Pro Mathematica Vol. XVIII, Nos. 35-36, 2004

A STUDY OF q-LAGRANGES POLYNOMIALS OF THREE VARIABLES

Mumtaz Ahmad Khan *and* ***Abdul Rahman Khan***

Abstract

The present paper introduces a q-analogue of Lagranges polynomials of three variables due to Khan and Shukla and gives certain results involving these polynomials.

Department of Applied Mathematics, Aligarh Muslim University, India

Introduction

Lagranges polynomials arise in certain problems in statistics. In literature they are denoted by symbol $g_n^{(\alpha,\beta)}(x,y)$ and are defined by means of the following generating relation (see [11]-[12]):

$$(1-xt)^{-\alpha}(1-yt)^{-\beta} = \sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)t^n \tag{1.1}$$

Brenke Polynomials [3] are defined as

$$A(t)\,B(xt) = \sum_{n=0}^{\infty} P_n(x)t^n \tag{1.2}$$

So that

$$P_n(x) = \sum_{k=0}^{n} a_k b_{n-k}\ x^k \tag{1.3}$$

where a_n and b_n are arbitrary constants and where $A(t) = \sum a_k\ t^k$, $B(x) = \sum b_k x^k$.

Furthermore, with the proper choice of the parameters, they form some interesting sets of orthogonal polynomials. These were first encountered by Al-Salam and Chihara [2].

$$P_n(x;q;a,b,c) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \left(q^{\frac{\alpha}{\lambda}}\right)_k \left(q^{\frac{\beta}{\mu}}\right)_{n-k} \alpha^{-k}\beta^{-n+k} \tag{1.4}$$

where $1-(a+1)xt+at^2 = \left(1-\frac{t}{\alpha}\right)\left(1-\frac{t}{\beta}\right)$ and

$1-(c+1)xt+ct^2 = \left(1-\frac{t}{\mu}\right)\left(1-\frac{t}{v}\right)$. Next they appeared as the q-Random Walk Polynomials of Askey and Ismail [1].

$$F_n(x;q;a,c) = \left\{\frac{1}{(q)_n}\right\} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right] \left(q^{\mu/\alpha}\right)_k \left(q^{v/\beta}\right)_{n-k} v^{k-n} \mu^{-k} \quad (1.5)$$

where $1-xt+ct^2 = \left(1-\frac{t}{\alpha}\right)\left(1-\frac{t}{\beta}\right)$ and $1-at+bt^2 = \left(1-\frac{t}{\lambda}\right)\left(1-\frac{t}{\mu}\right)$.

In 1991, M. A. Khan and A. K. Sharma [9] considered an interesting special case of Brenke Polynomials in the form of q-analogue of Lagranges polynomials (1.1). They defined q-Lagranges polynomials by means of the following generating relation using the notations of Slater [14]:

$${}_1\phi_0 \left[\begin{array}{cc} q^\alpha; & \\ & xt \\ -; & \end{array}\right] {}_1\phi_0 \left[\begin{array}{cc} q^\beta; & \\ & yt \\ -; & \end{array}\right] = \sum_{n=0}^{\infty} g_{n,q}^{(\alpha,\beta)}(x,y)t^n \quad (1.6)$$

where

$${}_1\phi_0 \left[\begin{array}{cc} q^\alpha; & \\ & z \\ -; & \end{array}\right] = \frac{(1-q^\alpha z)_\infty}{(1-z)_\infty} = \frac{1}{(1-z)_{\alpha,q}} \quad (1.7)$$

Recently in 1998, M.A. Khan and A.K. Shukla [10] studied Lagranges Polynomials of three variables. They defined the three variable analogue of Lagranges Polynomials $g_n^{(\alpha,\beta,\gamma)}(x,y,z)$ by means of the following generating relation:

$$(1-xt)^{-\alpha}(1-yt)^{-\beta}(1-zt)^{-\gamma} = \sum_{n=0}^{\infty} g_n^{(\alpha,\beta,\gamma)}(x,y,z)t^n \quad (1.8)$$

In order to study the above polynomials they introduced the following lemma:

Lemma:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} A(k,m,n) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{m} A(k,m-k,n-m) \quad (1.9)$$

and

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{m} A(k,m,n) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} A(k,m+k,n+m) \qquad (1.10)$$

In this paper we consider the q-analogue of Lagranges polynomials of three variables.

2 q-Lagranges Polynomials of three Variables

We define the q-analogue of Lagranges polynomials of three variables $g_n^{(\alpha,\beta,\gamma)}(x,y,z)$ by means of the following generating relation using the notations of Gasper and Rahman [6]:

$$\begin{aligned} &{}_1\phi_0\left(q^{\alpha};-;q,xt\right)\ {}_1\phi_0\left(q^{\beta};-;q,yt\right)\ {}_1\phi_0\left(q^{\gamma};-;q,zt\right) \\ &\qquad = \sum_{n=0}^{\infty} g_{n,q}^{(\alpha,\beta,\gamma)}(x,y,z)t^n \end{aligned} \qquad (2.1)$$

In order to study such a polynomial we need the following lemma:

Lemma 1. *We have*

$$\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} A(k,j,n) = \sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{k=0}^{n-j} A(k,j,n-j-k) \qquad (2.2)$$

and

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{k=0}^{n-j} A(k,j,n) = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} A(k,j,n+j+k) \qquad (2.3)$$

The above lemma is a special case of the following lemma due to Srivastava and Monacha [15]

Lemma 2. *For positive integers $m_1, \ldots, m_r (r \geq 1)$,*

$$\sum_{n=0}^{\infty} \sum_{k_1,\ldots,k_r=0}^{\infty} \theta(k_1 \ldots, k_r; n)$$
$$= \sum_{n=0}^{\infty} \sum_{k_1,\ldots,k_r=0}^{m_1k_1+\ldots+m_rk_r \leq n} \theta(k_1 \ldots, k_r; n - m_1k_1 - \ldots - m_rk_r) \tag{2.4}$$

and

$$\sum_{n=0}^{\infty} \sum_{k_1,\ldots,k_r=0}^{m_1k_1+\ldots+m_rk_r \leq n} \phi(k_1 \ldots, k_r; n)$$
$$= \sum_{n=0}^{\infty} \sum_{k_1,\ldots,k_r=0}^{\infty} \varphi(k_1 \ldots, k_r; n + m_1k_1 + \ldots + m_rk_r). \tag{2.5}$$

Expanding the L. H. S. of (2.1) using Lemma 1 and finally equating the coefficient of t^n on both sides, we get

$$g_{n,q}^{(\alpha,\beta,\gamma)}(x, y, z) = \sum_{j=0}^{n} \sum_{k=0}^{n-j} \frac{(q^\alpha; q)_{n-j-k}(q^\beta; q)_j(q^\gamma; q)_k x^{n-j-k} y^j z^k}{(q; q)_{n-j-k}(q; q)_j(q; q)_k}. \tag{2.6}$$

Polynomials $g_{n,q}^{(\alpha,\beta,\gamma)}(x, y, z)$ can be regarded as a generalization of q-Lagranges Polynomials of Khan and Sharma [9] from two to three variables as it can easily be seen that

$$g_{n,q}^{(\alpha,\beta,\gamma)}(x, y, 0) = g_{n,q}^{(\alpha,\beta)}(x, y). \tag{2.7}$$

Replacing z by yq^β in (2.1), we also get

$$g_{n,q}^{(\alpha,\beta,\gamma)}(x, y, q^\beta y) = g_{n,q}^{(\alpha,\beta+\gamma)}(x, y). \tag{2.8}$$

Similarly,

$$g_{n,q}^{(\alpha,\beta,\gamma)}(x, q^\alpha x, z) = g_{n,q}^{(\alpha+\beta,\gamma)}(x, y) \tag{2.9}$$

$$g_{n,q}^{(\alpha,\beta,\gamma)}(x,y,q^{\alpha}x) = g_{n,q}^{(\alpha+\gamma,\beta)}(x,y) \tag{2.10}$$

$$g_{n,q}^{(\alpha,\beta,0)}(x,y,z) = g_{n,q}^{(\alpha,\beta)}(x,y). \tag{2.11}$$

Now consider

$$\sum_{n=0}^{\infty} q_{n,q}^{(a+\alpha+\lambda,b+\beta+\mu,c+\gamma+\eta)}(x,y,z)t^{n}$$

$$={}_1\phi_0\left(q^{a+\alpha+\lambda};-;q,xt\right)\,{}_1\phi_0\left(q^{b+\beta+\mu};-;q,yt\right)\,{}_1\phi_0\left(q^{c+\gamma+\eta};-;q,zt\right)$$

$$={}_1\phi_0\left(q^{a};-;q,xt\right)\,{}_1\phi_0\left(q^{\alpha};-;q,q^{a}xt\right)\,{}_1\phi_0\left(q^{\lambda};-;q,q^{a+\alpha}xt\right)$$

$${}_1\phi_0\left(q^{b};-;q,yt\right)\,{}_1\phi_0\left(q^{\beta};-;q,q^{b}yt\right)\,{}_1\phi_0\left(q^{\mu};-;q,q^{b+\beta}yt\right)$$

$${}_1\phi_0\left(q^{c};-;q,zt\right)\,{}_1\phi_0\left(q^{\gamma};-;q,q^{c}zt\right)\,{}_1\phi_0\left(q^{\eta};-;q,q^{c+\gamma}zt\right)$$

$$={}_1\phi_0\left(q^{a};-;q,xt\right)\,{}_1\phi_0\left(q^{b};-;q,yt\right)\,{}_1\phi_0\left(q^{c};-;q,zt\right)$$

$${}_1\phi_0\left(q^{\alpha};-;q,q^{a}xt\right)\,{}_1\phi_0\left(q^{\beta};-;q,q^{b}yt\right)\,{}_1\phi_0\left(q^{\gamma};-;q,q^{c}zt\right)$$

$${}_1\phi_0\left(q^{\lambda};-;q,q^{a+\alpha}xt\right)\,{}_1\phi_0\left(q^{\mu};-;q,q^{b+\beta}yt\right)\,{}_1\phi_0\left(q^{\eta};-;q,q^{c+\gamma}zt\right)$$

$$=\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} g_{n,q}^{(a,b,c)}(x,y,z)g_{j,q}^{(\alpha,\beta,\gamma)}(q^{a}x,q^{b}y,q^{c}z)g_{k,q}^{(\lambda,\mu,\eta)}(q^{a+\alpha}x,q^{b+\beta}y,q^{c+\gamma}z)t^{n+j+k}$$

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{k=0}^{n-j} g_{n-j-k,q}^{(a,b,c)}(x,y,z)g_{j,q}^{(\alpha,\beta,\gamma)}(q^{a}x,q^{b}y,q^{c}z)g_{k,q}^{(\lambda,\mu,\eta)}(q^{a+\alpha}x,q^{b+\beta}y,q^{c+\gamma}z)t^{n}$$

using (2.2).

Now equating the coefficient of t^n, we get

$$g_{n,q}^{(a+\alpha+\lambda,b+\beta+\mu,c+\gamma+\eta)}(x,y,z) =$$

$$\sum_{j=0}^{n}\sum_{k=0}^{n-j} g_{n-j-k,q}^{(a,b,c)}(x,y,z)g_{j,q}^{(\alpha,\beta,\gamma)}(q^a x,q^b y,q^c z)g_{k,q}^{(\lambda,\mu,\eta)}(q^{a+\alpha}x,q^{b+\beta}y,q^{c+\gamma}z). \tag{2.12}$$

As a particular case of (2.12) it can easily be verified that

$$g_{n,q}^{(\alpha+\alpha',\beta+\beta',\gamma+\gamma')}(x,y,z) = \sum_{r=0}^{n} g_{n-r,q}^{(\alpha,\beta,\gamma)}(x,y,z)g_{r,q}^{(\alpha',\beta',\gamma')}(q^\alpha x,q^\beta y,q^\gamma z). \tag{2.13}$$

3 Main Results

The first formula to be proved is

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\binom{m+n}{n} g_{m+n,q}^{(\alpha,\beta,\gamma)}(x,y,z)t^n v^m$$
$$= \frac{(x(v+t)q^\alpha;q)_\infty(y(v+t)q^\beta;q)_\infty(z(v+t)q^\gamma;q)_\infty}{(x(v+t);q)_\infty(y(v+t);q)_\infty(z(v+t);q)_\infty}. \tag{3.1}$$

Proof:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\binom{m+n}{n} g_{m+n,q}^{(\alpha,\beta,\gamma)}(x,y,z)t^n v^m$$

$$= \sum_{n=0}^{\infty} g_{n,q}^{(\alpha,\beta,\gamma)}(x,y,z)(v+t)^n$$

$$ {}_1\phi_0\left(q^\alpha;-;q,x(v+t)\right)\,{}_1\phi_0\left(q^\beta;-;q,y(v+t)\right)\,{}_1\phi_0\left(q^\gamma;-;q,z(v+t)\right) $$

$$ =\frac{(x(v+t)q^\alpha;q)_\infty(y(v+t)q^\beta;q)_\infty(z(v+t)q^\gamma;q)_\infty}{(x(v+t);q)_\infty(y(v+t);q)_\infty(z(v+t);q)_\infty}. $$

□

The second formula to be proved is

$$ \sum_{n=0}^{\infty}{}_1F_0\left[\begin{matrix}-n; & \\ & \omega \\ -; & \end{matrix}\right]g_{n,q}^{(\alpha,\beta,\gamma)}(x,y,z)t^n $$

$$ =\frac{(xt(1-\omega)q^\alpha;q)_\infty(yt(1-\omega)q^\beta;q)_\infty(zt(1-\omega)q^\gamma;q)_\infty}{(xt(1-\omega);q)_\infty(yt(1-\omega);q)_\infty(zt(1-\omega);q)_\infty}. \tag{3.2} $$

Proof:

$$ \sum_{n=0}^{\infty}{}_1F_0\left[\begin{matrix}-n; & \\ & \omega \\ -; & \end{matrix}\right]g_{n,q}^{(\alpha,\beta,\gamma)}(x,y,z)t^n $$

$$ =\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\binom{n+r}{n}g_{n+r,q}^{(\alpha,\beta,\gamma)}(x,y,z)t^n(-\omega t)^r $$

□

from which the result follows by using (3.1).

The third formula to be proved is

$$ \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(n+m+k)!}{n!\;m!\;k!}g_{n+m+k,q}^{(\alpha,\beta,\gamma)}(x,y,z)t^nv^m\omega^k $$

$$ =\frac{(x(t+v+\omega)q^\alpha;q)_\infty(y(t+v+\omega)q^\beta;q)_\infty(z(t+v+\omega)q^\gamma;q)_\infty}{(x(t+v+\omega);q)_\infty(y(t+v+\omega);q)_\infty(z(t+v+\omega);q)_\infty} \tag{3.3} $$

The proof of (3.3) is similar to that of (3.1).

The fourth formula to be proved is

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\begin{pmatrix} m+n \\ n \end{pmatrix} g_{m+n,q}^{(\alpha,\beta,\gamma)}(x,y,z)t^n v^m q^{\frac{1}{2}m(m-1)}$$

$$= \phi^{(3)}\left[\begin{array}{ll} (v+t) :: -;-;- : q^{\alpha};q^{\beta};q^{\gamma}; & \\ & q,x,y,z \\ - :: -;-;- : -;-;-; & \end{array}\right]. \tag{3.4}$$

Proof:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\begin{pmatrix} m+n \\ n \end{pmatrix}_q g_{m+n,q}^{(\alpha,\beta,\gamma)}(x,y,z)t^n v^m q^{\frac{1}{2}m(m-1)}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(q)_n}{(q)_m(q)_{n-m}}\, g_{n,q}^{(\alpha,\beta,\gamma)}(x,y,z)t^{n-m}v^m q^{\frac{1}{2}m(m-1)}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(q^{-n})_m(-1)^m q^{mn}}{(q)_m}\, g_{n,q}^{(\alpha,\beta,\gamma)}(x,y,z)t^{n-m}v^m$$

$$= \sum_{n=0}^{\infty} g_{n,q}^{(\alpha,\beta,\gamma)}(x,y,z)t^n\, {}_1\phi_0\left[q^{-n};-;q,-\frac{vq^n}{t}\right]$$

$$\sum_{n=0}^{\infty} g_{n,q}^{(\alpha,\beta,\gamma)}(x,y,z)((v+t);q)_n$$

$$= \phi^{(3)}\left[\begin{array}{ll} (v+t) :: -;-;- : q^{\alpha};q^{\beta};q^{\gamma}; & \\ & q,x,y,z \\ - :: -;-;- : -;-;-; & \end{array}\right].$$

□

References

[1] R. ASKEY AND M.E.H. ISMAIL: *Recurrence relations, continued fractions and orthogonal polynomials.* Memoir 300, American Mathematical Society, (1984).

[2] W.A. ALSALAM AND T.S. CHIHARA: *Convolution of orthogonal polynomials.* SAIM J. Math. Anal. Vol. 7 (1976) 16-28.

[3] T.S. CHIHARA: *An introduction to orthogonal polynomials.* Gordon and Breach, New York, London and Paris, (1978).

[4] H. EXTON: *q-Hypergeometric functions and Applications.* Ellis Horwood Limited, Chichester, (1983).

[5] H. EXTON: *Multiple Hypergeometric Functions and Applications.* John Wiley and Sons (Halsted Press), New York; Ellis Horwood, Chichester, (1976).

[6] G. GASPER AND M. RAHMAN: *Basic Hypergeometric Series.* Encyclopedia of Mathematics and its applications, Vol. 35, Cambridge University Press, Cambridge, New York, Port Chester, Melbourne, Sydney, (1990).

[7] M.A. KHAN: *On q-Laguerre Polynomials.* Ganita, Vol. 34 (1983) 111-123.

[8] M.A. KHAN AND A.H. KHAN: *A note mixed Hypergeometric series.* Acta Mathematica Vietnamica, Vol. 14, No 1 (1989) 95-98.

[9] M.A. KHAN AND A.K. SHARMA: *A note on q-Lagranges polynomials.* Acta Ciencia Indica, Vol. XVII, M, 3 (1991) 475-478.

[10] M.A. KHAN AND A.K. SHUKLA: *On Lagranges polynomials of three variables.* Proyecciones, Vol.17, No 2 (1998) 227-235.

[11] K.C. MITTAL: *Some new generating functions for Lagrange's polynomials.* Mathematica (Cluj), Vol. 44 (1979) 57-62.

[12] K.C. MITTAL AND B.L. SHARMA: *Some formulae for Lagranges polynomials.* I and II. Mathematica (Cluj), Vol. 32 (1967) 311-319.

[13] E. D. RAINVILLE: *Special Functions.* MacMillan, New York; Reprinted by Chelsea Publ. Co. Bron., New York, (1971).

[14] L.J. SLATER: *Generalized hypergeometric Functions.* Cambridge University Press, (1966).

[15] H. M. SRIVASTAVA AND P. W. KARLSOON: *Multiple Gaussian Hypergeometric series.* John Wiley and Sons (Halsted Press), New York; Ellis Horwood Chichester, (1985).

[16] H.M. SRIVASTAVA AND H.L. MANOCHA: *A treatise on generating functions.* John Wiley and Sons (Halsted Press), New York; Ellis Horwood, Chichester, (1984).

Mumtaz Ahmad Khan
Department of Applied Mathematics
Faculty of Engineering and Technology
Aligarh Muslim University,
Aligarh-202 001, (U.P)., India