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APPROXIMATING SQUARE ROOTS

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Abstract

For a positive non square integer n, we give an algorithm for approximating $\sqrt{n}$ by rationals. As input we take such square root approximated to one decimal place, and from there on, we construct better and better approximations using only elementary operations of integers. Our methods are based on basic properties of the dynamics of polynomials.

1. Basic Facts of Complex Dynamics used.

We use only elementary facts about the Mandelbrot set. For a complex number c to belong to the Mandelbrot set, simply means that the iterates of $z=0$, under the iteration $z \to z^2 + c$, remaind bounded. That is, the set

$$\{0,\ c,\ c^2+c,\ (c^2+c)^2+c,\ \ldots\}$$

is bounded. In particular, any parameter in the real segment [-3/4,1/4] has this property. In fact, this segment is just the intersection of the main component of the Mandelbrot Set with the real axis. On the other side, if $x > 1/4$ is real, then x does not belong to the Mandelbrot Set (compare [D]).

We write this in the form to be used later.

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Lemma 1. *Let* $I_c = \{0, c, c^2 + c, (c^2 + c)^2 + c, \dots\}$ *denote the set of iterates of* 0 *under* $z \to z^2 + c$.

a) *If* $c \in [-3/4, 1/4]$, *then* I_c *is bounded.*
b) *If* $c > 1/4$ *is real, then* I_c *is unbounded. Furthermore, every sequence of different elements converges to* ∞.

2. Approximating Square Roots.

Now let n be a non square positive integer. We denote by α the unique positive integer such that $\alpha - \sqrt{n} \in [-3/4, 1/4]$. Then the algebraic conjugate of $\alpha - \sqrt{n}$ is $\alpha + \sqrt{n}$, which is clearly bigger than 1/4. Both numbers satisfy the equation

$$c^2 = 2\,\alpha\, c + n - \alpha^2 \qquad (1)$$

with integer coeficients, and roots $c(1) = \alpha - \sqrt{n}$ and $c(2) = \alpha + \sqrt{n}$.

Now it is clear that in order to find a sequence of rational numbers converging to $\sqrt{n}$, it is enough to find such sequence converging to $c(1)$.

From now on, we denote by c_k the k-th iterate of 0 under $z \to z^2 + c$; in this way $c_0 = 0$, $c_1 = c$, ..., $c_{k+1} = c_k^2 + c$.

Proposition 1. *The sequence* $\{ c(1)_k \}$ *is bounded. On the other hand* $c(2)_k \to \infty$.

Remark. In fact, $| c(1)_k | \leq 2$ (compare [D]).

Proof. This follows inmediately from Lemma 1 and the definition of $c(i)$. □

We note now that the iterates $c(i)_k$ can be expresed in the form

$$c(i)_k = a_k\, c(i) + b_k$$

with a_k, b_k integers and independent of i. In fact, as $c(i)^2 = 2\alpha\, c(i) + n - \alpha^2$; we can prove this by induction.

For $k = 0$ we have $c(i)_0 = 0$ and we should take $a_0 = b_0 = 0$. Next,

$$\begin{aligned} c(i)_{k+1} &= c(i)_k^{\,2}+c(i) \\ &= a_k^{\,2}\, c(i)^2 + 2a_k\, b_k\, c(i) + b_k^{\,2} + c(i) \\ &= a_k^{\,2}\,(2\,\alpha\, c(i) + n - \alpha^2)+ 2\, a_k\, b_k\, c(i) + b_k^{\,2} + c(\mathrm{i}). \end{aligned}$$

From which follows that

$$(2) \qquad \begin{aligned} a_{k+1} &= 2\,\alpha\, a_k^{\,2} + 2\, a_k\, b_k + 1 \\ b_{k+1} &= (n- \alpha^2)\, a_k^{\,2} + b_k^{\,2} \end{aligned}$$

proving the assertion.

Lemma 2. $|a_k|$, $|b_k| \to \infty$.

Proof. As a_k and b_k satisfy $c(i)_k= a_k\, c(i) + b_k$, and $\{c(i)_k\}$ is bounded (Proposition 1), we have that a_k and b_k are simultaneously bounded or unbounded. But if a_k , b_k (or any subsequence) were bounded, then $c(2)_k = a_k\, c(2) + b_k$ will also be bounded. But this is imposible by Proposition 1. □

Theorem. $b_k / a_k \to - c(1)$.

Proof. As $c(1)_k = a_k\, c(1) + b_k$ is bounded, the result follows dividing by a_k and using Lemma 2. □

Remark. In fact, it follows from the proof of the Theorem and the remark following Proposition 1, that

$$| c(1) + b_k / a_k \,| \le \,| 2 / a_k \,| .$$

3. Example.

We have that $1-\sqrt{2} \in [-0.75,0.25]$. So that we take $\alpha = 1$ and $n = 2$. The first few terms in the expression (2) become then, $a_0 = 0$, $b_0 = 0$; $a_1 = 1$, $b_1=0$; $a_2 = 3$, $b_2 = 1$; from there on, we use the recursive formulas

$$\begin{aligned} a_{k+1} &= 2\, a_k^{\,2}+ 2_k\; b_k +1 \\ b_{k+1} &= a_k^{\,2} + b_k^{\,2} \end{aligned}$$

From which we get $a_3 = 25$, $b_3 = 10$; $a_4 = 1751$, $b_4 = 725$; $a_5 = 8670953$, $b_5=3591626$. We leave for the reader the task of finding (for example) a_{12} and b_{12}, and verifying that $\sqrt{2}$ -1 and b_{12}/a_{12} agree up to 945 decimal places.

4. Final Remarks

If we look carefully at our methods, we see that the methods used, also extend to the case where c(1) and c(2) are (real) solutions of a monic equation with real coeficientes. The only requirement is really that c(1) belongs to the closed interval from -2 to 1/4 (which is the intersection of the Mandelbrot set with real axis), while c(2) to the complement of this interval. However, this gain of generality takes us out of the realm of elementary arithmetic. This can be recovered by considering c(1), c(2) $= m \pm k\sqrt{n}$. With values satisfying the requirements of the paragraph above. (And the pertinent modifications to equation (2)).

References

[D] Devaney, R; *An introduction to chaotic dynamical systems.* Addison Wesley 1989.

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