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A CRITERIA OF COMPLETENESS FOR COMPACT OPERATORS IN HILBERT SPACE

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Abstract

A necessary and sufficient condition is given for completeness of the set of eigenfunctions and generalized eigenfunctions associated to the non zero eigenvalues of a compact operator on a Hilbert Space.

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Introduction and Main Result

We begin reviewing briefly some properties of compact operators on a Hilbert space H that will be needed later on [4], [5], [6], [7], [8]. If H is a Hilbert space $K(H), K_1(H)$ and $K_2(H)$ will denote, respectively the set of compact, nuclear and Hilbert-Schmidt operators in $B(H)$, the set of bounded operators in H. If $\dim H = \infty$, then $I \in B(H)\backslash K(H)$, where I is the identity operator on H. If $A \in K(H)$ and $\sigma(A)$ denotes the spectrum of A, then every $\mu \in \sigma(A)\backslash\{0\}$ is an eigenvalue, that is $\mathrm{Ker}(\mu I - A) \neq \{0\}$, and every element of $\mathrm{Ker}(\mu I - A)\backslash\{0\}$ is called an eigenvector of A corresponding to the eigenvalue μ, in this case the number $m_\mu = \dim \mathrm{Ker}(\mu I - A)$ is finite and it is called the geometric multiplicity of the eigenvalue μ. For such a μ the number $p_\mu = \dim \bigcup_{k=1}^{\infty} \mathrm{Ker}(\mu I - A)^k$ is also finite and it is by definition the algebraic multiplicity of μ. Evidently $p_\mu \geq m_\mu$, but for compact normal operators $p_\mu = m_\mu$. The inequality $p_\mu > m_\mu$ holds if and only if $\bigcup_{k=1}^{\infty} \mathrm{Ker}(\mu I - A)^k \neq \mathrm{Ker}(\mu I - A)$; in this case the elements of $\bigcup_{k=1}^{\infty} \mathrm{Ker}(\mu I - A)^k \backslash \mathrm{Ker}(\mu I - A)$ are called generalized eigenvectors or root vectors of A associated to the eigenvalue μ. Evidently $\varphi \in H$ is a generalized eigenvector of A corresponding to the eigenvalue $\mu \in \sigma(A)\backslash\{0\}$ if and only if there is an $\ell \in \mathbb{N}\backslash\{1\}$ such that $(A - \mu I)^\ell \varphi = 0$ and $(A - \mu I)^{\ell-1}\varphi \neq 0$, let's call such an ℓ the height of the generalized eigenvector φ; by definition the height of an eigenvector is 1.

An important problem is to determine conditions on $A \in K(H)$, so that the smallest closed linear subspace that contains the set of eigenvectors and generalized eigenvectors associated to its non-zero eigenvalues coincides with $\overline{R(A)}$, the closure of the range of A. In the present note we give a necessary and sufficient condition for this to occur, that we have found useful in some problems arising from Number Theory.

If $A \in B(H)$, the resolvent set $\rho(A)$ of A is the set $\{\lambda \in \mathbb{C} : (\lambda I - A)^{-1} \in B(H)\}$ and by definition $\sigma(A) = \mathbb{C}\backslash\rho(A); \sigma(A)$ is compact and non-empty. The function $\lambda \longrightarrow (\lambda I - A)^{-1}$ is an analytic function from $\rho(A)$ into $B(H)$ and it is called the resolvent of A. If $A \in K(H), \sigma(A)$ is at most countable with 0 as its only possible limit point, moreover $0 \in \sigma(A)$ if $\dim H = \infty$. In this case the resolvent of A has poles at every point of $\sigma(A)\backslash\{0\}$, more precisely if $\mu \in \sigma(A)\backslash\{0\}$ and $\varepsilon > 0$ is chosen in such a way that $\Omega = \{\lambda \in \mathbb{C} : 0 < |\lambda - \mu| < \varepsilon\}$ does not meet $\sigma(A)$, then in Ω is valid the following Laurent series expansion for the resolvent [6], [9].

$$(\lambda I - A)^{-1} = \sum_{n=1}^{q_\mu} (\lambda - \mu)^{-n} (A - \mu I)^{n-1} P_\mu + \sum_{\ell=0}^{\infty} (\lambda - \mu)^\ell Q_\mu^{\ell+1}$$

where $P_\mu = \frac{1}{2\pi i} \oint_{|\lambda-\mu|=\varepsilon'} (\lambda I - A)^{-1} dx$,

$$Q_\mu = \frac{1}{2\pi i} \oint_{|\lambda-\mu|=\varepsilon'} (\lambda - \mu)^{-1} (\lambda I - A)^{-1} dx,$$

$0 < \varepsilon' < \varepsilon$ and the contour integrations are counter clockwise. Moreover, it holds that

$$AP_\mu = P_\mu A, \quad P_\mu^2 = P_\mu, \quad (A - \mu I)^n P_\mu \neq 0$$

if

$$0 \leq n \leq q_\mu - 1, \quad (A - \mu I)^{q_\mu} P_\mu = 0, \quad AQ_\mu = Q_\mu A,$$
$$P_\mu Q_\mu = Q_\mu P_\mu = 0, \quad (A - \mu I) Q_\mu = P_\mu - I$$

and P_μ is the Riez projection onto $\bigcup_{k=1}^{\infty} \operatorname{Ker}(\mu I - A)^k$. The order q_μ of the pole of the resolvent $(\lambda I - A)^{-1}$ at μ is equal to the maximum height of an element in $\bigcup_{k=1}^{\infty} \operatorname{Ker}(\mu I - A)^k \backslash \{0\}$. If the geometric multiplicity of μ, $m_\mu = 1$, then $q_\mu = p_\mu$, the algebraic multiplicity of μ. For compact normal operators $q_\mu = 1$.

By Satz 5.15 in [6] there are elements $\varphi_{j,\ell}^{(\mu)}$ in $P_\mu H = \bigcup_{k=1}^{\infty} \operatorname{Ker}(\mu I - A)^k$, where $1 \leq j \leq m_\mu, \ell =$ height of $\varphi_{j,\ell}^{(\mu)}, 1 \leq \ell \leq n_j$, such that

$$(A-\mu I)\varphi_{j,\ell}^{(\mu)}=\begin{cases}0 & if \quad \ell=1\\ \varphi_{j,\ell-1}^{(\mu)} & if \quad \ell>1\end{cases}$$

$\sum_{j=1}^{m_\mu} n_j = p_\mu, q_\mu = \max_{1\le j\le m_\mu}$. Similarly, one can choose elements $\psi_{r,s}^{(\overline{\mu})}$ in $P_\mu^* H = \bigcup_{k=1}^{\infty} \mathrm{Ker}(\overline{\mu}I - A^*)^k$, where $1 \le r \le m_{\overline{\mu}}, s =$ height of $\psi_{r,s}^{(\overline{\mu})}, 1 \le s \le n_r, \sum_{r=1}^{m_{\overline{\mu}}} n_r = p_{\overline{\mu}}, q_{\overline{\mu}} = \max_{1\le r\le m_{\overline{\mu}}}$, such that

$$(A^*-\overline{\mu}I)\psi_{r,s}^{(\overline{\mu})}=\begin{cases}0 & if \quad s=1\\ \psi_{r,s-1}^{(\overline{\mu})} & if \quad s>1\end{cases}$$

and $<\varphi_{j,\ell}^{(\mu)}, \psi_{r,s}^{(\overline{\mu})}>=\delta_{j,r}\delta_{\ell,n_r-s+1}$ (note that $m_\mu = m_{\overline{\mu}}, p_\mu = p_{\overline{\mu}}, q_\mu = q_{\overline{\mu}}$). In this case we have

$$P_\mu\xi = \sum_{j=1}^{m_\mu}\sum_{\ell=1}^{n_j} <\xi, \psi_{j,n_j-\ell+1}^{(\overline{\mu})}> \varphi_{j,\ell}^{(\mu)} \qquad \forall\xi\in H$$

Therefore if $\omega \in H$ is orthogonal to all the eigenfunctions and generalized eigenfunctions of $A \in K(H)$ corresponding to its non-zero eigenvalues, we will have $P_\mu^*\omega = 0, \forall\mu \in \sigma(A)\backslash\{0\}$ and therefore the function defined by

$$<\xi, (\overline{\lambda}^{-1}I - A^*)^{-1}\omega>=<(\lambda^{-1}I - A)^{-1}\xi, \omega>$$

is entire $\forall\xi \in H$.

We have found this last condition useful for a certain family of compact operators that arise in Number Theory. To state our results we need further information about compact operators. If $A \in K(H)$, let $\{s_n^2(A) : n \in \mathbb{N}\}$, be the sequence of eigenvalues of A^*A, each one of them repeated a number of times equal to its algebraic multiplicity and ordered in such a way that

$$s_n(A) \ge s_{n+1}(A) \ge 0 \quad \forall n \in \mathbb{N}.$$

If $0 < p < \infty$ we say that $A \in K_p(H)$ when $\sum_{n=1}^{\infty} s_n(A)^p < \infty; K_p(H)$ is a two sided ideal in $B(H)$, until now the more useful of these ideals have been $K_1(H)$, the ideal of nuclear operators, and $K_2(H)$, the ideal of Hilbert-Schmidt operators. If $A, B \in K_2(H)$ then $AB \in K_1(H)$. Let $\{\mu_n(A)\}_{n\geq 1}$ he the sequence of eigenvalues of the compact operator A, ordered in such a way that $|\mu_n(A)| \geq |\mu_{n+1}(A)| \forall n \in \mathbb{N}$ and each one of them being counted according to its algebraic multiplicity, then $\forall A \in K_p(H)$ hold the inequalities of Weyl [4], [5], [7], [8].

$$\sum_{j=1}^{n} |\mu_j(A)|^p \leq \sum_{j=1}^{n} s_j(A)^p \qquad \forall n \geq 1$$

Therefore if $A \in K_1(H)$ the series $\sigma_1(A) = \sum_{j=1}^{\infty} \mu_j(A)$ converges absolutely and it is called the trace of A. The trace of $A^r, r \in \mathbb{N}$, is denoted by $\sigma_r(A)$. There are two results of Lidskii [5], [8] that enable us in some cases to find the trace of a nuclear operator without prior knowledge of its spectrum: if $\{\varphi_n\}_{n\geq 1}$ is a complete orthonormal set in H then $\sigma_r(A) = \sum_{n=1}^{\infty} < A^r\varphi_n, \varphi_n >$ and when A is an integral operator $\sigma_r(A)$ is given as an integral of the "kernel" of the operators A^r. If $A \in K_1(H)$ then $\mu \in \sigma(A)\backslash\{0\}$ if and only if $D(\mu^{-1}) = \det_1(I - \mu^{-1}A) = 0$ where D is the entire function, known as the Fredholm determinant of $I - \mu^{-1}A$ given by the following formulae of Plemelj [4], [5], [7].

$$D(\lambda) = \sum_{n=0}^{\infty} d_n\lambda^n, \qquad d_0 = 1$$

$$d_n = \frac{(-1)^n}{n!} \begin{vmatrix} \sigma_1(A) & n-1 & 0 & \cdots & 0 \quad 0 \\ \sigma_2(A) & \sigma_1(A) & n-2 & \cdots & 0 \quad 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_{n-1}(A) & \sigma_{n-2}(A) & \sigma_{n-3}(A) & \cdots & \sigma_1(A) 1 \\ \sigma_n(A) & \sigma_{n-1}(A) & \sigma_{n-2}(A) & \cdots & \sigma_2(A)\sigma_1(A) \end{vmatrix}$$

$n \geq 1$

If k is the algebraic multiplicity of μ then $D^{(\ell)}(\mu^{-1}) = 0$ for $0 \leq \ell \leq k-1$ and $D^{(k)}(\mu^{-1}) \neq 0$. The Hadamard factorization of the entire function $\det_1(I - \lambda A)$ is $\prod_{i=1}^{\infty}[1 - \lambda\mu_i(A)]$.

Hilbert observed that if $A \in K_2(H)$ the above formulae for the Fredholm determinant still make sense if we take $\sigma_1(A) = 0$ (if $A \in K_2(H)\backslash K_1(H)$ the series that defines $\sigma_1(A)$ may be divergent). The function defined in this way is called the modified or renormalized Fredholm determinant for $A \in K_2(H)$ and it is denoted by $\det_2(I - \lambda A)$.

The Hadamard factorization of the entire function $\det_2(I - \lambda A)$ is $\prod_{i=1}^{\infty}[1 - \lambda\mu_i(A)]e^{\lambda\mu_i(A)}$ (the exponential factors $e^{\lambda\mu_i(A)}$ make the infinite product $\prod_{i=1}^{\infty}[1 - \lambda\mu_i(A)]$ convergent).

A similar procedure is used to define renormalized Fredholm determinants of operators in $K_p(H), p > 2$.

In previous work [1], [2], [3] we have reformulated the Riemann Hypothesis (RH) as a problem of Functional Analysis by means of the Hilbert Schmidt (non nuclear, non normal) integral operators on $L^2[0,1]$,

$$[A_\rho(\alpha)f](\theta) = \int_0^1 \rho\left(\frac{\alpha\theta}{x}\right) f(x)dx$$

where $\alpha \in]0,1]$: RH holds if and only if

$$\overline{R(A_\rho(\alpha)^*)} \supset L^2(0,\alpha), \quad \alpha \in]0,1]$$

(if it holds for one α, it holds for all others α in this interval).

In [2] we have evaluated the modified Fredholm determinants

$$\det_2[I - \mu A_\rho(\alpha)] = e^{\alpha\mu} T_\alpha(\mu)$$

where

$$T_\alpha(\mu) = 1 - \alpha\mu + \sum_{r=1}^{\infty} \frac{(-1)^{r+1}\alpha^{(r+1)(r+2)/2}}{(r+1)!(r+1)} \prod_{\ell=1}^{r} \zeta(\ell+1)\mu^{r+1}$$

We also have closed formulae for the eigenfunctions and generalized eigenfunctions of the operators $A_\rho(\alpha), \alpha \in]0,1]$. This information has been used in [3] to prove that the set of eigenvectors and generalized eigenvectors associated to the non-zero eigenvalues of $A_\rho(\alpha), \alpha \in]0,1[$ is total in $L^2(0,1)$, but it is not part of a Markushevich basis in $L^2(0,1)$. We have not been able to extend this result to A_ρ, since the proof depends essentially on the fact that T_α is an entire function of order zero for $\alpha \in]0,1[$, but T_1 has order 1.

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