Pro Mathematica, Vol. IX, Nos. 17-18, 1995

ON THE JUSTIFICATION OF THE LEAST SQUARE METHOD FOR NONPOTENTIAL, NONLINEAR OPERATORS

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Suppose in a Hilbert (possibly, negative) space H with inner product (.,.) there is given an equation

(1) $$N(u) = f.$$

N is an operator with the domain D(N). We assume that D(N) is dense and convex in some Hilbert space $H_1 \subseteq H$.

$$\overline{R(N)} = H;\ N(0) = 0;$$

(2) $\varphi_1, \varphi_2, \ldots \varphi_n \ldots$ - *basis in* H_1.

(3) $$u_n = \sum_{k=1}^{n} a_k \varphi_k$$

The method of least squares (LSM) is to take as an approximate solution of (1) a linear combination of the given elements $\varphi_1, \varphi_2, \ldots, \varphi_n$ whose unknown coefficients a_k are determined from the condition

(4) $$\|N(u_n) - f \mid H\|^2 \Rightarrow \min .$$

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The most general result in theoretical justification of LSM has been obtained by A. Langenbach [1]. It is formulated in

Theorem 1. *Suppose the condition (2) and*

$$[N(u_1) - N(u_2), u_1 - u_2] \geq C_1 \| u_1 - u_2 | H_1 \|^2$$

(5) $$\forall\, u_1 \in D(N),\ u_2 \in D(N)$$

(6) $$\| u \,|\, H \| \leq C_2 \| u \,|\, H_1 \| \quad \forall\, u \in D(N)$$

are satisfied. Then the sequence $\{u_n\}$ (3), (4) *converges in* H_1.

(Here and then constants C_i do not depend on u).

If N is a linear operator, the condition of monotony (5) implies that N is positive definite operator. So, the theorem 1 can't justificate LSM for the number of linear and nonlinear diferential equations. (See, for example, the parabolic problems). We shall generalize the result of Langenbach in the following theorems.

Theorem 2. *Let N satisfies (2) and let B be an operator with* $D(B) \supseteq D(N)$; $R(B) \subseteq H$.

If

$$[N(u_1) - N(u_2),\ B(u_1 - u_2)] \geq C_3 \| u_1 - u_2 | H_1 \|^2$$

(7) $$\forall u_1 \in D(N),\ u_2 \in D(N)$$

and

(8) $$\|B(u) \,|\, H \| \leq C_4 \| u \,|\, H_1 \| \qquad \forall u \in D(N),$$

then the sequence $\{u_n\}$ *converges in* H_1 *for any* $f \in H$.

Proof. The conditions (2), (3), (4) imply that for every fixed element f in H there exists at least one sequence $\{u_n\}$ such that

(9) $$\| N(u_n) - N(u_m) \,|\, H \| \to 0 \quad as \quad n \to \infty \ \text{ and } \ m \to \infty.$$

With the help of (7) and (8) we get

$$\|u_n - u_m \,|\, H_1 \|^2 \leq \frac{1}{C_3} \| N(u_n) - N(u_m) \,|\, H \| \ ,\ \| B(u_n - u_m) \,|\, H \| \leq$$
$$\leq \frac{C_4}{C_3} \| N(u_n) - N(u_m) \,|\, H \| \, , \| u_n - u_m \,|\, H_1 \|$$

and thus, also

$$\| u_n - u_m |_{H_1} \| \le \frac{C_4}{C_3} \| N(u_n) - N(u_n) - N(u_m) | _H \| \tag{10}$$

Now, (9), (10) imply the convergence of $\{u_n\}$. □

Theorem 3.

a) Let N satisfy (2) and have a Gateaux differential

$$N'(u)h \equiv \left\{\frac{d}{d\alpha} N(u + \alpha h)\right\}_{\alpha=0}$$

continuous on any line.

If

$$(N'(u)h, Bh) \ge C_3 \|h |_{H_1} \|^2 \quad \forall u \in D(N), \forall h \in D(N), \tag{11}$$

then (7) is valid.

b) If (7) and (8) are held, then N has inverse operator N^{-1} on $R(N)$ and N^{-1} satisfies the Lipschitz condition.

Proof.

a) On one hand for any $u_1 \in D(N)$, $u_2 \in D(N)$

$$\big(N(u_1) - N(u_2), B(u_1 - u_2)\big) = \int_0^1 \big(N'(tu_1 + (1-t)u_2)(u_1 - u_2), B(u_1 - u_2)\big)t.$$

(12)

With (11) on the other hand we get

$$\big(N(u_1) - N(u_2), B(u_1 - u_2)\big) \ge C_3 \|u_1 - u_2|_{H_1} \|^2 .$$

b) Let v_1 and v_2 are arbitrary elements in $R(N)$.

(7) implies that there exist uniquely determined elements $u_1 \in D(N)$ and $u_2 \in D(N)$ such that $v_1 = N(u_1)$, $v_2 = N(u_2)$. According to (10)

$$\| N^{-1}(v_1) - N^{-1}(v_2) |_{H_1} \| \le \frac{C_4}{C_3} \| v_1 - v_2 |_H \|, \ \forall\, v_1, v_2 \in R(N). \tag{13}$$

This inequality gives the Lipshitz constant estimate. □

Thus, the operator N^{-1} with the domain $R(N)$ dense in H, could be extended continuously to the whole H. Now we are to introduce

Definition 1. *Element* $u_0 \in H_1$ *is said to be the generalized solution of* (1) *if there exists a sequence* $\{u_n\}$, $u_n \in D(N)$, *such that*

$$(14) \qquad \| N(u_n) - f \,|\, H \| \to 0 \qquad \text{as } n \to 0$$

and

$$(15) \qquad \| u_n - u_0 \,|\, H_1 \| \to 0 \qquad \text{as } n \to 0$$

Corollary 1. *If the nonlinear operator N satisfies* (2), (7), (8) *then, for any* $f \in H$, *there exist unique* $u_0 \in H_1$ *- generalized solution of* (1).

According to LSM, the approximate solution is determined from (3), (4) which are equivalent to the system of the algebraic equations:

$$(16) \qquad \frac{\partial}{\partial a_k} \|N_n(u) - f \,|\, H \|^2 = 0 \ , \quad k = 1,2,...,n.$$

After the differentiation (16) transformes (according to [1]) into

$$(17) \qquad (N'(u_n)\varphi_k \, , N(u_n) - f) = 0, \quad k = 1,2,...,n.$$

Let's establish, in view of completeness, the solvability of the system (17) under the conditions (7), (8).

Lemma 1. *Let the operator N has a continuous Gateaux derivative N'*, $N(0)=0$, *and let basis* $\{\varphi_n\}$ *be orthonormal in* H_1

If the conditions (7), (8) *are satisfied, then LSM system of algebraic equations* (17) *is solvable for any* n=1,2,...

Proof. Using (7), (8) we get $\forall u_n \in D(N)$

$$C_4 \| N(u_n) \,|\, H \| \cdot \| u_n \,|\, H_1 \| \geq (N(u_n), B(u_n)) \geq C_3 \| u_n \,|\, H_1 \|^2 .$$

That leads to

$$(18) \qquad C_4 \| N(u_n) \,|\, H \| \geq C_3 \| u_n \,|\, H_1 \|$$

$\{\varphi_n\}$ is an orthonormal set in H_1 so, for u_n derived from (3) we obtain

$$\| u_n \mid H_1 \| = \left(\sum_{k=1}^{n} a_k^2 \right)^{1/2} .$$

Under the assumptions of the theorem the function

$F(a_1,a_2,...,a_n) = \| N(u_n) - f \mid H \|^2$ has continuous partial derivatives of the first order in an arbitrary ball

$$\sum_{k=1}^{n} a_k^2 \le R^2 .$$

Furthermore,

$$C_4 \| N(u_n) \mid H \| \ge C_3 \| u_n \mid H_1 \| = C_3 \left(\sum_{k=1}^{n} a_k^2 \right)^{1/2} \ge 2C_4 \| f \mid H \| .$$

The last inequality takes place on the surface of the ball as soon as R is great enough. That implies

$$\|N(u_n) - f \mid H \|^2 \ge \| N(u_n) \mid H \| \cdot \{\| N(u_n) \mid H \| - 2 \| f \mid H \|\} +$$

(19) $$+ \| f \mid H \|^2 \ge \| f \mid H \|^2$$

At the same time there exist at least one point ($u_n = 0$) where

$$\| N(u_n) - f \mid H \|^2 = \| f \mid H \|^2 .$$

Thus, the function f has its minimun into the ball and in this point (17) is held. Therefore, (17) is solvable. □

Remark 1. It is rather difficult to set up the conditions for (17) to have the unique solution. We are not familiar with any results of that kind formulated in terms of operators investigated in this paper.

Remark 2. An auxiliary operator B, figurated in (7), (8), exists for an arbitrary linear operator $N \equiv A$ with a bounded inverse.

Really, let $B = (A^{-1})^* S$, where S is an arbitrary operator such that $D(S)=D(A)$ and

(20) $$(Su,u) \ge C_7 \| u \mid H_1 \|^2 , \qquad \forall u \in D(S)$$

(21) $$\| Su \mid H \| \le C_8 \| u \mid H_1 \|^2 , \quad \forall u \in D(S)$$

For any u, u_1, u_2 - elements of $D(A)$, are evident the inequalities

$$(7_1) \qquad \big(A(u_1 - u_2), B(u_1 - u_2)\big) \geq C_7 \| u_1 - u_2 \,|_{H_1} \|^2$$

and

$$(8_1) \qquad \| B(u) \,|\, H \| = \|(A^{-1})^* Su \,|\, H \| \leq C_9 \| Su \,|\, H \| \leq C_{10} \| u \,|_{H_1} \|.$$

Whether the nonlinear operator N has the continuous Gateaux derivative $N'(x)$, we are able to construct the operator B.

Let $B = ([N'(0)]^{-1})^* S$, where S satisfies (20), (21) and $([N'(0)]^{-1})^*$ is supposed to exist. If

$$(22) \qquad \| ([N'(0)]^{-1})^* v \,|\, H \| \leq C_{11} \| v \,|\, H \|$$

$$(23) \qquad (N'(x)u, Bu) \geq C_{12}(N'(0)u, Bu),$$

then

$$\big(N(u_1) - N(u_2), B(u_1 - u_2)\big) = \big(N'(u_1 + \theta(u_1 - u_2))\,(u_1 - u_2), B(u_1 - u_2)\big) \geq$$

$$\geq C_{12}\big(N'(0)(u_1 - u_2), B(u_1\, u_2)\big) \equiv C_{12}\big(N'(0)(u_1 - u_2), ([N'(0)]^{-1})^* S(u_1 - u_2)\big)$$
$$= C_{12}\big((u_1 - u_2), S(u_1 - u_2)\big) \geq C_{12}\, C_7 \| u_1 - u_2 \,|_{H_1} \|^2$$

(7_2)

$$(8_2) \qquad \| B(u) |\, H \| \equiv \|([N'(0)]^{-1})^* Su |\, H \| \leq C_{11} \|Su|\, H \| \leq C_{11}\, C_8 \| u|_{H_1} \|^2 .$$

Thus, the operators N and B satisfy the assumptions of the theorem 2.

Let's note, that inequality (23) was cosidered in [4]. It turned out to be usable for extanding the direct variational method to nonlinear equations with nonpotential operators.

Remark 3. If B is symmetric and positive in H, we are able to define Hilbert space H_B as a completation of $D(B)$ in metric

$$(24) \qquad \| u \,|_{H_B} \| = [u,u]^{1/2}$$

$$(25) \qquad [u,v] = (u,Bv) .$$

In view of (24), (25) we can rewrite (7), (8) as

$$(7_3) \quad [N(u_1) - N(u_2), u_1 - u_2] \geq C_3 \| u_1 - u_2 \,|_{H_1} \|^2 \quad \forall u_1 \in D(N),\ u_2 \in D(N)$$

and

$$(8_3) \qquad [Bu,u] \le C_4 \| u |_{H_1} \|^2 \quad \forall u \in D(N)$$

Example: Let's justify LSM for the following nonlinear parabolic problem:

$$(26) \qquad N(u) \equiv \frac{\partial u}{\partial t} - \sum_{i=1}^{n-1} \frac{\partial^2 u}{\partial x_i^2} - \frac{\partial^2 (x,t,u)}{\partial x_n^2} = g(x,t), \ (x,t) \in \Omega,$$

where the bounded domain

$$\Omega = \{(x,t) \equiv (x_1,\dots,x_n,t): x_i \in (a_i,b_i),\ i=1,\dots,n,\ t \in (0,T)\}.$$

$$(27) \qquad u(x,t) = 0 \text{ as } t = 0, \ x_i \in (a_i,b_i),\ i=1,\dots,n,$$

$$(28) \qquad u(x,t) = 0 \text{ as } x_j = b_j,\ x_i \in (a_i,b_i),\ i=1,\dots,n,\ i \ne j, \quad t \in (0,T).$$

$$(29) \qquad \frac{\partial u(x,t)}{\partial n} = 0 \text{ as } x_j = a_j,\ x_i \in (a_i,b_i),\ i=1,\dots,n,\ i \ne j, \quad t \in (0,T).$$

$$(30) \quad g(x,t) \in L_2(\Omega); \ (\xi,\eta,\zeta) \in C^2(R^3); \ \exists \alpha \equiv \frac{\partial}{\partial u} \ge \alpha^2 > 0, \ \forall u \in D(N).$$

Here $D(N) = \{u(x,t) \mid u(x,t) \in C^{2,1}_{x,t}(\bar{\Omega})$ satisfying (27), (29)$\}$. We define the auxiliary operator B by the equality

$$(31) \quad Bh(x,t) = -\int_{b_n}^{x_n} d\xi \int_{a_n}^{\xi} h(x_1,x_2,\dots,x_{n-1},\theta,t)d\theta, \qquad D(B) = D(T).$$

Remark 4. It is easy to see, that operator B, defined by (31) is symmetric and positive in $L_2(\Omega)$. Really, integrating by parts we obtain

$$(32) \quad \begin{aligned} (Bu,v)_{L_2(\Omega)} &\equiv \int_\Omega \left(-\int_{b_n}^{x_n} d\xi \int_{a_n}^{\xi} u(x_1,x_2,\dots,x_{n-1},\theta,t)d\theta \right) v\, d\Omega = \\ &= \int_\Omega \left\{ \int_{a_n}^{x_n} u(x_1,x_2,\dots,x_{n-1},\theta,t)d\theta, \int_{a_n}^{x_n} v(x_1,x_2,\dots,x_{n-1},\theta,t)d\theta, \right\} d\Omega. \end{aligned}$$

That equality makes the assertion trivial.

Lemma 2. *If H_1 is the completion of $D(N)$ in metric*

$$\| u |_{H_1} \| = [u,u]^{1/2} \tag{33}$$

$$[u,v] = \int_\Omega \{\alpha^2 uv + \sum_{i=1}^{n-1} [\int_{a_n}^{x_n} \frac{\partial u}{\partial x_i}(x_1, x_2, \ldots, x_{n-1}, \theta, t) d\theta, \int_{a_n}^{x_n} \frac{\partial v}{\partial x_i}(x_1, x_2, \ldots, x_{n-1}, \theta, t)] d\theta \} d\Omega \tag{34}$$

and (30) is satisfied, then the conditions of the theorem 2 are held for the operators N (26) - (30) and B (31).

Proof. Integrating by parts on *D(N)*, we obtain, by (30)

$$(N'_u h, Bh) \equiv \int_\Omega \left\{ \frac{\partial h}{\partial t} - \sum_{i=1}^{n-1} \frac{\partial^2 h}{\partial x_i^2} - \frac{\partial^2}{\partial x_n^2} (\frac{\partial}{\partial u} h) \right\},$$

$$, \left\{ -\int_{b_n}^{x_n} d\xi \int_{a_n}^{\xi} h(x_1, x_2, \ldots, x_{n-1}, \theta, t) d\theta \right\} d\Omega =$$

$$= \int_\Omega \{ \int_{a_n}^{x_n} h(x_1, \ldots, x_{n-1}, \theta, t)\theta, \int_{a_n}^{x_n} \frac{\partial h}{\partial t} (x_1, \ldots, x_{n-1}, \theta, t)\theta +$$

$$+ \frac{\partial}{\partial u} h^2 + \sum_{i=1}^{n-1} [\int_{a_n}^{x_n} \frac{\partial h}{\partial x_i} (x_1, x_2, \ldots, x_{n-1}, \theta, t) d\theta]^2 \} d\Omega \geq \| h |_{H_1} \|^2 .$$

According to the theorem 3 (7) is satisfied. The validity of (8) can be established by direct computation: for all $h \in D(N)$

$$\| Bh | H^2 = \int_\Omega \left(-\int_{b_n}^{x_n} d\xi \int_{a_n}^{\xi} h(x_1, x_2, \ldots, x_{n-1}, \theta, t) d\theta \right)^2 d\Omega \leq \frac{(b_n - a_n)^2}{a^2} \| h |_{H_1} \|^2$$

The theorems prooved above make it possible to formulate

Corollary 2. *If* $N(0) = 0$; $\overline{R(N)} = H$; *basis* $\{\varphi_n\}$ *is orthonormal in* H_1, *then the following assertions are true.*

a) *For any* $g(x,t) \in L_2(\Omega)$ *there exists the unique element* $u_0 \in H_1$ and sequence $\{u_n\}$, $u_n \in D(N)$ *such that, as* $n \to \infty$,

$$\|u_n - u_0 |_{H_1}\| \to 0 ; \quad \|N(u_n) - g |_{L_2}(\Omega)\| \to 0.$$

b) *The sequence* $\{u_n\}$; $u_n = \sum_{k=1}^{n} a_k \varphi_k$; *can be found from the sufficient condition of LSM functional's minimum:*

$$F(a_1,...,a_n) \equiv \| N(u_n) - g |_{L_2}(\Omega)\| \to \min , \tag{35}$$

leading to the system of algebraic equations

$$\int_\Omega \{N'(u_n)\varphi_k [N(u_n) - g]\} d\Omega = 0, k = 1,\ldots,n. \tag{36}$$

c) The system of nonlinear algebraic equations (36) is solvable for all $g(x,t) \in L(\Omega)$, *for any* $n=1,2,\ldots$.

Remark 5. The Least Square Method for linear parabolic operators has been developed by R. S. Anderssen [2] and A. Carasso [3].

The symmetrizing operators looking like (31) were used in [4] for the investigation of linear parabolic problems. Results based upon the variational theory has been developed by Martynyuk A., Petryshyn W.V. and Shalov V. M. [5]. For the complete list of references see [4].

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