^{-2} \int_{I_k} \int_{I_k} \left| |g(x)|^r - |g(y)|^r \right| dx dy.$$

If $0 < r \leq 1$, then

$$\left| |g(x)|^r - |g(y)|^r \right| \leq |g(x) - g(y)|^r \leq |g(x) - c|^r + |g(y) - c|^r$$

holds for every constant c. Thus

$$\lambda^r |I_k|^{-1} \sum_{J_h \subset I_k} |J_h| \leq 2 \inf_{c \in R} |I_k|^{-1} \int_{I_k} |g(x) - c|^r dx \leq 2[f_r^{\#,Q}(z)]^r$$

holds for every $z \in I_k$. If $1 < r < \infty$, since

$$\left| |g(x)|^r - |g(y)|^r \right| \leq r(|g(x) - c| + |g(y) - c|)(|g(x)|^{r-1} + |g(y)|^{r-1})$$

holds for every constant c, by Hölder's inequality, we get:

$$\lambda^{r}|I_k|^{-1}\sum_{J_h\subset I_k}|J_h|\le C_r[f_r^{\#,Q}(z)]\lambda^{r-1}$$

for every $z\in I_k$. Then, for $0<r<\infty$, if there is a point $z\in I_k$ such that $f_r^{\#,Q}(z)<\gamma\lambda$, it follows that

$$|I_k|^{-1}\sum_{J_h\subset I_k}|J_h|\le C_r\gamma^{\min(r,1)},$$

where C_r depends on r and n but not on Q. Thus we have shown that for $\lambda\ge\lambda_0$ and $\omega\in A_\infty$

$$\omega\left(\left\{x\in Q:\tilde{M}_r^Q(g)(x)>3^{n/r}\lambda,\ f_r^{\#,Q}(x)<\gamma\,\lambda\right\}\right)$$
$$\le C\gamma^{\delta\min(r,1)}w\left(\left\{x\in Q:\tilde{M}_r^Q(g)(x)>\lambda\right\}\right) \tag{3.3}$$

holds. Here δ is the exponent in the A_∞ condition for ω. For $\lambda\le\lambda_0$, we use the obvious estimate

$$\omega\left(\left\{x\in Q:\tilde{M}_r^Q(g)(x)>3^{n/r}\lambda\right\}\right)\le\omega(Q).$$

Since

$$\lambda_0=(|Q|^{-1}\int_Q|f(y)-c_Q|^r\,dy)^{1/r}\le 2f_r^{\#,Q}(x)$$

holds for every $x\in Q$, we get:

$$\lambda_0\le 2(w(Q)^{-1}\int_Q[f_r^{\#,Q}(x)]^p\,w(x)dx)^{1/p}. \tag{3.4}$$

From (3.3) and (3.4), the proof of the treorem follows as usual, namely:

$$\int_Q \tilde{M}_r^Q(g)(x)^p w(x)dx$$

$$= C\int_0^\infty p\lambda^{p-1} w(\{x \in Q : \tilde{M}_r^Q(g)(x) > 3^{n/r}\lambda\})d\lambda$$

$$\leq C\int_0^{\lambda_0} p\lambda^{p-1} d\lambda\, w(Q) + C\int_{\lambda_0}^\infty p\lambda^{p-1} w\left(\left\{x \in Q : \tilde{M}_r^Q(g)(x) > 3^{n/r}\lambda, f_r^{\#,Q}(x) < \gamma\lambda\right\}\right)d\lambda$$

$$+ C\int_{\lambda_0}^\infty p\lambda^{p-1} w\left(\left\{f_r^{\#,Q}(x) > \gamma\lambda\right\}\right)$$

$$\leq Cw(Q)\lambda_0^p + C\gamma^{\delta \min(r,1)} \int_{\lambda_0}^\infty p\lambda^{p-1} w\left(\left\{x \in Q : \tilde{M}_r^Q(g)(x) > \lambda\right\}\right)$$

$$+ C\int_{\lambda_0}^\infty p\lambda^{p-1} w\left(\left\{x \in Q : f_r^{\#,Q}(x) > \gamma\lambda\right\}\right)$$

$$\leq C\int_Q (f_r^{\#,Q}(x))^p w(x)dx + C\gamma^{\delta \min(r,1)} \int_Q \tilde{M}_r^Q(g)(x)^p w(x)dx.$$

By taking γ small enough, we obtain the required inequality. ❑

Let $Q \subset Q'$, then

$$|c_Q - c_{Q'}|^p \leq 2^p |f(x) - c_Q|^p + 2^p |f(x) - c_{Q'}|^p .$$

Averaging on Q,

$$|c_Q - c_{Q'}|^p \leq 2^p w(Q)^{-1} \int_Q (|f(x) - c_Q|^p + |f(x) - c_{Q'}|^p) w(x)dx$$

$$\leq 2^p w(Q)^{-1} \left(\int_Q |f(x) - c_Q|^p w(x) + \int_{Q'} |f(x) - c_{Q'}|^p w(x)dx\right).$$

By Theorem 3.2, we get

$$|c_Q - c_{Q'}|^p \leq 2^p c_p w(Q)^{-1} \int f_r^{\#}(x)^p w(x)dx.$$

This shows that c_Q tends to a finite limit c when the cube Q increases to R^n. Moreover,

$$|c_Q - c|^p w(Q) \leq 2^p c_p w(Q)^{-1} \int f_r^{\#}(x)^p w(x)dx.$$

Since

$$M_r^Q(f - c)(x) \leq 2M_r^Q(f - c_Q)(x) + 2|c_Q - c|,$$

we have

$$\int_Q M_r^Q(f-c)(x)^p w(x)dx \le 2^p \int_Q M_r^Q(f-c_Q)(x)^p w(x)dx + 2^p |c_Q - c|^p w(Q)w(Q)$$

$$\le 2 2^p c_p \int f_r^{\#}(x)^p w(x)dx.$$

Therefore,

$$\int M_r(f-c)(x)^p w(x)dx \le c_p' \int f_r^{\#}(x)^p w(x)dx.$$

If in addition we assume that $w(\{x:|f(x)|>\lambda\})$ is finite for every $\lambda>0$, it turns out that c=0. Thus we have proved the following theorem:

Theorem 3.5 *Let* $w\in A_\infty$. *Then, for* $0<p<\infty$,

$$\int |f(x)-c|^p w(x)dx \le \int M_r(f-c)(x)^p w(x)dx$$

$$\le c_{p,r} \int f_r^{\#}(x)^p w(x)dx$$

holds with a constant $c_{p,r}$ *depending on p,r and w only. If in addition we assume that* $w(\{x:|f(x)|>\lambda\})$ *is finite for every* $\lambda>0$ *then c is equal to zero.*

4. B.M.O Spaces

Next we shall consider some B.M.O. spaces that are both relevant and natural in the theory of commutators.

Definition 4.1 *Let v be a weight and* $0<s<\infty$. *We shall say that a function a belongs to the class BMO (v,s) is there exists a finite constant C such that for every cube Q:*

$$\inf_{c\in R} \int_Q |a(x)-c|^s dx \le C \int_Q v(x)^s dx.$$

The smallest such constant C is denoted by $||a||_{*,v,s}^s$.

Next we shall apply Theorem 3.2 to derive some properties of the classes $BMO(v,s)$, which we believe have an independent interest.

Proposition 4.2 *If $a \in BMO(v,s)$, and $r > s$ then $a \in BMO(v,r)$. If $r < s$ and $\upsilon^s \in A_\infty$, then $a \in BMO(v,s)$, implies that $a \in BMO(v,r)$.*

Proof: Let a_Q satisfy

$$\int_Q |a(x)-a_Q|^s\,dx \le 2\inf_{c\in R}\int_Q |a(x)-c|^s\,dx.$$

Then by Theorem 3.2 and the definition of $BMO(\upsilon,s)$, we have:

$$\int_Q |a(x)-a_Q|^r\,dx \le C\int_Q [a_s^{\#,Q}(x)]^r\,dx \le C\|a\|_{*,\nu,s}^r \int_Q [M_1^Q(\nu^s)(x)]^{r/s}\,dx,$$

where C is independent of Q. if $r > s$, by the Hardy-Littlewood maximal theorem if follows that

$$\int_Q |a(x)-a_Q|^r\,dx \le C\|a\|_{*,\nu,s}^r \int_Q \nu(x)^r\,dx.$$

If $\nu^s \in A_\infty$ and $r>s$, there exists p such that $rp/s > 1$ and $\nu^{rp} \in A_\infty$. This implies (see [S-W]) that ν^r satisfies a reverse Hölder's inequality with exponent p. Thus, by Hölder's inequality and the Hardy-Littlewood maximal theorem, we get:

$$\int_Q |a(x)-a_Q|^r\,dx \le C\|a\|_{*,\nu,s}^r \int_Q [M_1^Q(\nu^s)(x)]^{r/s}\,dx$$

$$\le C\|a\|_{*,\nu,s}^r \left(\int_Q [M_1^Q(\nu^s)(x)]^{rp/s}\,dx\right)^{1/p} |Q|^{1/p'} \qquad \square$$

$$\le C\|a\|_{*,\nu,s}^r \left(\int_Q \nu(x)^{rp}\,dx\right)^{1/p} |Q|^{1/p'} \le C\|a\|_{*,\nu,s}^r \left(\int_Q \nu(x)^r\,dx\right)$$

Proposition 4.3 *Let $a \in BMO(\upsilon,r)$, let Q be a cube and let k be a positive integer. If for every integer h such that $0 \le h \le k$, a_{2^hQ} is a number satisfying:*

$$\int_{2^hQ} |a(x)-a_{2^hQ}|^r\,dx \le 2\inf_{c\in R}\int_{2^hQ} |a(x)-c|^r\,dx,$$

then:

$$|a_Q - a_{2^hQ}|^r \le C_r k^r \|a\|_{*,\nu,r}^r \max_{0\le h\le k}\left(|2^hQ|^{-1}\int_{2^kQ} \nu(x)^r\,dx\right).$$

Proof: We have

$$|a_Q - a_{2^k Q}| \le \sum_{h=0}^{k-1} |a_{2^h Q} - a_{2^{h+1} Q}|.$$

Since

$$|a_{2^h Q} - a_{2^{h+1} Q}|^r \le 2^r (|a_{2^h Q} - a(x)|^r + |a(x) - a_{2^{h+1} Q}|^r),$$

we obtain:

$$|a_{2^h Q} - a_{2^{h+1} Q}| \le 2^{\frac{1}{r}+1} \left(|2^h Q|^{-1} \int_{2^k Q} |a(x) - a_{2^h Q}|^r \, dx \right)^{1/r}$$

$$+ 2^{\frac{n+1}{r}+1} \left(|2^{h+1} Q|^{-1} \int_{2^{h+1} Q} |a(x) - a_{2^{h+1} Q}|^r \, dx \right)^{1/r}.$$

Therefore:

$$|a_Q - a_{2^k Q}|^r \le C_r k^r \|a\|^r_{*,\nu,r} \max_{0 \le h \le k} \left(|2^h Q|^{-1} \int_{2^h Q} \nu(x)^r \, dx \right),$$

as we wanted to show. ❑

Lemma 4.4 *Let* $a \in BMO(\nu,r)$, $\alpha^{-1}, \beta^{-1} \in A_1$, and $\alpha\beta^{-1} = \nu^r$. *Then, if* s *is such that* $1 \le s < \infty$ *and* $\alpha^{-s}, \beta^{-s} \in A_1$, *we have:*

$$|Q|^{-1} \int_Q |a(y) - a_Q|^{rs} \alpha(y)^{-s} \, dy \le C \operatorname{ess\,inf}_{x \in Q} . \beta(x)^{-s},$$

for any constant a_Q *satistying:*

$$\int_Q |a(y) - a_Q|^r \, dy \le 2 \inf_{c \in R} \int_Q |a(y) - c|^r \, dy,$$

where C is finite and independent of Q.

Proof: By Jensen's inequality, we may assume $s>1$. We have:

$$\int_Q |a(y) - a_Q|^{rs} \alpha(y)^{-s} \, dy \le \int_Q M_r^Q (a - a_Q)(y)^{rs} \alpha(y)^{-s} \, dy.$$

Then by theorem 3.2, we get:

$$\int_Q M_r^Q(a-a_Q)(y)^{rs}\alpha(y)^{-s}dy \le C\int_Q [a_r^{\#,Q}(y)]^{rs} a(y)^{-s}dy. \qquad (4.5)$$

Since by definition of $BMO(v,r)$ we have, for $y \in Q$:

$$a_r^{\#,Q}(y) \le CM_r^Q(v)(y) = C(M_1^Q(v^r)(y))^{1/r},$$

we see that the second member of (4.5) is bounded by:

$$C\int_Q M_1^Q(v^r)(y)^s\alpha(y)^{-s}dy \le C\int_Q v(y)^{rs}\alpha(y)^{-s}dy = C\int_Q \beta(y)^{-s}dy.$$

Thus, by our assumption that $\beta^{-s} \in A_1$, the conclusion of the lemma holds. ❑

Lemma 4.6 *Let* $1<p<\infty$. *Suppose that* $(\alpha,\beta) \in A_p(v^r)$ *and* $a \in BMO(v,r)$, *then with* a_Q *as before:*

$$\int_Q |a(x)-a_Q|^{rps}\beta(x)^{ps}dx \le C\int_Q \alpha(x)^{ps}dx$$

holds for all $s \ge 1$, *such that* $\beta^{sp} \in A_\infty$.

Proof: Since $|a(x)-a_Q| \le M_r^Q(a-a_Q)(x)$, a.e. for $x \in Q$, we have:

$$\int_Q |a(x)-a_Q|^{rps}\beta(x)^{ps}dx \le \int_Q M_r^Q(a-a_Q)(x)^{rps}\beta(x)^{ps}dx.$$

By theorem 3.2, the last integral is bounded by constant times

$$\int_Q [a_r^{\#,Q}(x)]^{rps}\beta(x)^{ps}dx,$$

and, by definition of the class $BMO(v,r)$, this integral is bounded by constant times

$$\int_Q M_1^Q(v^r)(x)^{ps}\beta(x)^{ps}dx \le C\int_Q v(x)^{rps}\beta(x)^{ps}dx = C\int_Q \alpha(x)^{ps}dx. \ ❑$$

5. The commutator of the Hardy-Littlewood maximal function

We define commutator of the Hardy-Littlewood maximal function as follows:

Given a function $a(x)$ and $0< r< \infty$

$$C_a^r(f)(x) = \sup_{x\in Q}|Q|^{-1}\int_Q |a(x)-a(y)|^r\,|f(y)|dy,$$

where Q denotes a cube.

The L^p estimates needed for $C_a^r(f)$ are a consequence of the estimates for a smooth variant of $C_a^r(f)$: Let $\phi(x)\geq 0$ be a smooth and rapidly decreasing function and $\phi_\epsilon(x) = \epsilon^{-n}\,\phi(x/\epsilon)$. For $0 < r < \infty$, we define

$$\Phi_a^r(f)(r) = \sup_{\epsilon>0}\int_{R^n} |a(x)-a(y)|^r\,\phi_\epsilon(x-y)|\,f(y)|dy.$$

These operators C_a^r and Φ_a^r have been studied in [S-T 2] and [S-T 3] for integral values of r.

Now we are ready to state the results on $C_a^r(f)$ and $\Phi_a^r(f)$.

Theorem 5.1 *Given a weight v and a function a in BMO(v,r),* $0<r<\infty$, *then the operator* Φ_a^r *is bounded from* $L^p(\alpha^p)$ *to* $L^p(\beta^p)$ *provided* (α,β) *belongs to* $A_p(v^r)$ *and* $1 < p < \infty$.

Theorem 5.2 *Let v be a weight such that* $v^r\in A_2$. *Then, the following statements are equivalent:*

(i) *For some p,* $1<p<\infty$, *the operator* C_a^r *is bounded from* $L^p(\alpha^p)$ *to* $L^p(\beta^p)$ *provided* (α,β) *belongs to* $A_p(v^r)$.

(ii) *For every p,* $1<p<\infty$, *the operador* C_a^r *is bounded from* $L^p(\alpha^p)$ *to* $L^p(\beta^p)$ *provided* (α,β) *belongs to* $A_p(v^r)$.

(iii) *a belongs to BMO(v,r).*

We begin by proving a lemma on weights that shall be needed.

Lemma 5.3 *Let* $r = r_1 + r_2, r_i > 0$ *for i = 1,2 and* $1\leq p<\infty$. *If* α^p *and* β^p *belong to* A_p *and* $\alpha\beta^{-1} = v^r$, *then* $\gamma = \alpha^{r_2/r}\beta^{r_1/r}$ *satisfies:*

(i) γ^p *belongs to* A_p and

(ii) $\alpha\gamma^{-1} = v^{r_1}$ *. and* $\gamma\beta^{-1} = v^{r_2}$.

Proof: If $1 < p < \infty$, by Hölder's inequality, we have:

$$\left(|Q|^{-1}\int_Q \gamma^{\,p} dx\right)^{1/p} \le \left(|Q|^{-1}\int_Q \alpha^{\,p} dx\right)^{\frac{r_2}{r\,p}} \left(|Q|^{-1}\int_Q \beta^{\,p} dx\right)^{\frac{r_1}{r\,p}}$$

and

$$\left(|Q|^{-1}\int_Q \gamma^{\,-p'} dx\right)^{1/p'} \le \left(|Q|^{-1}\int_Q \alpha^{\,-p'} dx\right)^{\frac{r_2}{r\,p'}} \left(|Q|^{-1}\int_Q \beta^{\,-p'} dx\right)^{\frac{r_1}{r\,p'}}.$$

By multiplying these two inequalities we get that $\gamma^{p} \in A_p$. If $p = 1$, the proof is even simpler. ❑

Proof of Theorem 5.1: First of all, we observe that if there is a pair of weights $(\alpha,\beta) \in A_p(v^r)$, then $v^r \in A_2 \subset A_\infty$ (see [S-T 1]). Thus, if $r \ge 1$, by Proposition 4.2, $a \in BMO(v,r)$ is equivalent to $a \in BMO(v,1)$. We shall estimate the sharp maximal function of the following auxiliary maximal function: Let $N > 0$, then set

$$\Phi^r_{a,N}(f)(x) = \sup_{0<\in<N} \int_{R^n} |a(x)-a(y)|^r \phi_\in(x-y)|f(y)|dy.$$

Let Q be a cube and $x_0 \in Q$. Let a_Q be a number satisfying

$$\int_Q |a(x)-a_Q|^r\, dx \le 2 \inf_{c\in R}\int_Q |a(x)-c|^r\, dx,$$

and

$$cQ = \sup_{0<\in<N} |Q|^{-1}\int_Q \left(\int_{R^n} |a(y)-a_Q|^r\, \phi_\in(z-y)|f(y)\chi_{R^n\setminus 4Q}(y)dy\right)dz.$$

Then, for $x\in Q$ we have

$$|\Phi^r_{a,N}(f)(x) - c_Q|$$

$$\leq \sup_{\in>0} \Big| \int_{R^n} |a(x) - a(y)|^r \phi_\in(x-y)|f(y)|dy$$

$$-|Q|^{-1} \int_Q \left(\int_{R^n} |a(y) - a_Q|^r \phi_\in(z-y)|f(y)|\chi_{R^n \backslash 4Q}(y)dy \right) dz \Big|$$

$$\leq \sup_{\in>o} \int_{R^n} \Big| |a(x) - a(y)|^r - |a(y) - a_Q|^r \Big| \phi_\in(x-y)|f(y)|dy$$

$$+ \sup_{\in>o} \int_{R^n} |a(y) - a_Q|^r \phi_\in(x-y)|f(y)|\chi_{4Q}(y)dy$$

$$+ \sup_{\in>o} |Q|^{-1} \int_Q \left(\int_{R^n} |a(y) - a_Q|^r |\phi_\in(x-y) - \phi_\in(z-y)| |f(y)|\chi_{R^n \backslash 4Q}(y)dy \right) dz$$

$$= A_1^Q(x) + A_2^Q(x) + A_3^Q(x).$$

Let us consider first the term $A_3^Q(x)$. Since $x, x_o, z \in Q$ and $y \notin 4Q$, the mean value theorem can be used to obtain:

$$\sup_{\in>0} |\phi_\in(x-y) - \phi_\in(z-y)| \leq C\delta \,/\, |x_0 - y|^{n+1},$$

where δ is the sidelength of Q. Therefore:

$$A_3^Q(x) \leq C \sum_{k=0}^{\infty} 2^{-k} |2^k Q|^{-1} \int_{2^k Q} |a(y) - a_Q|^r |f(y)|dy.$$

Thus, if we define

$$\sigma_3(f)(x_0) = \sup_{x_0 \in Q} |Q|^{-1} \int_Q A_3^Q(x)dx,$$

we get:

$$\sigma_3(f)(x_0) \leq C \sup_{x_0 \in Q} \sum_{k=0}^{\infty} 2^{-k} |2^k Q|^{-1} \int_{2^k Q} |a(y) - a_Q|^r |f(y)|dy.$$

We shall show that Theorem 2.2 applies to σ_3. Let α and β be two weights such that $\alpha^{-1}, \beta^{-1} \in A_1$ and $\alpha\beta = v^r$. Then

$$|2^k Q|^{-1} \int_{2^k Q} |a(y) - a_Q|^r \, |f(y)| dy$$

$$\leq C \| f\alpha \|_\infty |2^k Q|^{-1} \int_{2^k Q} |a(y) - a_{2^k Q}|^r \alpha(x)^{-1} dy$$

$$+ C \| f\alpha \|_\infty |a_Q - a_{2^k Q}|^r \, |2^k Q|^{-1} \int_{2^k Q} \alpha(x)^{-1} dy.$$

By Lemma 4.4 the first term is bounded by $C\| f\alpha \|_\infty \, \beta(x_0)^{-1}$. Since $\alpha^{-1} \in A_1$. Proposition 4.3 implies that the second term above is bounded by:

$$C\| f\alpha \|_\infty k^r \max_{0 \leq h \leq k} \left(|2^h Q|^{-1} \int_{2^h Q} v(y)^r dy \right) \inf_{v \in 2^k q} \alpha(y)^{-1} \leq C \| f\alpha \|_\infty k^r \beta(x_0)^{-1}.$$

After adding up in k, we obtain

$$\sigma_3(f)(x_0) \leq C \| f\alpha \|_\infty \, \beta(x_0)^{-1}.$$

Let us consider next the term $A_2^Q(x)$. We have:

$$A_2^Q(x) \leq CM(|a - a_Q|^r \, |f| \chi_{4Q})(x).$$

Thus, defining $\sigma_2(f)(x_0)$ by

$$\sigma_2(f)(x_0) = \sup_{x_0 \in Q} |Q|^{-1} \int_Q A_2^Q(x) dx,$$

we can write:

$$\sigma_2(f)(x_0) \leq C \sup_{x_0 \in Q} |Q|^{-1} \int_Q M(|a - a_Q|^r \, |f| \chi_{4Q})(x) dx.$$

We shall show that the extrapolation Theorem 2.2 can also be applied to σ_2. Let α, β and v as before. Then, for $s > 1$, but close enough to 1,

$$|Q|^{-1} \int_Q M(|a - a_Q|^r \, |f| \chi_{4Q})(x) dx \leq C \| f\alpha \|_\infty \left(|Q|^{-1} \int_{4Q} |a(x) - a_Q|^{rs} \alpha(x)^{-s} dx \right)^{1/s}.$$

Arguing as in the case of σ_3 for k=2, we obtain:

$$\sigma_2(f)(x_0) \le C\| f\alpha \|_\infty \, \beta(x_0)^{-1}.$$

Finally, we consider the term $A_1^Q(x)$. Here, we shall distinguish two cases: $0 < r \le 1$ and $1 < r < \infty$. In the first case, we have:

$$\left| |a(x)-a(y)|^r - |a(y)-a_Q|^r \right| \le |a(x)-a_Q|^r .$$

Thus,

$$A_1^Q(x) \le |a(x)-a_Q|^r \, M(f)| f(x).$$

Then, defining

$$\sigma_1(f)(x_0) = \sup_{x_0 \in Q} |Q|^{-1} \int_Q A_1^Q(x) dx,$$

we get

$$\sigma_1(f)(x_0) \le C \sup_{x_0 \in Q} |Q|^{-1} \int_Q |a(x)-a_Q|^r \, M(f)(x) dx.$$

We shall show that σ_1 satisfies the conditions for extrapolation in Theorem 2.2 . Let α, β and v be as before. Then since $\|\alpha M(f)\|_\infty \le C\| f\alpha \|_\infty$, we obtain

$$\int_Q |a(x)-a_Q|^r \, M(f)(x) dx \le C\| f\alpha \|_\infty \int_Q |a(x)-a_Q|^r \, \alpha(x)^{-1} dx.$$

Thus, by Lemma 4.4, it follows that

$$\sigma_1(f)(x_0) \le C\| f\alpha \|_\infty \, \beta(x_0)^{-1}.$$

For the case $1 < r < \infty$, we use the inequality

$$\left| |a(x)-a(y)|^r - |a(y)-a_Q|^r \right| \le C_r \left(|a(x)-a_Q|^r + |a(x)-a_Q| \; |a(x)-a(y)|^{r-1} \right).$$

Thus,

$$A_1^Q(x) \le C\left(|a(x)-a_Q|^r \, M(f)(x) + |a(x)-a_Q| \Phi_a^{r-1}(f)(x) \right).$$

By averaging $A_1^Q(x)$ over Q, we get:

$$\sup_{x_0 \in Q} |Q|^{-1} \int_Q A_1^Q(x)dx \le C \sup_{x_0 \in Q} \left(|Q|^{-1} \int_Q |a(x) - a_Q|^r \, M(f)(x)dx \right)$$

$$+ C \sup_{x_0 \in Q} \left(|Q|^{-1} \int_Q |a(x) - a_Q| \Phi_a^{r-1}(f)(x)dx \right) = C(\sigma_{11}(f)(x_0) + \sigma_{12}(\Phi_a^{r-1}(f))(x_0)).$$

We can deal with σ_{11} as we did with σ_1 in the case $0 < r \le 1$, obtaining

$$\sigma_{11}(f)(x_0) \le C \| f\alpha \|_\infty \, \beta(x_0)^{-1}.$$

As for σ_{12}, we observe that it coincides with σ_1 in case $r = 1$, therefore

$$\sigma_{12}(g)(x_0) \le C \| g\gamma \|_\infty \, \beta(x_0)^{-1},$$

if $\gamma\beta^{-1} = v$.

We have shown that all the σ's defined can be extrapolated from infinity. Thus, for $1 < p < \infty$ and $0 < r \le 1$, we get:

$$\|\Phi_{a,N}^r(f)\|_{L^p(\beta^p)} \le C \|(\Phi_{a,N}^r(f))^\#\|_{L^p(\beta^p)} \le C \sum_{i=1}^{3} \|\sigma_i(f)\|_{L^p(\beta^p)} \le C \|f\|_{L^p(\alpha^p)},$$

whenever $\alpha^p, \beta^p \in A_p$ and $\alpha\beta^{-1} = v$, provided $\Phi_{a,N}^r(f) \in L^p(\beta^p)$. For $1 < r < \infty$, let γ be the weight given in Lemma 5.3 with $r_2 = 1, r_1 = r - 1$. Then, since

$$(\Phi_{a,N}^r(f))^\#(x_0) \le C(\sigma_{11}(f)(x_0) + \sigma_{12}(\Phi_a^{r-1}(f))(x_0) + \sigma_2(f)(x_0) + \sigma_3(f)(x_0)),$$

we get

$$\|(\Phi_{a,N}^r(f))^\#\|_{L^p(\beta^p)} \le C(\|f\|_{L^p(\alpha^p)} + \|\Phi_a^{r-1}(f)\|_{L^p(\gamma^p)}).$$

If we assume that the theorem holds for Φ_a^{r-1}, then

$$\|(\Phi_{a,N}^r(f))^\#\|_{L^p(\beta^p)} \le C \|f\|_{L^p(\alpha^p)}.$$

In order to complete the proof of the theorem, we shall show that the Fefferman-Stein theorem 3.5 on the sharp maximal function can be applied if f is a bounded function with compact support. We observe that in this case $\Phi_{a,N}^r(f)$ is also a function with compact support. This was the main reason for introducing $\Phi^r_{a,N}$. Let Q be a cube containing the supports of both f and $\Phi^r_{a,N}(f)$. Then

$$\int_Q \Phi^r_{a,N}(f)(x)^p\, \beta(x)^p\, dx \le C\int_Q |a(x)-a_Q|^{rp}\, M(f)(x)^p\, \beta(x)^p\, dx$$
$$+ C\int_Q M(|a-a_Q|^r\,|f|)(x)^p\, \beta(x)^p\, dx.$$

Then $\Phi^r_{a,N}(f) \in L^p(\beta^p)$. Thus, if $0 < r \le 1$,

$$\int_{R^n} \Phi^r_{a,N}(f)(x)^p\, \beta(x)^p\, dx \le C\int_{R^n} \Phi^r_{a,N}(f)^{\#}(x)^p\, \beta(x)^p\, dx \le C\|f\|^p_{L^p(\alpha^p)},$$

where the constant C does not depend on N. Since $\Phi^r_{a,N}(f)(x)$ tends to $\Phi^r_a(f)(x)$ point-wisely as N tends to ∞. Fatou's lemma gives us:

$$\left(\int_{R^n} [\Phi^r_a(f)(x)]^p\, \beta(x)^p\, dx\right)^{1/p} \le C\|f\|_{L^p(\alpha^p)}$$

for every $(\alpha, \beta) \in A_p(v^r)$. This finishes the proof of the theorem. ❑

Proof of Theorem 5.2: Let us show that (i) implies (ii). Since the classes $A_p(v^r)$ allow extrapolation (see [S-T 1]), if the operator C^r_a is bounded from $L^p(\alpha^p)$ to $L^p(\beta^p)$ for every pair $(\alpha, \beta) \in A_p(v^r)$, for a given value of p, then by Theorem 2.1, the same is true for any p such that $1<p<\infty$. Let us prove that (ii) implies (iii). Since $v^r \in A_2$ (otherwise the classes $A_p(v^r)$ are empty, see [S-T 1]) we can factor $v^r = v_1 v_2^{-1}$ with $v_i \in A_1$ for $i = 1,2$. If $\alpha = (v_1 v_2^{-1})^{1/2}$ and $\beta = (v_2 v_1^{-1})^{1/2}$, then $(\alpha, \beta) \in A_2(v^r)$. We observe that $\alpha = \beta^{-1}$ and $\alpha^2 = v^r$. We have

$$\inf_{c\in R} \int_Q |a(x)-c|^r dx \le \inf_{y\in Q} \int_Q |a(x)-a(y)|^r dx$$
$$\le |Q|^{-1} \int_Q\int_Q |a(x)-a(y)|^r dxdy$$
$$\le \left(\int_Q \left(|Q|^{-1}\int_Q |a(x)-a(y)|^r dy\right)^2 \beta(x)^2 dx\right)^{1/2} \left(\int_Q \alpha(x)^2 dx\right)^{1/2}.$$

Since for $x \in Q$

$$|Q|^{-1} \int_Q |a(x) - a(y)|^r dy \leq C_a^r (\chi_Q)(x),$$

we get

$$\inf_{c \in R} \int_Q |a(x) - c|^r dx \leq \left(\int_Q C_a^r (\chi_Q(x)^2 \beta(x)^2 dx \right)^{1/2} \left(\int_Q \nu(x)^r dx \right)^{1/2}$$

Then, since we are assumming that (ii) holds, we get:

$$\inf_{c \in R} \int_Q |a(x) - c|^r dx \leq C \int_Q \nu(x)^r dx,$$

as we wanted to show. Finally, in order to prove that (iii) implies (i), we choose ϕ, a rapidly decreasing function such that $\chi_{\{x:|x| \leq 2\}} \leq \phi$. Then, $C_a^r(f)(x) \leq C \Phi_a^r(f)(x)$, and consequently, (iii) implies (i) by Theorem 5.2 ❑

6. Commutators of singular and Strongly singular integral operators

Given a positive real number b and a smooth radial cut-off function $\theta(x)$ supported in the ball $\{x: |x| \leq 2\}$, we consider the strongly singular kernel:

$$k(x) = \frac{e^{i|x|^{-h}}}{|x|^n} \theta(x).$$

Let us denote by Tf the corresponding singular integral operator:

$$Tf(x) = p.v. \int_{R^n} k(x-y) f(y) dy.$$

This operator has been studied by several authors. Among others, we mention I. I.Hirschman [H], S.Wainger [W], C.Fefferman and E.M. Stein [F-S] and S.Chanillo [C]. In particular, S.Chanillo developed the weighted L^p-theory using as a basic tool a result stated as Lemma 2.1 in [C, p.82]. For the commutator of the Hilbert transform, we mention R.Coifman, R.Rochberg and Guido Weiss [C-R-W] for the unweighted case, and

S.Bloom [B], who introduced the classes of weights we shall be dealing with. The case of general singular integrals with extrapolation methods was considered by C.Segovia and J.L.Torrea in [S-T 1] and [S-T 2].

Our purpose is to obtain weighted L^p estimates with pairs of weights for commutators of the strongly singular integral operator T. More precisely, we define the commutator of orden m,m a positive integer, with a function $a(x)$ as:

$$T_a^m(f)(x) = p.v.\int_{R^n}(a(x)-a(y))^m k(x-y)f(y)dy.$$

We shall reduce the study of this commutator to that of the commutator of fractional order $C_a^r(f)$ of the Hardy-Littlewood maximal operator defined in §5. More precisely, our technique consists in controlling the sharp maximal function of the commutator of the operator T by the commutator of the Hardy-Littlewood maximal function. This type of argument can also be applied to commutators of standard singular integrals and fractional integrals, providing proofs which we believe are simpler than those obtained before.

Now we are ready to state our results

Theorem 6.1 *Let $a(x)$ be a locally integrable function. For any number r such that $1< r < \infty$, the exists a constant C_r such that*

$$(T_a^m(f))^{\#}(x_0) \le C_r\left\{\sum_{h=0}^{m-1} M(C_a^{m-h}(T_a^h(f)))(x_0) + [M)c_a^{mr}(|f|^r))(x_0)]^{1/r}\right\}$$

holds for every $f \in C_0^\infty(R^n)$, almost everywhere in $x_0 \in R^n$.

Theorem 6.2 *Let $a \in BMO(\nu,1)$. Then, the commutator T_a^m is a bounded operator from $L^p(\alpha^p)$ to $L^p(\beta^p)$ provided (α,β) belongs to $A_p(\nu^m)$ and $1 < p < \infty$.*

Proff of Theorem 6.1: Our purpose is to estimate $(T_a^m(f))^{\#}(x_0), x_0 \in R^n$. Let Q be a cube with sidelength δ, such that $x_0 \in Q$. Let δ_0 be a number satisfying $4\delta_0 = \delta_0^{1/1+b}$. Obviously, $\delta_0 < 1$. Let us consider the case $\delta < \delta_0$. Let $\tilde{Q}$ be another cube with the same center as Q but with sidelength $\delta^{1/1+b}$.

We decompose f as $f = f_1 + f_2 + f_3$, where $f_1 = f\chi_{4Q}, f_2 = f\chi_{\tilde{Q}\setminus 4Q}$ and $f_3 = f\chi_{R^n\setminus\tilde{Q}}$. Since

$$(m_Q a - a(y))^m = \sum_{h=0}^{m}\binom{m}{h}(m_Q a - a(x))^{m-h}(a(x)-a(y))^h,$$

we have:

$$\begin{aligned} T_a^m(f)(x) = &-\sum_{h=0}^{m-1}\binom{m}{h}(m_Q a - a(x))^{m-h} T_a^h(f)(x) \\ &+\int_{R^n}(m_Q a - a(y))^m k(x-y) f(y)d, \end{aligned} \tag{6.3}$$

where $T_a^0(f) = Tf$. We define

$$c_Q = |Q|^{-1}\int_Q\left(\int_{R^n}(m_Q a - a(y))^m k(z-y) f_3(y)dy\right)dz.$$

Then:

$$\begin{aligned} &|Q|^{-1}\int_Q |T_a^m(f)(x) - c_Q|dx \\ &\le C\sum_{h=0}^{m-1}|Q|^{-1}\int_Q |m_Q a - a(x)|^{m-h}\,|T_a^h(f)(x)|dx \\ &+\sum_{i=1}^{2}|Q|^{-1}\int_Q\left|\int_{R^n}(m_Q a - a(y))^m k(x-y) f_i(y)dy\right|dx \\ &+|Q|^{-1}\int_Q\left(|Q|^{-1}\int_Q\left|\int_{R^n}(m_Q a - a(y))^m[k(x-y)-k(z-y)]f_3(y)dy\right|dz\right)dx. \end{aligned} \tag{6.4}$$

Then terms under the first summation sign are bounded by:

$$|Q|^{-1}\int_Q\left(|Q|^{-1}\int_Q |a(z)-a(x)|^{m-h}\,|T_a^h f(x)|dx\right)dz \le M\left(C_a^{m-h}(T_a^h f)\right)(x_0), x_0 \in Q.$$

The integral in (6.4) involving f_1 is bounded by:

$$|Q|^{-1}\int_Q |T((m_Q - a)^m f_1)(x)|dx \le \left(|Q|^{-1}\int_Q |T((m_Q - a)^m f_1)(x)|^r dx\right)^{1/r}, 1<r<\infty,$$

and, by the L^p boundedness of T with respect to Lebesgue measure, this is bounded by:

$$C\left(|Q|^{-1}\int_{4Q}|m_Q - a(x)|^{mr}\,|f(x)|^r\,dx\right)^{1/r}$$

$$\leq C\left(|Q|^{-1}\int_Q\left(|Q|^{-1}\int_{4Q}|a(z)-a(y)|^{mr}\,|f(x)|^r\,dx\right)dz\right)^{1/r}$$

$$\leq C\left[M(C_a^{mr}(|f|^r)(x_0)\right]^{1/r}$$

As for the integral in (6.4) involving f_2, we decompose it as the sum of two terms like in [C], p.88, obtaining:

$$\int_{R^n}(m_Q a - a(y))^m k(x-y) f_2(y)dy =$$

$$\int\frac{e^{i|x-y|^{-h}}\theta(x-y)}{|x-y|^{n(2+b)/r'}}\left(\frac{1}{|x-y|^{n(1-(2+b)/r')}}-\frac{1}{|x_o-y|^{n(1-(2+b)/r')}}\right)(m_Q a - a(y))^m f_2(y)dy$$

$$+\int\frac{e^{i|x-y|^{-b}}\theta(x-y)}{|x-y|^{n(2+b)/r'}}\frac{(m_Q a - a(y))^m f_2(y)}{|x_0-y|^{n(1-(2+b)/r')}}dy = A(x)+B(x),$$

where r is taken so close to 1 as to guarentee that $2+b<r'$. The term $A(x)$ is bounded in the following way:

$$|A(x)|\leq C\sum_{k=0}^{\infty}2^{-k}(2^{-k}\delta)^{-n}\int_{|y-x_0|\leq 2^{k+1}\delta}|m_Q a - a(y)|^m\,|f(y)|dy$$

$$\leq C|Q|^{-1}\int_Q dz\left(\sum_{k=0}^{\infty}2^{-k}(2^k\delta)^{-n}\int_{|y-x|\leq 2^{k+2}\delta}|a(z)-a(y)|^m\,|f(y)|dy\right)$$

$$\leq CM(C_a^m(f))(x_0).$$

For B we get, following Chanillo's argument:

$$|Q|^{-1}\int_Q |B(x)dx \le C|Q|^{-1/r'}\left(\int_{R^n}\frac{|m_Q a - a(y)|^{mr}\,|f_2(y)|^r}{|y-x_0|^{nr(1-(2+b)/r')}}dy\right)^{1/r}$$

$$\le C|Q|^{-1/r'}\left(\sum_{k=0}^{k_0}(2^k\delta)^{(r-1)(1+b)n}(2^k\delta)^{-n}\int_{|y-x_0|<2^{k+1}\delta}|m_Q a - a(y)|^{mr}\,|f(y)|^r\,dy\right)^{1/r},$$

where k_0 satisfies $2^{k_0}\delta < \delta^{\frac{1}{1+b}} \le 2^{k_0+1}\delta$. Thus we arrive at:

$$|Q|^{-1}\int_Q |B(x)|dx \le C\left[M(C_a^{mr}(|f|^r))\right]^{1/r}(x_0).$$

Let us consider now the integral in (6.4) involving f_3. It follows from the mean value theorem and the boundedness of the support of k that for $x\in Q, z\in Q$ and $y\in R^n\setminus\tilde{Q}$ there exists a constant C such that

$$|k(x-y)-k(z-y)|\le C\delta\;/\,|y-z|^{n+b+1}.$$

Therefore, the integral in (6.4) involving f_3 is bounded by:

$$C|Q|^{-1}\int_Q\left(\delta\int_{|y-x_0|>(1/2)\delta^{1/(1+b)}}|m_Q a - a(y)|^m\,|y-z|^{-(n+b+1)}\,|f(y)|dy\right)dz$$

$$\le C|Q|^{-1}\int_Q\left(\sum_{k=0}^{\infty}2^{-k}(2^k\delta)^{-\frac{n}{1+b}}\int_{|y-z|<2(2^k\delta)^{1/(1+b)}} m|_Q\, a - a(y)|^m\,|f(y)|dy\right)dz$$

$$\le CM(C_a^m(f))(x_0).$$

Finally, if $\delta \ge \delta_0$ we do not substract the constant c_Q. To estimate the averages of the terms under the summation sign in (6.3) we proceed as above. As for the average of the last term we observe that, for $\gamma = 4\delta_0^{-1}, x\in Q$ and $y\notin\gamma Q$ we have $|x - y| > 2$. Therefore, if we write $f = f\chi_{\gamma Q} + f\chi_{R^n\setminus\gamma Q}$, only the first term in this sum can contribute anything to this average, which is consequently bounded by

$$|Q|^{-1}\int_Q\left|\int_{\gamma Q}(m_Q a-a(y))^m k(x-y)f(y)dy\right|dx$$

$$\leq C\left(|Q|^{-1}\int_{\gamma Q}(m_Q a-a(y))^{mr}|f(y)|^r dy\right)^{1/r}$$

$$\leq C\left(|Q|^{-1}\int_Q(|Q|^{-1}\int_{\gamma Q}|a(z)-a(y)|^{mr}|f(y)|^r dy)dz\right)^{1/r}$$

$$\leq C\left[M(C_a^{mr}(|f|^r))(x_0)\right]^{1/r}.$$

Collecting our estimates, we have shown that

$$(T_a^m(f))^{\#}(x_0)\leq C\left\{\sum_{h=0}^{m-1}M(C_a^{m-h}(T_a^h f))(x_0)+[M(C_a^{mr}(|f|^r))(x_0)]^{1/r}\right\},$$

for any r such that $1<r<\infty$, and this completes the proof of the theorem. ❑

Proof of Theorem 6.2: We shall assume that Theorem 5.2 holds. By Proposition 4.2 $a\in BMO(\nu,mr)$ for any $r>1$. If $(\alpha,\beta)\in A_p(\nu^m)$ then $(\alpha^r,\beta^r)\in A_{p/r}(\nu^{mr})$ for $r>1$ and close enough to 1. Indeed, let r be such that $1<r<p$ and $\alpha^p,\beta^p\in A_{p/r}$, then $\alpha^r\beta^{-r}=\nu^{mr}$ and $(\alpha^r)^{p/r}$, $(\beta^r)^{p/r}\in A_{p/r}$. For this choice of r we have

$$\int_{R^n}\left[M(C_a^{mr}(|f|^r))\right]^{p/r}\beta(x)^p dx$$

$$=\int_{R^n}\left[M(C_a^{mr}(|f|^r))\right]^{p/r}[\beta(x)^r]^{p/r}dx\leq C\int_{R^n}[C_a^{mr}(|f|^r)(x)]^{p/r}[\beta(x)^r]^{p/r}dx.$$

Then, applying Theorem 5.2, this is bounded by

$$\int_{R^n}(|f|^r)^{p/r}(\alpha^r)^{p/r}dx=\int_{R^n}|f|^p\alpha^p dx.$$

On the other hand, by Lemma 5.3, talking $r_1=h$ and $r_2=m-h$, there exist γ_h such that $(\gamma_h,\beta)\in A_p(\nu^{m-h})$ and $(\alpha,\gamma_h)\in A_p(\nu^h)$. Thus, again by Theorem 5.2

$$\int_{R^n} M(C_a^{m-h}(T_a^h(f)))(x)^p \beta(x)^p dx$$

$$\leq C\int_{R^n} C_a^{m-h}(T_a^h(f))(x)^p \beta(x)^p dx \leq C\int_{R^n} \left|T_a^h(f)(x)\right|^p \gamma_h(x)^p dx.$$

If in Theorem 6.1 we take $m = 1$, then $h = 0$ and $\gamma_0 = \alpha$, thus by the weighted L^p estimates for T due to S. Chanillo [C], we get

$$\int_{R^n} \left[T_a^m(f)^\#(x)\right]^p \beta(x)^p dx \leq C\int_{R^n} |f(x)|^p \alpha(x)^p dx.$$

Assuming that we can apply the Fefferman-Stein theorem on the sharp maximal function, the last inequality would imply

$$\int_{R^n} \left|T_a^1(f)(x)\right|^p \beta(x)^p dx \leq C\int_{R^n} |f(x)|^p \alpha(x)^p dx,$$

which proves the theorem for $m = 1$. By induction on m and the arguments given above, the theorem follows for every m, provided that the conditions required to apply the aforemantioned Fefferman-Stein theorem are fulfilled. Let f be a bounded function with compact support. Since the kernel k has also compact support the same happens for the function $T_a^m(f)$. Let Q be a cube containing the supports of both f and $T_a^m(f)$. Then, since

$$(a(y) - a_Q)^m = \sum_{h=0}^{m} \binom{m}{h} (a(y) - a(x))^h (a(x) - a_Q)^{m-h},$$

we have

$$\begin{aligned} T_a^m(f)(x) &\leq C\sum_{h=1}^{m-1} |a(x) - a_Q|^{m-h} |T_a^h(f)(x)| \\ &+ |a(x) - a_Q|^m |T(f)(x)| + |T((a - a_Q)^m f)(x)|. \end{aligned} \tag{6.5}$$

Let us consider the last term. We have

$$\int_Q |T((a - a_Q)^m f)(x)|^p \beta(x)^p dx$$

$$\leq C\int_Q |a(x) - a_Q|^{mp} |f(x)|^p \beta(x)^p dx \leq C\|f\|_\infty^p \int_Q |a(x) - a_Q|^{mp} \beta(x)^p dx$$

By Lemma 4.6, the last term is bounded by

$$C\|f\|_\infty^p \int_Q \alpha(x)^p\,dx < \infty.$$

For the term $|a(x)-a_Q|^m\,|T(f)(x)|$ in (6.5), we have, if $r>1$ is close enough to 1

$$\int_Q |a(x)-a_Q|^{mp}\,|T(f)(x)|^p\,\beta(x)^p\,dx$$

$$\leq \left(\int_Q |a(x)-a_Q|^{mpr}\,\beta(x)^{pr}\,dx\right)^{1/r}\left(\int_Q |T(f)(x)|^{pr'}\,dx\right)^{1/r'}$$

$$\leq C\left(\int_Q \alpha(x)^{pr}\,dx\right)^{1/r}\|f\|_\infty^p\,|Q|^{1/r'} < \infty.$$

In the case $m=1$, this is all we need in order to apply the Fefferman-Stein theorem, obtaining:

$$\int_{R^n} |T_a^1(f)(x)|^p\,\beta(x)^p\,dx \leq C\int_{R^n} |f(x)|^p\,\alpha(x)^p\,dx,$$

for any bounded function f with compact support. If we assume that for $0<h<m$

$$\int_{R^n} |T_a^h(f)(x)|^p\,\beta(x)^p\,dx \leq C\int_{R^n} |f(x)|^p\,\alpha(x)^p\,dx,$$

holds for any $(\alpha,\beta)\in A_p(v^h)$, then for the terms under the summation sign in (6.5) we get

$$\int_Q |a(x)-a_Q|^{(m-h)p}\,|T_a^h(f)(x)|^p\,\beta(x)^p\,dx$$

$$\leq \left(\int_Q |a(x)-a_Q|^{mp}\,\beta(x)^p\,dx\right)^{(m-h)/m}\left(\int_Q |T_a^h(f)(x)|^{pm/h}\,\beta(x)^p\,dx\right)^{h/m}$$

$$\leq C\left(\int_Q \alpha(x)^p\,dx\right)^{(m-h)/m}\left(\int_Q |T_a^h(f)(x)|^{pm/h}\,(\beta(x)^{h/m})^{pm/h}\,dx\right)^{h/m}. \tag{6.6}$$

Since $\alpha\beta^{-1} = v^m$ and $\alpha^p, \beta^p \in A_p$, we have $\alpha^{h/m}(\beta^{h/m})^{-1} = v^h$ and $\alpha^p, \beta^p \in A_{mp/h}$, i.e., $(\alpha^{h/m}, \beta^{h/m}) \in A_{mp/h}(v^h)$. Thus (6.6) is bounded by

$$C\left(\int_Q \alpha(x)^p\,dx\right)^{(m-h)/m}\left(\int_Q |f(x)|^{pm/h}\,\alpha(x)^p\,dx\right)^{h/m} \leq \|f\|_\infty^p \int_Q \alpha(x)^p\,dx,$$

and the proof is complete ❑

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