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ON CERTAIN FRACTIONAL INTEGRAL OPERATORS OF TWO VARIABLES AND INTEGRAL TRANSFORMS

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Abstract

The present paper is in continuation to authors earlier paper [9] where two variable analogues of certain fractional integral operators of M. Saigo were investigated. This paper deals with the effect of operating two variable analogues of Mellin and Laplace transforms on these two variable analogues of fractional integral operators of the earlier paper.

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Introduction

In 1978, M. Saigo [15] defined certain integral operator involving the Gauss hypergeometric function as follows:

Let $\alpha > \beta$ and η be real numbers. The fractional integral operator $I_x^{\alpha,\beta,\eta}$, which acts on certain functions $f(x)$ on the interval $(0,\infty)$ was defined as

$$I_x^{\alpha,\beta,\eta}\ f = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} F(\alpha+\beta,-\eta;1-\frac{t}{x}) f(t)\ dt \qquad (1.1)$$

Under the same assumptions in defining (1.1), he also defined the integral operator $J_x^{\alpha,\beta,\eta}$ as

$$J_x^{\alpha,\beta,\eta} f = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta} F\left(\alpha+\beta,-\eta;\alpha;1-\frac{x}{t}\right) f(t) dt. (1.2)$$

Later on in 1988, Saigo and Raina [17] obtained the generalized fractional integrals and derivatives introduced by Saigo [15]-[16] of the system $S_q^n(x)$, where the general system of polynomials.

$$S_q^n(x) = \sum_{r=0}^{[n/q]} \frac{(-n)_{qr}}{r!}\ A_{n,r}\ x^r$$

were defined by Srivastava [18], where $q > 0$ and $n \geq 0$ are integers, and $A_{n,r}$ are arbitrary sequence of real or complex numbers.

In an earlier communication the present authors [9] defined and studied certain two variables analogues of (1.1) and (1.2) which are as given below:

I. Let $c > 0$, a, b, b' be real numbers. A two variable analogue of fractional integral operator $I_{0,x}^{\alpha,\beta,\eta}$ due to M. Saigo i.e. of (1.1) is defined as

$$\begin{aligned} {}_1I_{0,x;0,y}^{a,b,b';c} f(x,y) &= \frac{x^{-a}y^{-a}}{\{\Gamma(c)\}^2} \int_0^x \int_0^y (x-u)^{c-1}(y-v)^{c-1} \times \\ & F_1\left[\begin{array}{c} a,b,b';1-\frac{u}{x},1-\frac{v}{y} \\ c; \end{array}\right] f(u,v) dv du. \end{aligned} \qquad (1.3)$$

SPECIAL CASES:

(i) For $a = b = b' = 0, c = \alpha$, (1.3) reduces to

$$
\begin{aligned}
{}_1I^{0,0,0;\alpha}_{0,x;0,y}\ f(x,y) &= {}_1R^{\alpha}_{0,x;0,y}\ f(x,y) \\
&= \frac{1}{\{\Gamma(\alpha)\}^2}\int_0^x\int_0^y (x-u)^{\alpha-1}\times \\
&\quad (y-v)^{\alpha-1}\ f(u,v)\ dvdu. \qquad (1.4)
\end{aligned}
$$

Here (1.1) may be considered as a two variable analogue of Riemann - Liouville fractional integral operator $R^{\alpha}_{0,x}$.

(ii) For $a = c = \alpha,\ b = -\eta,\ b' = 0$, (1.3) becomes

$$
\begin{aligned}
{}_1I^{\alpha,-\eta,0;\alpha}_{0,\,x;\,0,\,y}\ f(x,y) &= {}^x_1E^{\alpha,\eta}_{0,x;0,y}\ f(x,y) \\
&= \frac{x^{-\alpha-\eta}y^{-\alpha}}{\{\Gamma(\alpha)\}^2}\int_0^x\int_0^y (x-u)^{\alpha-1}\times \\
&\quad (y-v)^{\alpha-1}u^{\eta}f(u,v)dvdu. \qquad (1.5)
\end{aligned}
$$

(iii) For $a = c = \alpha,\ b = 0,\ b' = -\eta$, (1.3) gives

$$
\begin{aligned}
{}_1I^{\alpha,0,-\eta;\alpha}_{0,\,x;\,0,\,y}\ f(x,y) &= {}^y_1E^{\alpha,\eta}_{0,x;0,y}\ f(x,y) \\
&= \frac{x^{-\alpha}y^{-\alpha-\eta}}{\{\Gamma(\alpha)\}^2}\int_0^x\int_0^y (x-u)^{\alpha-1}\times \\
&\quad (y-v)^{\alpha-1}v^{\eta}f(u,v)dvdu. \qquad (1.6)
\end{aligned}
$$

Here (1.5) and (1.6) may be considered as two-variable analogues of Erdélyi-Kober fractional integral operator $E^{\alpha,\eta}_{0,\,x}$.

Under the same conditions of (1.3), a two variable analogue of another fractional integral operator $J^{\alpha,\beta,\eta}_{x,\infty}$ due to M. Saigo i.e. of (1.2) is defined as follows:

$$
\begin{aligned}
{}_1J^{a,b,b';c}_{x,\infty;y,\infty}\ f(x,y) &= \frac{1}{\{\Gamma(c)\}^2}\times \\
&\quad \int_x^{\infty}\int_y^{\infty}(u-x)^{c-1}(v-y)^{c-1}\times \\
&\quad F_1\left[\begin{matrix} a,b,b';1-\frac{x}{u},1-\frac{y}{v} \\ c; \end{matrix}\right]\times \\
&\quad u^{-a}v^{-a}f(u,v)dvdu. \qquad (1.7)
\end{aligned}
$$

SPECIAL CASES:

(i) For $a = b = b' = 0,\ c = \alpha$, (1.7) reduces to

$$\begin{aligned} {}_1J^{0,0,0;\alpha}_{x,\infty;y,\infty} f(x,y) &= {}_1L^{\alpha}_{x,\infty;y,\infty} f(x,y) \\ &= \frac{1}{\{\Gamma(\alpha)\}^2} \int_x^\infty \int_y^\infty (u-x)^{\alpha-1} \times \\ &\quad (v-y)^{\alpha-1} f(u,v)dvdu. \qquad (1.8) \end{aligned}$$

It can be considered as a two variable analogue of Weyl fractional integral operator $L^{\alpha}_{x,\infty}$.

(ii) For $a = c = \alpha,\ b = -\eta,\ b' = 0$, (1.7) becomes

$$\begin{aligned} {}_1J^{\alpha,-\eta,0;\alpha}_{x,\infty;y,\infty} f(x,y) &= {}^x_1K^{\alpha,\eta}_{x,\infty;y,\infty} f(x,y) \\ &= \frac{x^\eta}{\{\Gamma(\alpha)\}^2} \int_x^\infty \int_y^\infty (u-x)^{\alpha-1} \times \\ &\quad (v-y)^{\alpha-1} u^{-\alpha-\eta} v^{-\alpha} f(u,v)dvdu. \qquad (1.9) \end{aligned}$$

(iii) For $a = c = \alpha,\ b = 0,\ b' = -\eta$ (1.7) gives

$$\begin{aligned} {}_1J^{\alpha,0,-\eta;\alpha}_{x,\infty;y,\infty} f(x,y) &= {}^y_1K^{\alpha,\eta}_{x,\infty;y,\infty} f(x,y) \\ &= \frac{y^\eta}{\{\Gamma(\alpha)\}^2} \int_x^\infty \int_y^\infty (u-x)^{\alpha-1} \times \\ &\quad (v-y)^{\alpha-1} u^{-\alpha} v^{-\alpha-\eta} f(u,v)dvdu. \qquad (1.10) \end{aligned}$$

Here (1.9) and (1.10) may be considered as two variable analogues of Erdélyi-Kober fractional integral operator $K^{\alpha,\eta}_{x,\infty}$.

II. Let $c > 0,\ c' > 0,\ a,\ b,\ b'$ be real numbers. Then a second two variable analogue of $I^{\alpha,\beta,\eta}_{0,x}$ is as given below:

$$\begin{aligned} {}_2I^{a,b,b';c,c'}_{0,x;0,y} f(x,y) &= \frac{x^{-a}y^{-a}}{\Gamma(c)\Gamma(c')} \int_0^x \int_0^y (x-u)^{c-1}(y-v)^{c'-1} \times \\ &\quad F_2\left[\begin{matrix} a,b,b';1-\frac{u}{x},1-\frac{v}{y} \\ c,c'; \end{matrix}\right] f(u,v)dvdu. \qquad (1.11) \end{aligned}$$

SPECIAL CASES:

(i) For $a = b = b' = 0,\ c = \alpha,\ c' = \alpha'$, (1.11) reduces to

$$
\begin{aligned}
{}_2I^{0,0,0;\alpha,\alpha'}_{0,x;0,y} f(x,y) &= {}_2R^{\alpha,\alpha'}_{0,x;0,y} f(x,y) \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\alpha')} \int_0^x \int_0^y (x-u)^{\alpha-1} \times \\
&\quad (y-v)^{\alpha'-1} f(u,v)\,dv du. \qquad (1.12)
\end{aligned}
$$

Here (1.12) may be taken as second two variable analogues of Riemann-Liouville fractional integral operator $R^{\alpha}_{0,x}$. For $\alpha' = \alpha$, (1.12) reduces to (1.4).

(ii) For $a = c = \alpha,\ b = -\eta,\ b' = 0,\ c' = \alpha'$, (1.11) becomes

$$
\begin{aligned}
{}_2I^{\alpha,-\eta,0;\alpha,\alpha'}_{0,x;0,y} f(x,y) &= {}^x_2E^{\alpha,\alpha',\eta}_{0,x;0,y} f(x,y) \\
&= \frac{x^{-\alpha-\eta} y^{-\alpha}}{\Gamma(\alpha)\Gamma(\alpha')} \int_0^x \int_0^y (x-u)^{\alpha-1} \times \\
&\quad (y-v)^{\alpha'-1} u^{\eta} f(u,v)\,dv du. \qquad (1.13)
\end{aligned}
$$

For $\alpha' = \alpha$, (1.13) reduces to (1.5).

(iii) For $a = c = \alpha,\ b = 0,\ b' = -\eta,\ c' = \alpha$, (1.11) gives

$$
\begin{aligned}
{}_2I^{\alpha,0,-\eta;\alpha,\alpha'}_{0,x;0,y} f(x,y) &= {}^y_2E^{\alpha,\alpha',\eta}_{0,x;0,y} f(x,y) \\
&= \frac{x^{-\alpha} y^{-\alpha-\eta}}{\Gamma(\alpha)\Gamma(\alpha')} \int_0^x \int_0^y (x-u)^{\alpha-1} \times \\
&\quad (y-v)^{\alpha'-1} v^{\eta} f(u,v)\,dv du. \qquad (1.14)
\end{aligned}
$$

For $\alpha' = \alpha$, (1.14) reduces to (1.6).

Here (1.13) and (1.14) may be taken as second two variable analogues of Erdélyi-Kober fractional integral operator $E^{\alpha,\eta}_{0,x}$.

Under the same conditions of (1.11), a second two variable analogue of $J_{x,\infty}^{\alpha,\beta,\eta}$ is as defined below:

$$
\begin{aligned}
{}_2J_{x,\infty;y,\infty}^{a,b,b';c,c'}\ f(x,y) \quad = \quad & \frac{1}{\Gamma(c)\Gamma(c')}\int_x^\infty\int_y^\infty (u-x)^{c-1}\times \\
& (v-y)^{c'-1}F_2\left[\begin{array}{c} a,b,b';1-\frac{x}{u},1-\frac{y}{v} \\ c,c'; \end{array}\right]\times \\
& u^{-a}v^{-a}f(u,v)\ dvdu. \qquad (1.15)
\end{aligned}
$$

SPECIAL CASES:

(i) For $a=b=b'=0,\ c=\alpha,\ c'=\alpha'$, (1.15) reduces to

$$
\begin{aligned}
{}_2J_{x,\infty;y,\infty}^{0,0,0;\alpha,\alpha'}f(x,y) \quad &= \quad {}_2L_{x,\infty;y,\infty}^{\alpha,\alpha'}\ f(x,y) \\
&= \quad \frac{1}{\Gamma(\alpha)\Gamma(\alpha')}\int_x^\infty\int_y^\infty (u-x)^{\alpha-1}\times \\
& \qquad (v-y)^{\alpha'-1}\ f(u,v)\ dvdu. \qquad (1.16)
\end{aligned}
$$

We may consider (1.16) as second two variable analogues of Weyl fractional integral operator $L_{x,\infty}^{\alpha}$. For $\alpha'=\alpha$, (1.16) reduces to (1.8).

(ii) For $a=c=\alpha,\ b=-\eta,\ b'=0, c'=\alpha'$, (1.15) becomes

$$
\begin{aligned}
{}_2J_{x,\infty;y,\infty}^{\alpha,-\eta,0;\alpha,\alpha'}f(x,y) \quad &= \quad {}_2^{x}K_{x,\infty;y,\infty}^{\alpha,\alpha',\eta}\ f(x,y) \\
&= \quad \frac{x^\eta}{\Gamma(\alpha)\Gamma(\alpha')}\int_x^\infty\int_y^\infty (u-x)^{\alpha-1}\times \\
& \qquad (v-y)^{\alpha'-1}u^{-\alpha-\eta}v^{-\alpha}\ f(u,v)\ dv\,du. \\
& \qquad (1.17)
\end{aligned}
$$

For $\alpha'=\alpha$, (1.17) reduces to (1.9).

(iii) For $a=c=\alpha,\ b=0,\ b'=-\eta,\ c'=\alpha'$, (1.15) gives

$$
\begin{aligned}
{}_2J_{x,\infty;y,\infty}^{\alpha,0,-\eta;\alpha,\alpha'}f(x,y) \quad &= \quad {}_2^{y}K_{x,\infty;y,\infty}^{\alpha,\alpha',\eta}\ f(x,y) \\
&= \quad \frac{y^\eta}{\Gamma(\alpha)\Gamma(\alpha')}\int_x^\infty\int_y^\infty (u-x)^{\alpha-1}\times \\
& \qquad (v-y)^{\alpha'-1}u^{-\alpha}v^{-\alpha-\eta}\ f(u,v)\ dvdu. \\
& \qquad (1.18)
\end{aligned}
$$

For $\alpha'=\alpha$, (1.18) reduces to (1.10).

We may consider (1.17) and (1.18) as second two variable analogues of Erdélyi-Kober fractional integral operator $K_{x,\infty}^{\eta,\alpha}$.

III. Let $c > 0, a,\ a',\ b,\ b'$ be real numbers. Then a third two variable analogue of $I_{0,x}^{\alpha,\beta,\eta}$ is as follows:

$$
{}_3I_{0,x;0,y}^{a,a',b,b';c}\ f(x,y)
$$

$$
= \frac{x^{-a}y^{-a'}}{\{\Gamma(c)\}^2}\int_0^x\int_0^y (x-u)^{c-1}(y-v)^{c'-1}\times
$$

$$
F_3\left[\begin{array}{c} a,a',b,b';1-\frac{u}{x},1-\frac{v}{y} \\ c\ ; \end{array}\right] f(u,v)\,dvdu. \tag{1.19}
$$

Special Cases:

(i) For $a = a' = 0,\ c = \alpha$, (1.19) reduces to

$$
{}_3I_{0,x;0,y}^{0,0,b,b';\alpha}\ f(x,y) = {}_1R_{0,x;0,y}^{\alpha}\ f(x,y)
$$

$$
= \frac{1}{\{\Gamma(\alpha)\}^2}\int_0^x\int_0^y (x-u)^{\alpha-1}(y-v)^{\alpha-1}\ f(u,v)\ dv\,du. \tag{1.20}
$$

which is (1.4).

(ii) For $a = c = \alpha,\ a' = 0,\ b = -\eta$, (1.19) becomes

$$
{}_3I_{0,x;0,y}^{\alpha,0,-\eta,b';\alpha} f(x,y) = {}_3^xE_{0,x;0,y}^{\alpha,\eta} f(x,y)
$$

$$
= \frac{x^{-\alpha-\eta}}{\{\Gamma(\alpha)\}^2}\int_0^x\int_0^y (x-u)^{\alpha-1}(y-v)^{\alpha-1}u^{\eta} f(u,v)\,dv\,du. \tag{1.21}
$$

(iii) For $a = 0,\ a' = c = \alpha,\ b' = -\eta$, (1.19) gives

$$
{}_3I_{0,x;0,y}^{0,\alpha,b,-\eta;\alpha} f(x,y) = {}_3^yE_{0,x;0,y}^{\alpha,\eta}\ f(x,y)
$$

$$
= \frac{y^{-\alpha-\eta}}{\{\Gamma(\alpha)\}^2}\int_0^x\int_0^y (x-u)^{\alpha-1}(y-v)^{\alpha-1}v^{\eta}\ f(u,v)dv\,du. \tag{1.22}
$$

Here (1.21) and (1.22) may be thought of as the third two variable analogues of Erdélyi-Kober fractional integral operator $E_{0,x}^{\alpha,\eta}$.

Under the same conditions of (1.19), a third two variable analogue of $J_{x,\infty}^{\alpha,\beta,\eta}$ is as defined below:

$$
\begin{aligned}
&{}_3J_{x,\infty;y,\infty}^{a,a',b,b';c} f(x,y) \\
&= \frac{1}{\{\Gamma(c)\}^2} \int_x^\infty \int_y^\infty (u-x)^{c-1}(v-y)^{c-1} \times \\
&\quad F_3\left[\begin{matrix} a, a', b, b'; 1-\frac{x}{u}, 1-\frac{y}{v} \\ c; \end{matrix}\right] u^{-a} v^{-a'} f(u,v)\, dv\, du.
\end{aligned} \tag{1.23}
$$

Special Cases:

(i) For $a = a' = 0$, and $c = \alpha$, (1.23) reduces to

$$
\begin{aligned}
&{}_3J_{x,\infty;y,\infty}^{0,0,b,b';\alpha}\ f(x,y) = {}_1L_{x,\infty;y,\infty}^{\alpha}\ f(x,y) \\
&= \frac{1}{\{\Gamma(\alpha)\}^2} \int_x^\infty \int_y^\infty (u-x)^{\alpha-1}(v-y)^{\alpha-1} \times \\
&\quad f(u,v)\ dv\, du
\end{aligned} \tag{1.24}
$$

which is (1.8).

(ii) For $a' = 0,\ a = c = \alpha,\ b = -\eta$, (1.23) becomes

$$
\begin{aligned}
&{}_3J_{x,\infty;y,\infty}^{\alpha,0,-\eta,b';\alpha}\ f(x,y) = {}_3^{x}K_{x,\infty;y,\infty}^{\alpha,\eta}\ f(x,y) \\
&= \frac{x^\eta}{\{\Gamma(\alpha)\}^2} \int_x^\infty \int_y^\infty (u-x)^{\alpha-1}(v-y)^{\alpha-1} u^{-\alpha-\eta} \times \\
&\quad f(u,v)\ dv\, du.
\end{aligned} \tag{1.25}
$$

(iii) For $a = 0,\ a' = c = \alpha,\ b' = -\eta$, (1.23) gives

$$
{}_3J^{0,\alpha,b,-\eta;\alpha}_{x,\infty;y,\infty}\ f(x,y) = {}^{y}_{3}K^{\alpha,\eta}_{x,\infty;y,\infty}\ f(x,y)
$$

$$
= \frac{y^{\eta}}{\{\Gamma(\alpha)\}^2}\int_x^{\infty}\int_y^{\infty}(u-x)^{\alpha-1}(v-y)^{\alpha-1}v^{-\alpha-\eta}\times \tag{1.26}
$$

$$
f(u,v)\,dv\,du.
$$

Here (1.25) and (1.26) may be taken as the third two variable analogues of Erdélyi-Kober fractional integral operator $K^{\eta,\alpha}_{x,\infty}$.

IV. Let $c > 0,\ c' > 0,\ a,\ b$ be real numbers. Then a fourth two variable analogue of $I^{\alpha,\beta,\eta}_{0,x}$ is as defined below:

$$
{}_4I^{a,b;c,c'}_{0,x;0,y}\ f(x,y) = \frac{x^{-a}y^{-a}}{\Gamma(c)\Gamma(c')}\int_0^x\int_0^y (x-u)^{c-1}(y-v)^{c'-1}\times
$$

$$
F_4\left[\begin{array}{l} a,b;1-\frac{u}{x},1-\frac{v}{y} \\ c,c'; \end{array}\right] f(u,v)dv\,du. \tag{1.27}
$$

Under the same conditions of (1.27), a fourth two variable analogue of $J^{\alpha,\beta,\eta}_{x,\infty}$ is as given below:

$$
{}_4J^{a,b;c,c'}_{x,\infty;y,\infty}\ f(x,y) = \frac{1}{\Gamma(c)\Gamma(c')}\int_x^{\infty}\int_y^{\infty}(u-x)^{c-1}\times
$$

$$
(v-y)^{c'-1}F_4\left[\begin{array}{l} a,b;1-\frac{x}{u},1-\frac{y}{v} \\ c,c'; \end{array}\right] u^{-a}v^{-a}\times
$$

$$
f(u,v)\,dv\,du. \tag{1.28}
$$

The aim of the present paper is to study the effects of integral transforms say the Mellin and Laplace transforms on the two variable analogues of fractional integral operators introduced in [9] and reproduced here through (1.3), (1.7), (1.11), (1.15), (1.19), (1.23), (1.27) and (1.28).

We shall also need here the definitions of certain generalizations of Appell's functions studied by the present authors in a separate communication [8].

For convenience we reproduce the definitions of only those functions of [8] which we need in this paper. Two such functions M_1 and M_4 needed in this paper are defined as

$$M_1(a,a',b,b',c,c',d,e,e';x,y)$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_m(a')_n(b)_m(b')_n(c)_m(c')_n}{(d)_{m+n}(e)_m(e')_n} \frac{x^m}{m!}\frac{y^n}{n!}, \tag{1.29}$$

$$Max\{|x|,|y|\} < 1;$$

$$M_4(a,b,b',c,c',d,e,e';x,y)$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n(c)_m(c')_n}{(d)_{m+n}(e)_m(e')_n} \frac{x^m}{m!}\frac{y^n}{n!}, \tag{1.30}$$

$$Max\{|x|,|y|\} < 1.$$

We also need the results of the following theorems of [9] in this paper:

Theorem 1.1. *For functions of two variables $f(x,y)$ and $g(x,y)$ defined in the positive quadrant of the xy-plane and $c > 0$, we have*

$$\begin{aligned}&\int_0^\infty\int_0^\infty f(x,y)\,{}_1I_{0,x;0,y}^{a,b,b';c}\,g(x,y)\,dy\,dx\\ &= \int_0^\infty\int_0^\infty g(x,y)\,{}_1J_{x,\infty;y,\infty}^{a,b,b';c}\,f(x,y)\,dy\,dx\end{aligned} \tag{1.31}$$

provided that each double integral exists.

Theorem 1.2. *For functions of two variables $f(x,y)$ and $g(x,y)$ defined in the positive quadrant of the xy-plane and $c > 0, c' > 0$, we have*

$$\begin{aligned}&\int_0^\infty\int_0^\infty f(x,y)\,{}_2I_{0,x;0,y}^{a,b,b';c,c'}\,g(x,y)\,dy\,dx\\ &= \int_0^\infty\int_0^\infty g(x,y)\,{}_2J_{x,\infty;y,\infty}^{a,b,b';c,c'}\,f(x,y)\,dy\,dx\end{aligned} \tag{1.32}$$

provided that each double integral exists.

Theorem 1.3. *Under the conditions stated in theorem 1.1, we have*

$$\int_0^\infty \int_0^\infty f(x,y)\ {}_3I_{0,x;0,y}^{a,a',b,b';c}\ g(x,y)\ dy\,dx$$
$$= \int_0^\infty \int_0^\infty g(x,y)\ {}_3J_{x,\infty;y,\infty}^{a,a',b,b';c}\ f(x,y)\ dy\,dx \tag{1.33}$$

provided that each double integral exists.

Theorem 1.4. *Under the conditions stated in theorem 1.2, we have*

$$\int_0^\infty \int_0^\infty f(x,y)\ {}_4I_{0,x;0,y}^{a,b;c,c'}\ g(x,y)\ dy\,dx$$
$$= \int_0^\infty \int_0^\infty g(x,y)\ {}_4J_{x,\infty;y,\infty}^{a,b;c,c'}\ f(x,y)\ dy\,dx \tag{1.34}$$

provided that each double integral exists.

The present paper also indicates representation of the results of the following theorems of [9] in terms of double Mellin transforms.

Theorem 1.5. *For functions* $f(x,y), g(x,y), f\left(\frac{1}{x},\frac{1}{y}\right)$ *and* $g\left(\frac{1}{x},\frac{1}{y}\right)$ *defined for* $0 \le x < \infty$, $0 \le y < \infty$ *and* $c > 0$, *we have*

$$\int_0^\infty \int_0^\infty (xy)^{a-c-1} f\left(\frac{1}{x},\frac{1}{y}\right)\ {}_1I_{0,x;0,y}^{a,b,b';c} g(x,y)\ dy\,dx$$
$$= \int_0^\infty \int_0^\infty (xy)^{a-c-1} g\left(\frac{1}{x},\frac{1}{y}\right)\ {}_1I_{0,x;0,y}^{a,b,b';c} f(x,y)\ dy\,dx \tag{1.35}$$

provided that each double integral exists.

Theorem 1.6. *Under the conditions stated in theorem 1.5, we have*

$$\int_0^\infty \int_0^\infty (xy)^{a-c-1}\ f\left(\frac{1}{x},\frac{1}{y}\right)\ {}_1J_{x,\infty;y,\infty}^{a,b,b';c}\ g(x,y)\ dy\,dx$$
$$= \int_0^\infty \int_0^\infty (xy)^{a-c-1}\ g\left(\frac{1}{x},\frac{1}{y}\right)\ {}_1J_{x,\infty;y,\infty}^{a,b,b';c}\ f(x,y)\ dy\,dx \tag{1.36}$$

provided that each double integral exists.

Theorem 1.7. *For functions* $f(x,y)$, $g(x,y)$, $f\left(\frac{1}{x},\frac{1}{y}\right)$ *and* $g\left(\frac{1}{x},\frac{1}{y}\right)$ *defined for* $0 \le x < \infty, 0 \le y < \infty$ *and* $c > 0$, $c' > 0$, *we have*

$$\begin{aligned}&\int_0^\infty \int_0^\infty x^{a-c-1} y^{a-c'-1} f\left(\frac{1}{x},\frac{1}{y}\right) \; {}_2I_{0,x;0,y}^{a,b,b';c,c'} g(x,y)\, dy\, dx \\ &= \int_0^\infty \int_0^\infty x^{a-c-1} y^{a-c'-1} g\left(\frac{1}{x},\frac{1}{y}\right) \; {}_2I_{0,x;0,y}^{a,b,b';c,c'} f(x,y)\, dy\, dx\end{aligned} \tag{1.37}$$

provided that each double integral exists.

Theorem 1.8. *Under the conditions stated in theorem 1.7, we have*

$$\begin{aligned}&\int_0^\infty \int_0^\infty x^{a-c-1} y^{a-c'-1} f\left(\frac{1}{x},\frac{1}{y}\right) \; {}_2J_{x,\infty;y,\infty}^{a,b,b';c,c'} g(x,y)\, dy\, dx \\ &= \int_0^\infty \int_0^\infty x^{a-c-1} y^{a-c'-1} \; g\left(\frac{1}{x},\frac{1}{y}\right) \; {}_2J_{x,\infty;y,\infty}^{a,b,b';c,c'} f(x,y)\, dy\, dx\end{aligned} \tag{1.38}$$

provided that each double integral exists.

Theorem 1.9. *For functions* $f(x,y)$, $g(x,y)$, $f\left(\frac{1}{x},\frac{1}{y}\right)$ *and* $g\left(\frac{1}{x},\frac{1}{y}\right)$ *defined for* $0 \le x < \infty$, $0 \le y < \infty$ *and* $c > 0$, *we have*

$$\begin{aligned}&\int_0^\infty \int_0^\infty x^{a-c-1} y^{a'-c-1} f\left(\frac{1}{x},\frac{1}{y}\right) \; {}_3I_{0,x;0,y}^{a,a',b,b';c} \; g(x,y)\, dy\, dx \\ &= \int_0^\infty \int_0^\infty x^{a-c-1} y^{a'-c-1} \; g\left(\frac{1}{x},\frac{1}{y}\right) \; {}_3I_{0,x;0,y}^{a,a',b,b';c} \; f(x,y)\, dy\, dx\end{aligned} \tag{1.39}$$

provided that each double integral exists.

Theorem 1.10. *Under the conditions stated in theorem 1.9, we have*

$$\begin{aligned}&\int_0^\infty \int_0^\infty x^{a-c-1} y^{a'-c-1} f\left(\frac{1}{x},\frac{1}{y}\right) \; {}_3J_{x,\infty;y,\infty}^{a,a',b,b';c} \; g(x,y)\, dy\, dx \\ &= \int_0^\infty \int_0^\infty x^{a-c-1} y^{a'-c-1} g\left(\frac{1}{x},\frac{1}{y}\right) \; {}_3J_{x\infty;y,\infty}^{a,a',b,b';c} \; f(x,y)\, dy\, dx\end{aligned} \tag{1.40}$$

provided that each double integral exists.

Theorem 1.11. *For functions* $f(x,y), g(x,y), f\left(\frac{1}{x},\frac{1}{y}\right)$ *and* $g\left(\frac{1}{x},\frac{1}{y}\right)$ *defined for* $0 \le x < \infty, 0 \le y < \infty$ *and* $c > 0, c' > 0$, *we have*

$$\int_0^\infty \int_0^\infty x^{a-c-1} y^{a-c'-1} f\left(\frac{1}{x},\frac{1}{y}\right) \; {}_4I^{a,b,c,c'}_{0,x;0,y} \; g(x,y)\; dy\, dx$$
$$= \int_0^\infty \int_0^\infty x^{a-c-1} y^{a-c'-1} g\left(\frac{1}{x},\frac{1}{y}\right) \; {}_4I^{a,b;c;c'}_{0x;0,y} \; f(x,y)\; dy\, dx \tag{1.41}$$

provided that each double integral exists.

Theorem 1.12. *Under the conditions stated in theorem 1.11, we have*

$$\int_0^\infty \int_0^\infty x^{a-c-1} y^{a-c'-1} f\left(\frac{1}{x},\frac{1}{y}\right) \; {}_4J^{a,b;c,c'}_{x,\infty;y,\infty} \; g(x,y)\; dy\, dx$$
$$= \int_0^\infty \int_0^\infty x^{a-c-1} y^{a-c'-1} g\left(\frac{1}{x},\frac{1}{y}\right) \; {}_4J^{a,b;c,c'}_{x\infty;y,\infty} \; f(x,y)\; dy\, dx \tag{1.42}$$

provided that each double integral exists.

2 Mellin Transformation

In this section we shall study the effect of operating two variable analogues of Mellin transform on the above defined operators. A two variable analogue of Mellin transform of a function $f(x, y)$ of two variables x and y is defined as follows:

$$M\{f(u,v) : s,t\} = \int_0^\infty \int_0^\infty u^{s-1} v^{t-1} f(u,v)\; dv\, du. \tag{2.1}$$

The effects of operating (2.1), on the operators (1.3), (1.7), (1.11), (1.15), (1.19), (1.23), (1.27) and (1.28) are given in the form of the following theorems:

Theorem 2.1. *For $c > 0, Rl(1+a-c-s) > 0, Rl(1+a-c-t) > 0$, we have*

$$M\left\{ {}_1I_{0,x;0,y}^{a,b,b';c} \ f(x,y) : s,t\right\} = \frac{\Gamma(1+a-c-s)\Gamma(1+a-c-t)}{\Gamma(1+a-s)\Gamma(1+a-t)} \times$$

$$M_4\left[\begin{array}{l} a; \quad b \ \ b'; \ c, c; 1, 1 \\ c; 1+a-s, 1+a-t; \end{array}\right] M\{(xy)^{c-a} f(x,y) : s,t\} \tag{2.2}$$

provided that term by term integration is valid and M_4 is given by (1.30).

Theorem 2.2. *For $c > 0, Rl(s) > 0, Rl(t) > 0$, we have*

$$M\left\{ {}_1J_{x,\infty;y,\infty}^{a,b,b';c} \ f(x,y) : s,t\right\} = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+c)\Gamma(t+c)} \times$$

$$M_4\left[\begin{array}{l} a; \quad b, \ \ b'; c, c; 1, 1 \\ c; s+c, t+c; \end{array}\right] M\{(xy)^{c-a} \ f(x,y) : s,t\} \tag{2.3}$$

provided that term by term integration is valid and M_4 is given by (1.30).

Theorem 2.3. *For $c > 0, c' > 0, Rl(1+a-c-s) > 0, Rl(1+a-c'-t) > 0$, we have*

$$M\left\{ {}_2I_{0,x;0,y}^{a,b,b';c,c'} \ f(x,y) : s,t\right\}$$

$$= \frac{\Gamma(1+a-c-s)\Gamma(1+a-c'-t)\Gamma(1-t-b')}{\Gamma(1+a-s)\Gamma(1+a-t-b')\Gamma(1-t)} \times$$

$${}_3F_2\left[\begin{array}{l} a, \ b \ , \quad t \ ; \ 1 \\ 1+a-s, t+b; \end{array}\right] M\{x^{c-a}y^{c'-a} f(x,y) : s,t\} \tag{2.4}$$

provided that term by term integration is valid.

Theorem 2.4. *For $c > 0, c' > 0$*

$$M\left\{ {}_2J^{a,b,b';c,c'}_{x,\infty;y,\infty}\ f(x,y) : s,t\right\}$$

$$= \frac{\Gamma(s)\Gamma(t)\Gamma(t+c'-b'-a)}{\Gamma(s+c)\Gamma(t+c'-b')\Gamma(t+c'-a)}\times$$

$${}_3F_2\left[\begin{matrix} a, & b, & 1+a-c'-t; & 1 \\ s+c, & 1+a-b'-c'-t; & & \end{matrix}\right] M\{x^{c-a}y^{c'-a}f(x,y) : s,t\}$$
(2.5)

provided that term by term integration is valid.

Theorem 2.5. *For $c > 0, Rl(1+a-c-s) > 0, Rl(1+a'-c-t) > 0$, we have*

$$M\left\{ {}_3I^{a,a',b,b';c}_{0,x;0,y}\ f(x,y) : s,t\right\}$$

$$= \frac{\Gamma(1+a-c-s)\Gamma(1+a'-c-t)}{\Gamma(1+a-s)\Gamma(1+a'-t)}\times$$

$$M_1\left[\begin{matrix} a,a';b,b';c,c;\ 1,1 \\ c;1+a-c,1+a'-t; \end{matrix}\right] M\{x^{c-a}y^{c-a'}f(x,y) : s,t\}$$
(2.6)

provided that term by term integration is valid and M_1 is given by (1.29).

Theorem 2.6. *For $c > 0, Rl(s) > 0, Rl(t) > 0$, we have*

$$M\left\{ {}_3J^{a,a',b,b';c}_{x,\infty;y,\infty}\ f(x,y) : s,t\right\}$$

$$= \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+c)\Gamma(t+c)}\times$$

$$M_1\left[\begin{matrix} a,a';b,b';c,c;1,1 \\ c;\ s+c,\ t+c; \end{matrix}\right] M\{x^{c-a}y^{c-a'}f(x,y) : s,t\}$$
(2.7)

provided that term by term integration is valid.

Theorem 2.7. *For* $c > 0, c' > 0, Rl(1+a-c-s) > 0, Rl(1+a-c'-t) > 0$, *we have*

$$M\left\{ {}_4I^{a,b,c,c'}_{0,x;0,y} f(x,y) : s,t\right\}$$

$$= \frac{\Gamma(1+a-c-s)\Gamma(1+a-c'-t)\Gamma(1-t-b)}{\Gamma(1+a-s)\Gamma(1-t)\Gamma(1+a-b-t)}\times$$

$${}_4F_3\left[\begin{matrix} a,\ b,\ t,\ b+t-a;\ \frac{1}{4} \\ 1+a-s, \frac{1}{2}(b+t), \frac{1}{2}(1+b+t); \end{matrix}\right] M\{x^{c-a}y^{c'-a}f(x,y) : s,t\} \tag{2.8}$$

provided that term by term integration is valid.

Theorem 2.8. *For* $c > 0, c' > 0, Rl(s) > 0, Rl(t) > 0$, *we have*

$$M\left\{ {}_4J^{a,b,c,c'}_{x,\infty;y,\infty} f(x,y)\right\} = \frac{\Gamma(s)\Gamma(t)\Gamma(t+c'-a-b)}{\Gamma(s+c)\Gamma(t+c'-a)\Gamma(t+c'-b)}\times$$

$${}_4F_3\left[\begin{matrix} a, b, 1+a-c'-t, 1+b-c'-t; \frac{1}{4} \\ s+c, \frac{1}{2}(1+a+b-c'-t), \frac{1}{2}(2+a+b-c'-t); \end{matrix}\right]\times$$

$$M\{x^{c-a}y^{c'-a}f(x,y) : s,t\} \tag{2.9}$$

provided that term by term integration is valid.

From the above theorems of this section certain interesting corollaries readily follow giving the effects of operating (2.1) on the operators (1.4), (1.5), (1.6), (1.8), (1.9), (1.10), (1.12), (1.13), (1.14), (1.16), (1.17), (1.18), (1.20), (1.21), (1.22), (1.24), (1.25) and (1.26).

Further, in view of results of theorems 1.1, 1.2, 1.3 and 1.4 we give some more theorems connecting double Mellin transform and the operators (1.3), (1.7), (1.11), (1.15), (1.19), (1.23), (1.27) and (1.28).

Theorem 2.9. *For a function of two variables* $f(x,y)$ *defined in the* *positive quadrant of the xy-plane and $c > 0$, we have*

$$\int_0^\infty \int_0^\infty f(x,y)\, {}_1I_{0,x;0,y}^{a,b,b';c}\{x^{s-1}y^{t-1}\}dy\,dx = M\{\, {}_1J_{x,\infty;y,\infty}^{a,b,b';c}\; f(x,y) : s,t\} \tag{2.10}$$

provided that the double integrals involved exist.

Theorem 2.10. *Under the conditions stated in theorem 2.9, we have*

$$\int_0^\infty \int_0^\infty f(x,y)\; {}_1J_{x,\infty;y,\infty}^{a,b,b';c}\{x^{s-1}y^{t-1}\}dy\,dx = M\{\, {}_1I_{0,x;0,y}^{a,b,b';c}\; f(x,y) : s,t\} \tag{2.11}$$

provided that the double integrals involved exist.

Theorem 2.11. *For a function of two variables $f(x,y)$ defined in the positive quadrant of the xy-plane and $c > 0, c' > 0$, we have*

$$\int_0^\infty \int_0^\infty f(x,y)\, {}_2I_{0,x;0,y}^{a,b,b';c,c'}\{x^{s-1}y^{t-1}\}dy\,dx = M\{\, {}_2J_{x,\infty;y,\infty}^{a,b,b';c,c'}\; f(x,y) : s,t\} \tag{2.12}$$

provided that the double integrals involved exist.

Theorem 2.12. *Under the conditions stated in theorem 2.11, we have*

$$\int_0^\infty \int_0^\infty f(x,y)\; {}_2J_{x,\infty;y,\infty}^{a,b,b';c,c'}\{x^{s-1}y^{t-1}\}dy\,dx = M\{\, {}_2I_{0,x;0,y}^{a,b,b';c,c'}\; f(x,y) : s,t\} \tag{2.13}$$

provided that the double integrals involved exist.

Theorem 2.13. *Under the conditions stated in theorem 2.9, we have*

$$\int_0^\infty \int_0^\infty f(x,y)\ {}_3I_{0,x;0,y}^{a,a',b,b';c}\{x^{s-1}y^{t-1}\}dy\,dx \\ = M\{\ {}_3J_{x\infty;y\infty}^{a,a',b,b';c}\ f(x,y) : s,t\} \tag{2.14}$$

provided that the double integrals involved exist.

Theorem 2.14. *Under the conditions stated in theorem 2.9, we have*

$$\int_0^\infty \int_0^\infty f(x,y)\ {}_3J_{x,\infty;y\infty}^{a,a',b,b';c}\{x^{s-1}y^{t-1}\}dy\,dx \\ = M\{\ {}_3I_{0,x;0,y}^{a,a',b,b';c}\ f(x,y) : s,t\} \tag{2.15}$$

provided that the double integrals involved exist.

Theorem 2.15. *Under the conditions stated in theorem 2.11, we have*

$$\int_0^\infty \int_0^\infty f(x,y)\ {}_4I_{0,x;0,y}^{a,b,c,c'}\{x^{s-1}y^{t-1}\}dy\,dx \\ = M\{\ {}_4J_{x,\infty;y\infty}^{a,b;c,c'}\ f(x,y) : s,t\} \tag{2.16}$$

provided that the double integrals involved exist.

Theorem 2.16. *Under the conditions stated in theorem 2.11, we have*

$$\int_0^\infty \int_0^\infty f(x,y)\ {}_4J_{x,\infty;y\infty}^{a,b,c,c'}\{x^{s-1}y^{t-1}\}dy\,dx \\ = M\{\ {}_4I_{0,x;0,y}^{a,b;c,c'}\ f(x,y) : s,t\} \tag{2.17}$$

provided that the double integrals involved exist.

Theorem 2.17. *For functions of two variables $f(x,y)$ and $g(x,y)$ defined in the positive quadrant of the xy-plane and $c > 0$, we have*

$$M\left[f(x,y)\,{}_1I_{0,x;0,y}^{a,b,b';c}\{x^{s-1}y^{t-1}g(x,y)\} : s,t\right] \\ = M\left[g(x,y)_1\,J_{x,\infty;y,\infty}^{a,b,b';c}\{x^{s-1}y^{t-1}f(x,y)\} : s,t\right]. \tag{2.18}$$

Theorem 2.18. *For functions of two variables $f(x,y)$ and $g(x,y)$ defined in the positive quadrant of the xy-plane and $c>0, c'>0$, we have*

$$M\left[f(x,y)\,{}_2I_{0,x;0,y}^{a,b,b';c,c'}\{x^{s-1}y^{t-1}g(x,y)\}:s,t\right] = M\left[g(x,y)_2J_{x,\infty;y,\infty}^{a,b,b';c,c'}\{x^{s-1}y^{t-1}f(x,y)\}:s,t\right]. \tag{2.19}$$

Theorem 2.19. *Under the conditions stated in theorem 2.17, we have*

$$M\left[f(x,y)\,{}_3I_{0,x;0,y}^{a,a',b,b';c}\{x^{s-1}y^{t-1}g(x,y)\}:s,t\right] = M\left[g(x,y)_3J_{x,\infty;y,\infty}^{a,a',b,b';c}\{x^{s-1}y^{t-1}f(x,y)\}:s,t\right]. \tag{2.20}$$

Theorem 2.20. *Under the conditions stated in theorem 2.18, we have*

$$M\left[f(x,y)\,{}_4I_{0,x;0,y}^{a,b;c,c'}\{x^{s-1}y^{t-1}g(x,y)\}:s,t\right] = M\left[g(x,y)_4J_{x,\infty;y,\infty}^{a,b;c,c'}\{x^{s-1}y^{t-1}f(x,y)\}:s,t\right]. \tag{2.21}$$

It is interesting to not that in terms of double Mellin transforms the results (1.35), (1.36), (1.37), (1.38), (1.39), (1.40), (1.41) and (1.42) respectively can be written as

$$M\left\{f\left(\tfrac{1}{x},\tfrac{1}{y}\right)\,{}_1I_{0,x;0,y}^{a,b,b';c}\,g(x,y):a-c,\,a-c\right\} = M\left\{g\left(\tfrac{1}{x},\tfrac{1}{y}\right)\,{}_1I_{0,x;0,y}^{a,b,b';c}\,f(x,y):a-c,\,a-c\right\}, \tag{2.22}$$

$$M\left\{f\left(\tfrac{1}{x},\tfrac{1}{y}\right)\,{}_1J_{x,\infty;y,\infty}^{a,b,b';c}\,g(x,y):a-c,\,a-c\right\} = M\left\{g\left(\tfrac{1}{x},\tfrac{1}{y}\right)\,{}_1J_{x,\infty;y,\infty}^{a,b,b';c}\,f(x,y):a-c,\,a-c\right\}, \tag{2.23}$$

$$M\left\{f\left(\tfrac{1}{x},\tfrac{1}{y}\right)\,{}_2I_{0,x;0,y}^{a,b,b';c,c'}\,g(x,y):a-c,\,a-c'\right\} = M\left\{g\left(\tfrac{1}{x},\tfrac{1}{y}\right)\,{}_2I_{0,x;0,y}^{a,b,b';c,c'}\,f(x,y):a-c,\,a-c'\right\}, \tag{2.24}$$

$$
\begin{aligned}
M\left\{f\left(\tfrac{1}{x},\tfrac{1}{y}\right)\ {}_2J^{a,b,b';c,c'}_{x,\infty;y,\infty}\ g(x,y) : a-c,\, a-c'\right\}& \\
= M\left\{g\left(\tfrac{1}{x},\tfrac{1}{y}\right)\ {}_2J^{a,b,b';c,c'}_{x,\infty;y,\infty}\ f(x,y) : a-c,\, a-c'\right\},&
\end{aligned} \tag{2.25}
$$

$$
\begin{aligned}
M\left\{f\left(\tfrac{1}{x},\tfrac{1}{y}\right)\ {}_3I^{a,a',b,b';c}_{0,x;0,y}\ g(x,y) : a-c,\, a'-c\right\}& \\
= M\left\{g\left(\tfrac{1}{x},\tfrac{1}{y}\right)\ {}_3I^{a,a',b,b';c}_{0,x;0,y}\ f(x,y) : a-c,\, a'-c\right\},&
\end{aligned} \tag{2.26}
$$

$$
\begin{aligned}
M\left\{f\left(\tfrac{1}{x},\tfrac{1}{y}\right)\ {}_3J^{a,a',b,b';c}_{x,\infty;y,\infty}\ g(x,y) : a-c,\, a'-c\right\}& \\
= M\left\{g\left(\tfrac{1}{x},\tfrac{1}{y}\right)\ {}_3J^{a,a',b,b';c}_{x,\infty;y,\infty}\ f(x,y) : a-c,\, a'-c\right\},&
\end{aligned} \tag{2.27}
$$

$$
\begin{aligned}
M\left\{f\left(\tfrac{1}{x},\tfrac{1}{y}\right)\ {}_4I^{a,b;c,c'}_{0,x;0,y}\ g(x,y) : a-c,\, a-c'\right\}& \\
= M\left\{g\left(\tfrac{1}{x},\tfrac{1}{y}\right)\ {}_4I^{a,b;c,c'}_{0,x;0,y}\ f(x,y) : a-c,\, a-c'\right\},&
\end{aligned} \tag{2.28}
$$

$$
\begin{aligned}
M\left\{f\left(\tfrac{1}{x},\tfrac{1}{y}\right)\ {}_4J^{a,b;c,c'}_{x,\infty;y,\infty}\ g(x,y) : a-c,\, a-c'\right\}& \\
= M\left\{g\left(\tfrac{1}{x},\tfrac{1}{y}\right)\ {}_4J^{a,b;c,c'}_{x,\infty;y,\infty}\ f(x,y) : a-c,\, a-c'\right\}.&
\end{aligned} \tag{2.29}
$$

3 Laplace Transformation

The double Laplace transform of a function of two variables $f(x,y)$ defined in the positive quadrant of the xy-plane is defined by the equation

$$
L\{f(x,y) : s,t\} = \int_0^\infty \int_0^\infty e^{-sx-ty}\ f(x,y)dy\,dx. \tag{3.1}
$$

Making use of results of theorems 1.1, 1.2, 1.3 and 1.4 the relationships of (3.1) with the operators (1.3), (1.7), (1.11), (1.15), (1.19), (1.23), (1.27) and (1.28) are given in the form of the following theorems:

Theorem 3.1. *For a function of two variables $f(x,y)$ defined in the positive quadrant of the xy-plane and $c > 0$, we have*

$$\int_0^\infty \int_0^\infty f(x,y)\, {}_1I_{0,x;0,y}^{a,b,b';c}[e^{-sx-ty}]dy\, dx = L\{\, {}_1J_{x,\infty;y,\infty}^{a,b,b';c}\; f(x,y) : s,t\} \tag{3.2}$$

provided that the double integrals involved exist.

Theorem 3.2. *Under the conditions of theorem 3.1, we have*

$$\int_0^\infty \int_0^\infty f(x,y)\; {}_1J_{x,\infty;y,\infty}^{a,b,b';c}[e^{-sx-ty}]dy\, dx = L\{\, {}_2I_{0,x;0,y}^{a,b,b';c,c'}\; f(x,y) : s,t\} \tag{3.3}$$

provided that the double integrals involved exist.

Theorem 3.3. *For a function of two variables $f(x,y)$ defined in the positive quadrant of the xy-plane and $c > 0, c' > 0$, we have*

$$\int_0^\infty \int_0^\infty f(x,y)\, {}_2I_{0,x;0,y}^{a,b,b';c,c'}[e^{-sx-ty}]dy\, dx = L\{\, {}_2J_{x,\infty;y,\infty}^{a,b,b';c,c'}\; f(x,y) : s,t\} \tag{3.4}$$

provided that the double integrals involved exist.

Theorem 3.4. *Under the conditions of theorem 3.3, we have*

$$\int_0^\infty \int_0^\infty f(x,y)\; {}_2J_{x,\infty;y,\infty}^{a,b,b';c,c'}[e^{-sx-ty}]dy\, dx = L\{\, {}_2I_{0,x;0,y}^{a,b,b';c,c'}\; f(x,y) : s,t\} \tag{3.5}$$

provided that the double integrals involved exist.

Theorem 3.5. *For a function of two variables $f(x,y)$ defined in the positive quadrant of the xy-plane and $c > 0$, we have*

$$\int_0^\infty \int_0^\infty f(x,y)\, {}_3I_{0,x;0,y}^{a,a',b,b';c}[e^{-sx-ty}]dy\,dx$$
$$= L\{\,{}_3J_{x,\infty;y,\infty}^{a,a',b,b';c}\ f(x,y) : s,t\} \tag{3.6}$$

provided that the double integrals involved exist.

Theorem 3.6. *Under the conditions stated of theorem 3.5, we have*

$$\int_0^\infty \int_0^\infty f(x,y)\ {}_3J_{x,\infty;y,\infty}^{a,a',b,b';c}\ [e^{-sx-ty}]dy\,dx$$
$$= L\{\,{}_3I_{0,x;0,y}^{a,a',b,b';c}\ f(x,y) : s,t\} \tag{3.7}$$

provided that the double integrals involved exist.

Theorem 3.7. *For a function of two variables $f(x,y)$ defined in the positive quadrant of the xy-plane and $c > 0, c' > 0$, we have*

$$\int_0^\infty \int_0^\infty f(x,y)\, {}_4I_{0,x;0,y}^{a,b;c,c'}\ [e^{-sx-ty}]dy\,dx$$
$$= L\{\,{}_4J_{x,\infty;y,\infty}^{a,b;c,c'}\ f(x,y) : s,t\} \tag{3.8}$$

provided that the double integrals involved exist.

Theorem 3.8. *Under the conditions stated of theorem 3.7, we have*

$$\int_0^\infty \int_0^\infty f(x,y)\ {}_4J_{x,\infty;y,\infty}^{a,b;c,c'}\ [e^{-sx-ty}]dy\,dx$$
$$= L\{\,{}_4I_{0,x;0,y}^{a,b;c,c'}\ f(x,y) : s,t\} \tag{3.9}$$

provided that the double integrals involved exist.

We further give relationships among double Laplace transform, double Mellin transform and the operators (1.3), (1.7), (1.11), (1.15), (1.19), (1.23), (1.27) and (1.28) in the form of the following theorems:

Theorem 3.9. *For functions of two variables $f(x,y)$ and $g(x,y)$ defined in the positive quadrant of the xy-plane and $c > 0$, we have*

$$M\left[f(x,y)\,{}_1I^{a,b,b';c}_{0,x;0,y}\ \{e^{-sx-ty}g(x,y)\} : s,t\right]$$
$$= L\left[g(x,y)\,{}_1J^{a,b,b';c}_{x,\infty;y,\infty}\ \{x^{s-1}y^{t-1}f(x,y)\} : s,t\right]. \tag{3.10}$$

Theorem 3.10. *For functions of two variables $f(x,y)$ and $g(x,y)$ defined in the positive quadrant of the xy-plane and $c > 0, c' < 0$, we have*

$$M\left[f(x,y)\,{}_2I^{a,b,b';c,c'}_{0,x;0,y}\ \{e^{-sx-ty}g(x,y)\} : s,t\right]$$
$$= L\left[g(x,y)\,{}_2J^{a,b,b';c,c'}_{x,\infty;y,\infty}\ \{x^{s-1}y^{t-1}f(x,y)\} : s,t\right]. \tag{3.11}$$

Theorem 3.11. *Under the conditions stated in theorem 3.9, we have*

$$M\left[f(x,y)\,{}_3I^{a,a',b,b';c}_{0,x;0,y}\ \{e^{-sx-ty}g(x,y)\} : s,t\right]$$
$$= L\left[g(x,y)\,{}_3J^{a,a',b,b';c}_{x,\infty;y,\infty}\ \{x^{s-1}y^{t-1}f(x,y)\} : s,t\right]. \tag{3.12}$$

Theorem 3.12. *Under the conditions stated in theorem 3.10, we have*

$$M\left[f(x,y)\,{}_4I^{a,b;c,c'}_{0,x;0,y}\ \{e^{-sx-ty}g(x,y)\} : s,t\right]$$
$$= L\left[g(x,y)\,{}_4J^{a,b;c,c'}_{x,\infty;y,\infty}\ \{x^{s-1}y^{t-1}f(x,y)\} : s,t\right]. \tag{3.13}$$

Further, we have

Theorem 3.13. *Under the conditions stated in theorem 3.9, we have*

$$L\left[f(x,y)\,{}_1I^{a,b,b';c}_{0,x;0,y}\ \{e^{-sx-ty}\,g(x,y)\} : s,t\right]$$
$$= L\left[g(x,y)\,{}_1J^{a,b,b';c}_{x,\infty;y,\infty}\ \{e^{-sx-ty}f(x,y)\} : s,t\right]. \tag{3.14}$$

Theorem 3.14. *Under the conditions stated in theorem 3.10, we have*

$$L\left[f(x,y)\,{}_2I^{a,b,b';c,c'}_{0,x;0,y}\ \{e^{-sx-ty}g(x,y)\} : s,t\right]$$
$$= L\left[g(x,y)\,{}_2J^{a,b,b';c,c'}_{x,\infty;y,\infty}\ \{e^{-sx-ty}f(x,y)\} : s,t\right]. \tag{3.15}$$

Theorem 3.15. *Under the conditions stated in theorem 3.9, we have*

$$L\left[f(x,y)\,{}_3I^{a,a',b,b';c}_{0,x;0,y}\ \{e^{-sx-ty}g(x,y)\} : s,t\right] \\ = L\left[g(x,y)\,{}_3J^{a,a',b,b';c}_{x,\infty;y,\infty}\ \{e^{-sx-ty}f(x,y)\} : s,t\right]. \tag{3.16}$$

Theorem 3.16. *Under the conditions stated in theorem 3.10, we have*

$$L\left[f(x,y)\,{}_4I^{a,b;c,c'}_{0,x;0,y}\ \{e^{-sx-ty}g(x,y)\} : s,t\right] \\ = L\left[g(x,y){}_4J^{a,b;c,c'}_{x,\infty;y,\infty}\ \{e^{-sx-ty}f(x,y)\} : s,t\right]. \tag{3.17}$$

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