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ON D-K-MACKEY LOCALLY K-CONVEX SPACES

Miguel ***Caldas***

Abstract:

D-K-Mackey locally K-convex spaces are introduced and a description of their topologies is obtained.

Introduction.

The non-Archimedean analogues of Mackey, d-barrelled and d-infrabarrelled locally convex spaces over **R** *or* **C** *were introduced by J. Van Tiel [8] and the author ([2], [3]) respectively. In the present article we define the larger class of D-K-Mackey non-Archimedean locally convex spaces over a spherically complete field K, which is an extension of the classical definition of J. Rojo [6]. The main goal of this paper is to give several characterizations of such space by means of topologies. Their relation with other significant class of non-Archimedean locally convex spaces over K (briefly locally K-convex) are stablished.*

Departamento de Matemática Aplicada - IMUFF, R.J. Brasil.

Terminology and Notation.

We shall adopt the notation and terminology of [8], [9] and [3]. Some of the notations and terminology used in the sequel are as follows: K will denote a non-trivial spherically complete non-Archimedean valued field and (E,τ) a locally K-convex space endowed with the locally K-convex topology τ. As in [9], if A is a subset of E the pseudo-polar A^p $\cdot$(respectively pseudo-bipolar A^{pp}) of A is defined as $A^p = \{g \in E'; |g(A)| < 1\}$ (respectively $A^{pp} = \{x \in E; |A^p(x)| < 1\}$). We have $A = A^{pp}$ if and only A is K-convex and Closed ([9] Proposition 2)

In this paper E will always stand for a separated locally K-convex space over a spherically complete field K.

Definition 1. *Let E be a locally K-convex space and E' its dual.*

(i) (E,τ) *is said to be K-Mackey if the topology* τ *coincides with* $\tau_c(E,E')$, *where* $\tau_c(E,E')$ *be the locally K-convex topology in E of uniform convergence on the collection of all K-convex bounded and c-compact subset of* $(E', \sigma(E',E))$ *and is the strongest* (E,E')*-compatible locally K-convex topology on E.*

(ii) (E,τ) *is said to be d-K-Mackey, if each* $\sigma(E',E)$*-bounded H of E' which is the countably union of equicontinuous subsets of E' and such that the K-convex hull of H is relatively c-compact for the topology* $\sigma(E',E)$, *is itself equicontinuous.*

(See [7], for the concept and property of an c-compact subset).

Lemma 1. *Let E be a locally K-convex space. Then the K-convex hull of a K-convex c-compact subset A of E is c-compact.*

Proof. Since K is spherically complete, the set $B = \{\lambda \in K; |\lambda| \leq 1\}$ is c-compact ([8] Theorem 2.6). Therefore it is enough to see that the K-convex hull of A is the image of $B \times A$ under the mapping $(\lambda, x) \rightarrow \lambda x$.

Proposition 1. *Let E be a locally K-convex space and E' its dual. Every K-convex subset of E' which is bounded and relatively c-compact for the topology* $\sigma(E',E)$ *is bounded for the topology* $b(E',E)$.

Proof. Let M be a k-convex bounded and relatively c-compact of $(E',\sigma(E',E))$. By ([8] Theorem 2.5 and 2.7) and Lemma 1 the k-convex closed hull $N = \overline{C(M)}$ of the closure of M is a k-convex bounded and c-compact subset of $(E',\sigma(E',E))$. It's pseudo-polar N^p is a neighborhood of zero in E for the topology $\tau_C\ (E,E')$. Let B be an arbitrary bounded subset of E. Then B is also bounded for the topology $\tau_C\ (E,E')$ ([1] p.70) and thus there exists $\lambda \in K^*$ such that $B \subseteq \lambda N^p$. But then $M \subseteq N = N^{pp} \subseteq \lambda B^p$. Hence by the definition of the topology $b(E',E)$, the set M is $b(E',E)$-bounded.

Proposition 2. *Let (E,τ) be a locally k-convex space with topology τ. Then τ coincides with the topology of uniform convergence on the equicontinuous subsets of E'.*

Proof. Let Θ the collection of all equicontinuous subsets of E' and τ_Θ be the locally k-convex topology on E of uniform convergence on Θ. If U is a k-convex τ-neighborhood of zero in E, then U^p is equicontinuous in E'. Hence $U = U^{pp}$ is a τ_Θ-neighborhood of zero in E. Thus τ_Θ is finer than the topology τ. Conversely, let H be a equicontinuous set in E', we can find a k-convex τ-neighborhood U of zero in E such that $|H(U)| < 1$. Then $H \subset U^p$. It follows that $H^p \supset U^{pp} = U$; i.e., H^p is a τ-neighborhood of zero in E. Thus τ is finer than the topology τ_Θ and the desired equality $\tau = \tau_\Theta$ is established.

Our next goal is to prove certain characterizations of d-k-Mackey spaces. In order to do so we shall the following.

Definition 2. *Let E be a locally k-convex space and let Γ be the collection of all k-convex bounded relatively c-compact subset of $(E',\sigma(E',E))$, which is the countably union of equicontinuous subset of E'. Then the corresponding Γ-topology on E of uniform convergence on Γ will be denoted by $\Gamma_d(E,E')$.*

Clearly $\sigma(E,E') \leq \tau_d(E,E') \leq \tau_C\ (E,E') \leq b(E,E')$. Therefore, the topology $\tau_d(E,E')$ is (E,E')-compatible.

The following proposition prove that the given topology of a d-k-infrabarrelled space E ([3]), over a spherically complete field k is the $\tau_d(E,E')$ locally k-convex topology on E.

Proposition 3. *If (E,τ) is d-k-infrabarrelled, then the topology τ coincides with the topology $\tau_d(E,E')$.*

Proof. If is enough to apply Proposition 1.

Theorem 1. *For a locally k-convex space (E,τ), the following conditions are equivalent:*

(i) (E,τ) *is d-k-Mackey.*

(ii) $\tau = \tau_d(E,E')$.

Proof. (i)→(ii): $\tau=\tau_\Theta$ (Proposition 2) where τ_Θ be the locally *k*-convex topology on *E* defined by the family $\Theta=\{H\subset E'$; *k*-convex equicontinuous$\}$. Since, every *k*-convex equicontinuous subset of *E'* is relatively *c*-compact for the topology $\sigma(E,E')$ ([8] Theorem 4.4 (b)). Then $\Theta\subset\Gamma$ (Γ as in definition 2). Hence τ_Θ is weaker than $\tau_d(E,E')$. Let now $H\in\Gamma$. By Lemma 1 and hypothesis, *H* is equicontinuous. Thus $H\in\Theta$. Hence $\tau_d(E,E')$ is weaker than τ_Θ and the desired equality $\tau=\tau_d(E,E')$ is established.

(ii)→(i): Let *H* $\sigma(E',E)$-bounded of *E'* which is the countably union of equicontinuous subsets of *E'* and such that the *k*-convex hull *C*(*H*) of *H* is relatively *c*-compact for the topology $\sigma(E',E)$. Then *C*(*H*) is a *k*-convex, bounded e relatively *c*-compact subset of $(E',\sigma(E',E))$. Therefore its pseudo-polar $(C(H))^P \subset (H)^P$ is a neighborhood of zero in *E* for the topology $\tau_d(E,E')$; i.e., by hipothesis a τ-neighborhood of zero. Hence *H* is equicontinuous.This proves (i).

As a direct consequence of Theorem 1. We have:

Corollary 1. *A k-Mackey space is always d-k-Mackey.*

Proof. Let (E,τ) be a *k*-Mackey space. We shall show that τ is the topology $\tau_d(E,E')$. Indeed. By definition 1(i) and remark of definition 2 implies that $\tau_d(E,E') \le \tau_C(E,E') = \tau$. On the other hand $\tau=\tau_\Theta \le \tau_d(E,E')$. Therefore $\tau=\tau_d(E,E')$. This prove that (E,τ) is a *d-k*-Mackey space (Theorem 1).

Remark 1.

(i) It follows from Proposition 3 and Theorem 1 that every *d-k*-infrabarrelled space is a *d-k*-Mackey.

(ii) The following diagram helps to remember some of the relations proved in this and their relation with other classes:

$$\begin{array}{ccc}
k\text{-barrelled} & \xrightarrow{[2]} & d\text{-}k\text{-barrelled} \\
\text{[clear]}\ \downarrow & & \downarrow\ \text{[clear]} \\
k\text{-infrabarrelled} & \xrightarrow{[3]} & d\text{-}k\text{-infrabarrelled} \\
\text{[5]}\ \downarrow & & \downarrow\ \text{[Proposition 3]} \\
k\text{-Mackey} & \xrightarrow{\text{[Corollary 1]}} & d\text{-}k\text{-Mackey}
\end{array}$$

Theorem 2. *Let E and F be two separated locally k-convex spaces. Then every linear mapping f: E→F which is continuous for the topologies* $\sigma(E,E')$ *and* $\tau_d(F,F')$, *is also continuous for the topologies* $\tau_d(E,E')$ *and* $\tau_d(F,F')$.

Proof. Let $V=H^p$ be a neighborhood of zero in F for the topology $\tau_d(F,F')$, where $H=\bigcup_{n\geq 1} H_n$ is a k-convex, bounded, relatively c-compact subset of $(F',\sigma(F',F))$ and H_n equicontinuous ($n\geq 1$). Since ${}^t f$: $F'\to E'$ (${}^t f$ transpose of f) is continuous for the topology $\sigma(F',F)$ and $\sigma(E',E)$ ([1] p.101), the set $X={}^t f(H)$ is a k-convex bounded relatively c-compact of $(E', \sigma(E',E))$ which is the countably union of ${}^t f(H_n)$ equicontinuous subsets and thus $U=X^p$ is a neighborhood of zero in E for the topology $\tau_d(E,E')$,. Since $X={}^t f(H)$, we have ${}^t({}^t f)(X^p)\subset H^p$, that is $f(U)\subset V$, which proves that f is continuous for the topologies $\tau_d(E,E')$ and $\tau_d(F,F')$.

Corollary 2. *Let* (E,τ_E) *and* (F, τ_F) *be locally k-convex spaces,* (E,τ_E) *d-k-Mackey. Then every linear mapping f: E→F which is continuous for the topologies* $\sigma(E,E')$ *and* $\sigma(F,F')$ *is also continuous for the topologies* τ_E *and* τ_F.

Proof. By the assumption and by Theorem 2 the mapping f is continuous for the topologies $\tau_E=\tau_d(E,E')$ and $\tau_d(F,F')$. But $\tau_d(F,F')$ is finer than τ_F. Then f is continuous for the topologies τ_E and τ_F.

Let us recall that the hypothesis of this corollary is satisfied if E is an d-k-infrabarrelled space (Proposition 3).

Theorem 3. *For a locally k-convex space* (E,τ_E) *the following conditions are equivalent:*

(i) (E,τ_E) *is d-k-Mackey.*

(ii) *For every locally k-convex space* (F,τ_F), *each linear mapping f:E→F which is continuous for the topologies* τ_E *and* $\sigma(F,F')$ *is also continuous for the topologies* τ_E *and* $\tau_d(F,F')$.

Proof.:

(i)→(ii): By ([1] p.103) the mapping f is continuous for the topology $\sigma(E,E')$ and $\sigma(F,F')$. By the Theorem 2 it is also continuous for $\tau_d(E,E')$ and $\tau_d(F,F')$. Finally since $\tau_E=\tau_d(E,E')$ (Theorem 1), f is continuous for τ_E and $\tau_d(F,F')$ (also, since $\tau_F\leq\tau_d(F,F')$ the mapping f is continuous for τ_E and τ_F).

(ii)→(i): Since $\sigma(E,E')\leq \tau_E$, the mapping canonical imbedding $j{:}E\to E$ is continuous for the topologies τ_E and $\sigma(E,E')$. By the assumption (ii) it is also continuous for the topologies τ_E and $\tau_d(E,E')$. Hence $\tau_d(E,E')\leq\tau_E$. Therefore $\tau_d(E,E') = \tau_E$.

d-k-Mackey spaces have remarkable stability properties which we list in the following Proposition and that reasoning as in [3] can be proved.

Proposition 4. *Let* (E,τ_E) *and* (F,τ_F) *be two locally k-convex spaces.*

(i) *Let D a dense k-subspace of E. Then* (E,τ_E) *is d-k-Mackey if* (D,τ_D) *is d-k-Mackey.*

(ii) *Let f be a linear continuous, almost open (a fortiori, open) and surjective mapping from E into F. Then* (F,τ_F) *is d-k-Mackey if* (E,τ_E) *is d-k-Mackey.*

(iii) *If* (E,τ_E) *is a d-k-Mackey space and M a closed k-subspace of E. Then the quotient space E/M is a d-k-Mackey space.*

(iv) *Let £ be the family of all d-k-Mackey. Then £ is stable under the formation of arbitrary direct sums, inductive limits, and arbitrary products.*

Finally we apply these notion of *d-k*-Mackey to the space of the continuous mappings.

We suppose that X is an ultraregular space, that is a separated topological space where every point has a filterbase of clopen neighborhoods. $C(X,E)$ the space of all continuous E-valued mappings on X, endowed with the compact-open topology. We call a topological space *w*-compact if every countable union of compact set is relatively compact.

Theorem 4. *If* $C(X,E)$ *is a d-k-*Mackey *space, then* $C(X,K)$ *and E are d-k-Mackey spaces.*

Proof. In ([4], Proposition 2.1 and 2.2) it has been show that $C(X,K)$ and E are closed complemented k-subspaces of $C(X,E)$. Therefore, there exist two separated quotients spaces of $C(X,E)$ which are isomorphous to $C(X,K)$ and E, respectively. Since by Proposition 4(iii) the property of being *d-k*-Makey is invariable under separated quotient formation, $C(X,K)$ and E are *d-k*-Mackey.

Theorem 5. *Let X be an ultraregular w-compact space and* (E_n,τ_n) *be a crescent sequence of locally k-convex spaces. If* $(E,\tau) = \underrightarrow{Lim}(E_n,\tau_n)$ *then the inductive limit* $\underrightarrow{Lim}\ C(X,E_n)$ *is a dense topological k-subspace of* $C(X,E)$.

Proof. See ([4] Theorem 2.5).

Corollary 3. *Let X be an ultraregular w-compact space and E be the inductive limit of* E_n, *where* $(E_n)_{n\in N}$ *is a crescent sequence of non-Archimedean normed spaces. Then* $C(X,E)$ *is an d-k-Mackey space.*

Proof. By Theorem 5, the inductive limit of spaces $C(X,E_n)$ is a dense topological k-subspace in $C(X,E)$. Since E_n is non -Archimedean normed and by ([4] Theorem 4.8) can be proved that the space $C(X,E_n)$ is d-k-infrabarrelled. Hence and Remark 1(i) $C(X,E_n)$ is d-k-Mackey. By Proposition 4(iv), the inductive limit of spaces $C(X,E_n)$ is d-k-Mackey and by the same Proposition 4(*i*) it results that $C(X,E)$ is d-k-Mackey.

References

[1] *N.* Bourbaki. Espaces Vectoriels Topologiques. Chap.3, 4 et 5, Hermann, Paris (1967).

[2] *M. Caldas.* Uma Generalização Não-Arquimediana do Teorema de Mahowald Para Espaços d-Tonelados. Portugaliae Math. V.49 (1992).

[3] *M. Caldas.* Stability of Barrelledness and Infrabarrelledness in Locally K-Convex Spaces. Bull. Cal. Math. Soc. 84 (1992), 97-102.

[4] *S. Navarro.* Espaços Vetoriais Topologicos de Funçoes Continuas. Tese de Doutorado. Brasil (1981).

[5] *D. Pombo Jr.* A note On Infrabarrelled Locally k-Convex Spaces. Indian J. Pure Appl. Math. 22 (1991), 575-578.

[6] *J. Rojo.* Diversas Caracterizaciones de la Noción Tonelaje. Rev. Contri. Cient. y Tec. 31 (1978), 47-61.

[7] *T.A. Springer.* Une Notion De Compacite Dans La Theorie des Espaces Vectoriels Topologiques. Indag. Math. 27 (1965), 182-189.

[8] *J. Van Tiel.* Espaces localement k-Convexes. Indag. Math. 27 (1965) 249-289.

[9] *J. Van Tiel.* Esembles Pseudo-Polaires Dans les Espaces Localement k-Convexes 28 (1966), 369-373.

gmamccs@vmhpo.uff.br
Departamento de Matemática Aplicada - IMUFF
Universidad Federal Fluminense
Rua São Paulo s/n. CEP: 24210
Niteroi, RJ - Brasil