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ON A GENERAL CLASS OF POLYNOMIALS $L_n^{(\alpha,\beta;\gamma,\delta)}(x,y)$ OF TWO VARIABLES SUGGESTED BY THE POLYNOMIALS $L_n^{(\alpha,\beta)}(x,y)$ OF RAGAB AND $L_n^{(\alpha,\beta)}(x)$ OF PRABHAKAR AND REKHA

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Abstract

The present paper is a study of a general class of polynomial $L_n^{(\alpha,\beta;\gamma,\delta)}(x,y)$ of two variables which include as particular cases the polynomials $L_n^{(\alpha,\beta)}(x,y)$ due to Ragab and $L_n^{\alpha,\beta}(x)$ due to Prabhakar and Rekha. Certain generating functions, a finite sum property, integral representations, Schläfli's contour integral, fractional integrals, Laplace transform and a series relation for this general class of polynomial have been obtained.

Key words: *Polynomial of two variables, generating functions, integral representations, fractional integrals.*

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1 Introduction

In 1972 T.R. Prabhakar and Suman Rekha [9] considered a general class of polynomials suggested by Laguerre polynomials as defined below:

$$L_n^{\alpha,\beta}(x) = \frac{\Gamma(\alpha n+\beta+1)}{n!}\sum_{k=0}^{n}\frac{(-n)_k\, x^k}{k!\,\Gamma(\alpha k+\beta+1)} \tag{1.1}$$

and proved certain results for these polynomial. Later, in 1978, they [10] derived an integral representation, a finite sum formula and a series relation for these polynomials. Evidently, if $\alpha = k$, a positive integer, then $L_n^{k,\beta}(x^k) = Z_n^{\beta}(x;k)$ which is Konhauser's polynomial and more particularly $L_n^{1,\beta}(x) = L_n^{(\beta)}(x)$ where $L_n^{(\beta)}(x)$ is the generalized Laguerre polynomials.

Recently, in 1991, S.F. Ragab [11] defined Laguerre polynomials of two variables $L_n^{(\alpha,\beta)}(x,y)$ as follows:

$$L_n^{(\alpha,\beta)}(x,y) = \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!}\sum_{r=0}^{n}\frac{(-y)^r\, L_{n-r}^{(\alpha)}(x)}{r!\,\Gamma(\alpha+n-r+1)\,\Gamma(\beta+r+1)} \tag{1.2}$$

where $L_n^{(\alpha)}(x)$ is the well known Laguerre polynomial of one variable defined by (E.D. Rainville [12], p. 200).

It may be noted that the definition (1.2) for $L_n^{(\alpha,\beta)}(x,y)$ is equivalent to

$$L_n^{(\alpha,\beta)}(x,y) = \frac{(\alpha+1)_n(\beta+1)_n}{(n!)^2}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}\, y^r\, x^s}{r!\,s!(\alpha+1)_s(\beta+1)_r} \tag{1.3}$$

where n is any positive integer or zero.

In this paper, we have defined and studied a general class of polynomials of two variables $L_n^{(\alpha,\beta;\gamma,\delta)}(x,y)$ suggested by (1.1) and (1.3) which in particular cases yield both (1.1) and (1.3) and in special case gives Konhauser's polynomials.

2 The Polynomial $L_n^{(\alpha,\beta;\gamma,\delta)}(x,y)$

The Polynomial $L_n^{(\alpha,\beta;\gamma,\delta)}(x,y)$ is defined as follows:

$$L_n^{(\alpha,\beta;\gamma,\delta)}(x,y) = \frac{\Gamma(\alpha n+\beta+1)\Gamma(\gamma n+\delta+1)}{(n!)^2} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s}\ x^s\, y^r}{r!s!\Gamma(\alpha s+\beta+1)\Gamma(\gamma r+\delta+1)}. \tag{2.1}$$

Putting $\alpha = \gamma = 1$, (2.1) reduces to

$$L_n^{(1,\beta;1,\delta)}(x,y) = \frac{(\beta+1)_n(\delta+1)_n}{(n!)^2} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s}\ x^s y^r}{r!\, s!(\beta+1)_s(\delta+1)_r} \tag{2.2}$$

which is Ragab's polynomial $L_n^{(\beta,\delta)}(x,y)$.

Putting $\gamma = 1$, $\delta = 0$ and $y = 0$ in (2.1), we obtain

$$L_n^{(\alpha,\beta;1,0)}(x,0) = \frac{\Gamma(\alpha n+\beta+1)}{n!} \sum_{s=0}^{n} \frac{(-n)_s\ x^s}{s!\ \Gamma(\alpha s+\beta+1)} \tag{2.3}$$

which is Prabhakar and Rekha's polynomial $L_n^{\alpha,\beta}(x)$, which in turn reduces to Laguerre polynomial $L_n^{\beta}(x)$ for $\alpha = 1$.

Putting $\alpha = k$, a positive integer, $\gamma = 1, \delta = 0$ and $\gamma = 0$ and replacing x by x^k in (2.1), we obtain

$$L_n^{(k,\beta;1,0)}(x^k, 0) = \frac{\Gamma(kn+\beta+1)}{n!} \sum_{s=0}^{n} (-1)^s \binom{n}{s} \frac{x^{ks}}{\Gamma(ks+\beta+1)} \tag{2.4}$$

which is Konhauser's polynomial $Z_n^{\beta}(x;k)$ (see [5], [6]).

Also, for $\alpha = k$, a positive integer, $\gamma = p$, a positive integer, β replaced by α, δ replaced by β, x replaced by x^k and y replaced by y^p, (2.1) gives a two variable analogue of Konhauser's polynomial $Z_n^{\alpha}(x;k)$ studied by the present authors [3] in a separate earlier communication. They defined it as

$$\begin{aligned} &Z_n^{\alpha,\beta}(x,y;k,p) \\ &= \frac{\Gamma(kn+\alpha+1)\Gamma(pn+\beta+1)}{(n!)^2} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s}\; x^{ks}\; y^{pr}}{r!\; s!\; \Gamma(ks+\alpha+1)\; \Gamma(pr+\beta+1)}. \end{aligned} \tag{2.5}$$

3 Generating Functions

The polynomials $L_n^{(\alpha,\beta;\gamma,\delta)}(x,y)$ have the generating function indicated in

$$e^t \phi(\alpha, \beta+1; -xt) \phi(\gamma, \delta+1; -yt) = \sum_{n=0}^{\infty} \frac{n!\; L_n^{(\alpha,\beta;\gamma,\delta)}(x,y)\; t^n}{\Gamma(\alpha n+\beta+1)\Gamma(\gamma n+\delta+1)} \tag{3.1}$$

where $\phi(\alpha, \beta; z)$ is the Bessel-Wright function [A. Erdelyi et. al., vol. 3 [2], p. 211].

For an arbitrary C, we easily obtain the following generating function for (2.1):

$$\sum_{n=0}^{\infty} \frac{n!\; (C)_n L_n^{(\alpha,\beta;\gamma,\delta)}(x,y)\, t^n}{\Gamma(\alpha n+\beta+1)\; \Gamma(\gamma n+\delta+1)} = (1-t)^{-C}\, E^{C}_{\alpha,\beta+1;\gamma,\delta+1}\left(\tfrac{-xt}{1-t}, \tfrac{-yt}{1-t}\right) \tag{3.2}$$

where $E^C_{a,b;d,e}(x,y)$ is defined by

$$E^C_{a,b;d,e}(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(C)_{r+s}\ x^s\ y^r}{r!\ s!\ \Gamma(as+b)\ \Gamma(dr+e)}. \tag{3.3}$$

It may be regarded as a two variable analogue of Wright's function $E^C_{a,b}(z)$ defined by

$$E^C_{a,b}(z) = \sum_{r=0}^{\infty} \frac{(C)_r\ z^r}{\Gamma(ar+b)\, r!}. \tag{3.4}$$

If we replace t by $\frac{t}{C}$ and let $|C| \longrightarrow \infty$ in (3.2) , we obtain the generating relation (3.1).

Before proceeding to obtain more general sets of generating functions for (2.1), we introduce here Wright's type double and triple hypergeometric series respectively as follows:

$$\psi^{(2)}\left[\begin{array}{ll} (a,A):(b,B);(d,D); & \\ & x,y \\ (f,F):(g,G);(h,H); & \end{array}\right]$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\Gamma\{a+A(m+n)\}\Gamma(b+Bm)\,\Gamma(d+Dn)}{\Gamma\{f+F(m+n)\}\,\Gamma(g+Gm)\,\Gamma(h+Hn)} \cdot \frac{x^m}{m!}\frac{y^n}{n!}. \tag{3.5}$$

$$\psi^{(3)}\left[\begin{array}{ll} (a,A)::(b,B);(b',B');(b'',B''):(d,D);(d',D');(d'',D''); & \\ & x,y,z \\ (f,F)::(g,G);(g',G');(g'',G''):(h,H);(h',H');(h'',H''); & \end{array}\right]$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{\Gamma\{a+A(m+n+p)\}\Gamma\{b+B(m+n)\}\Gamma\{b'+B'(m+p)\}}{\Gamma\{f+F(m+n+p)\}\Gamma\{g+G(m+n)\}\Gamma\{g'+G'(m+p)\}} \tag{3.6}$$

$$\frac{\Gamma\{b''+B''(n+p)\}\Gamma(d+Dm)\Gamma(d'+D'n)\Gamma(d''+D''p)}{\Gamma\{g''+G''(n+p)\}\Gamma(h+Hm)\Gamma(h'+H'n)\Gamma(h''+H''p)} \cdot \frac{x^m\, y^n\, z^p}{m!\, n!\, p!}.$$

Now for arbitrary λ and μ, one can easily obtain the following more general generating function for (2.1):

$$\sum_{n=0}^{\infty} \frac{n!\ \Gamma(\lambda n+\mu)\ L_n^{(\alpha,\beta;\gamma,\delta)}(x,y)t^n}{\Gamma(\alpha n+\beta+1)\ \Gamma(\gamma n+\delta+1)}$$

$$= \psi^{(3)} \left[\begin{matrix} (\mu,\lambda) :: -;-;- : \text{———} ; \text{———} ; -; \\ \text{——} :: -;-;- : (\beta+1,\alpha); (\delta+1,\gamma); -; \end{matrix} \quad -xt,-yt,t \right]. \tag{3.7}$$

Interesting special cases of (3.7) are as follows:

$$\sum_{n=0}^{\infty} \frac{n!\ L_n^{(\alpha,\beta;\gamma,\delta)}(x,y)t^n}{\Gamma(\alpha n+\beta+1)}$$

$$= \psi^{(3)} \left[\begin{matrix} (\delta+1,\gamma) :: -;-;- : \text{———} ; \text{———} ; -; \\ \text{———} :: -;-;- : (\beta+1,\alpha); (\delta+1,\gamma); -; \end{matrix} \quad -xt,-yt,t \right] \tag{3.8}$$

$$\sum_{n=0}^{\infty} \frac{n!\ L_n^{(\alpha,\beta;\gamma,\delta)}(x,y)t^n}{\Gamma(\gamma n+\delta+1)}$$

$$= \psi^{(3)} \left[\begin{matrix} (\beta+1,\alpha) :: -;-;- : \text{———} ; \text{———} ; -; \\ \text{———} :: -;-;- : (\beta+1,\alpha);(\delta+1,\gamma); -; \end{matrix} \quad -xt,-yt,t \right]. \tag{3.9}$$

Yet, another generating relation of interest for (2.1) is as follows:

$$\sum_{n=0}^{\infty} \frac{n!\,\Gamma(\lambda n\mu)\; L_n^{(\alpha,\beta;\gamma,\delta)}(x,y)t^n}{\Gamma(\delta n+\eta)\;\Gamma(\alpha n+\beta+1)\;\Gamma(\gamma n+\delta+1)}$$

$$= \psi^{(3)} \left[\begin{array}{l} (\mu,\lambda) :: -;-;- : \text{———};\text{———};-; \\ \\ (\eta,\rho) :: -;-;- : (\beta+1,\alpha);(\delta+1,\gamma);-; \end{array} \; -xt,-yt,t \right]. \tag{3.10}$$

In particular, when $\lambda=\rho=\alpha=\gamma=1$, (3.10) reduces to

$$\sum_{n=0}^{\infty} \frac{n!\,(\mu)_n\; L_n^{(\beta;\delta)}(x,y)t^n}{(\eta)_n\,(\beta+1)_n(\delta+1)_n}$$

$$= F^{(3)} \left[\begin{array}{l} \mu :: -;-;- : \text{———};\text{———};-; \\ \\ \eta :: -;-;- : \beta+1;\, \delta+1;-; \end{array} \; -xt,-yt,t \right],$$

where $L_n^{(\beta,\delta)}(x,y)$ is Ragab's Laguerre polynomial of two variables and $F^{(3)}[x,y,z]$ is a triple hypergeometric series [cf. Srivastava [14], p.428].

4 A Finite Sum Property of $L_n^{(\alpha,\beta;\gamma,\delta)}(x,y)$

It is easy to derive the following finite sum property of the polynomial $L_n^{(\alpha,\beta;,\gamma,\delta)}(x,y)$:

$$L_n^{(\alpha,\beta;\gamma,\delta)}(xz,yz)$$

$$= \sum_{k=0}^{n} \frac{k!\Gamma(\alpha n+\beta+1)\Gamma(\gamma n+\delta+1)(1-z)^{n-k}z^k}{n!\,(n-k)!\Gamma(\alpha k+\beta+1)\Gamma(\gamma k+\delta+1)} L_k^{(\alpha,\beta;\gamma,\delta)}(x,y). \tag{4.1}$$

Proof of (4.1): The generating relation (3.1) together with the fact that

$$e^t\phi(\alpha,\beta+1;-xzt)\phi(\gamma,\delta+1;-yzt)$$

$$=e^{(1-z)t}e^{zt}\phi(\alpha,\beta+1;-x(zt))\phi(\gamma,\delta+1;-y(zt))$$

yields

$$\sum_{n=0}^{\infty}\frac{n!\,L_n^{(\alpha,\beta;\gamma,\delta)}(xz,yz)t^n}{\Gamma(\alpha n+\beta+1)\Gamma(\gamma n+\delta+1)}$$

$$=\left(\sum_{n=0}^{\infty}\frac{(1-z)^n t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{n!\,L_n^{(\alpha,\beta;\gamma,\delta)}(x,y)z^n\,t^n}{\Gamma(\alpha n+\beta+1)\,\Gamma(\gamma n+\delta+1)}\right)$$

from which, on comparing the coefficients of t^n on both sides, we get (4.1). □

For $\alpha=\gamma=1,\beta$ replaced by α and δ replaced by β, we get the corresponding result for Ragab's Laguerre polynomials of two variables $L_n^{(\alpha,\beta)}(x,y)$ established by Khan and Shukla [4]. Some other particular cases of interest are as follows:

(i) For $\gamma=1,\delta=0,y=0$, (4.1) reduces to

$$L_n^{\alpha,\beta}(xz)=\sum_{k=0}^{n}\frac{\Gamma(\alpha n+\beta+1)(1-z)^{n-k}z^k}{(n-k)!\,\Gamma(\alpha k+\beta+1)}\,L_k^{\alpha,\beta}(x), \qquad (4.2)$$

which is a finite sum property for Prabhakar and Rekha's polynomial $L_n^{\alpha,\beta}(x)$ which for $\alpha = 1$ and β replaced α gives

$$L_n^{(\alpha)}(xz) = \sum_{k=0}^{n} \frac{(\alpha+1)_n(1-z)^n z^k}{(n-k)!\,(\alpha+1)_k} L_k^{(\alpha)}(x)$$

where $L_n^{(\alpha)}(x)$ is a well known Laguerre polynomial.

(ii) Putting $\alpha = k$, a positive integer, $\gamma = 1, \delta = 0, y = 0$ and β replaced by α and x replaced by x^k we get the following finite sum property for Konhauser's biorthogonal polynomial $Z_n^\alpha(x;k)$ of the second kind:

$$Z_n^\alpha(xz;k) = \sum_{r=0}^{n} \frac{\Gamma(kn+\alpha+1)(1-z)^{n-r} z^r}{(n-r)!\,\Gamma(kr+\alpha+1)} Z_r^\alpha(x;k). \qquad (4.3)$$

5 Integral Representations

Using the definition of Beta function it is easy to derive the following integral representantion of $L_n^{(\alpha,\beta;\gamma,\delta)}(x,y)$ (see Rainville [12]):

$$\int_0^t \int_0^s x^\beta (s-x)^{\beta'-1} y^\delta (t-y)^{\delta'-1} L_n^{(\alpha,\beta;\gamma,\delta)}(x^\alpha, y^\gamma)dxdy \qquad (5.1)$$

$$\frac{\Gamma(\alpha n+\beta+1)\Gamma(\gamma n+\delta+1)\Gamma(\beta')\Gamma(\delta')s^{\beta+\beta'}t^{\delta+\delta'}}{\Gamma(\alpha n+\beta+\beta'+1)\Gamma(\gamma n+\delta+\delta'+1)} L_n^{(\alpha,\beta+\beta';\gamma,\delta+\delta')}(s^\alpha, t^\gamma).$$

Using the integral (see Erdelyi et. al [1], vol.1, pp.14),

$$2i \sin \pi z \Gamma(z) = -\int_\infty^{(0+)} (-t)^{z-1} e^{-t} dt \qquad (5.2)$$

and the fact that

$$(1-x-y)^n = \sum_{r=0}^{n}\sum_{s=0}^{n-r} \frac{(-n)_{r+s}\ y^r\ x^s}{r!\ s!} \tag{5.3}$$

it is easy to derive the following integral representation for $L_n^{(\alpha,\beta;\gamma,\delta)}(x,y)$:

$$\begin{aligned} &-\int_{\infty}^{(0+)}\int_{\infty}^{(0+)} (-u)^{-\beta-1}(-v)^{-\delta-1}e^{-u-v}(1-x(-u)^{-\alpha}-y(-v)^{-\gamma})^n\, dudv \\ &= \frac{4\pi^2(n!)^2}{\Gamma(\alpha n+\beta+1)\Gamma(\gamma n+\delta+1)}\, L_n^{(\alpha,\beta;\gamma,\delta)}(x,y). \end{aligned} \tag{5.4}$$

We also have

$$\begin{aligned} &\frac{(n!)^2}{\Gamma(\alpha n+\beta+1)\Gamma(\gamma n+\delta+1)} \int_0^{\infty}\int_0^{\infty} u^{\beta}v^{\delta}e^{-u-v}L_n^{(\alpha,\beta;\gamma,\delta)}(xu^{\alpha},yv^{\gamma})dudv \\ &= (1-x-y)^n. \end{aligned} \tag{5.5}$$

6 Schläfli's Contour Integral

It is easy to show that

$$\begin{aligned} &L_n^{(\alpha,\beta;\gamma,\delta)}(x,y) \\ &\frac{\Gamma(\alpha n+\beta+1)\Gamma(\gamma n+\delta+1)}{(n!)^2\ (2\pi i)^2}\int_{-\infty}^{(0+)}\int_{-\infty}^{(0+)} \frac{(u^{\alpha}v^{\gamma}-xv^{\gamma}-yu^{\alpha})^n}{u^{\alpha n+\beta+1}v^{\gamma n+\delta+1}}e^{u+v}du\ dv. \end{aligned} \tag{6.1}$$

Proof of (6.1): The R.H.S. of (6.1) is equal to

$$\frac{\Gamma(\alpha n+\beta+1)\Gamma(\gamma n+\delta+1)}{(n!)^2}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}\ x^s y^r}{r!\ s!}\frac{1}{2\pi i}\int_{-\infty}^{(0+)} u^{-(\alpha s+\beta+1)}e^u\,du\times$$

$$\times\frac{1}{2\pi i}\int_{-\infty}^{(0+)} v^{-(\gamma r+\delta+1)}e^v\,dv$$

$$=\frac{\Gamma(\alpha n+\beta+1)\Gamma(\gamma n+\delta+1)}{(n!)^2}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}\ x^s\ y^r}{r!\ s!\ \Gamma(\alpha s+\beta+1)\ \Gamma(\gamma r+\delta+1)}$$

using Hankel's formula (see Erdelyi, A. et al. [1], 1.6(2)).

$$\frac{1}{\Gamma(z)}=\frac{1}{2\pi i}\int_{-\infty}^{0+} e^t\ t^{-z}dt, \tag{6.2}$$

finally (6.1) follows from (2.1). □

For $\alpha = k$, a positive integer, $\gamma = p$, a positive integer, β replaced by α, δ replaced by β, x replaced by x^k and y replaced by y^p, (6.1) reduces to the corresponding Schläfli's contour integral for two variable analogue of Konhauser's polynomial $Z_n^{\alpha,\beta}(x,y;k,p)$ obtained by Khan and Ahmad [3].

7 Fractional Integrals

Let L denote the linear space of (equivalent classes of) complex-valued functions $f(x)$ which are Lebesgue-integrable on $[0,\alpha], \alpha < \infty$. For $f(x) \in L$ and complex number μ with $R\ell\,\mu > 0$, the Riemann-Liouville fractional integral of order μ is defined as (see Prabhakar [7], p.72).

$$I^\mu f(x)=\frac{1}{\Gamma(\mu)}\int_0^x (x-t)^{\mu-1}f(t)dt \text{ for almost all } x\in[0,\alpha]. \tag{7.1}$$

Using the operator I^{μ}, Prabhakar [8] obtained the following result for $R\ell\,\mu > 0$ and $R\ell\,\alpha > -1$:

$$I^{\mu}[x^{\alpha} Z_n^{\alpha}(x;k)] = \frac{\Gamma(kn+\alpha+1)}{\Gamma(kn+\alpha+\mu+1)} x^{\alpha+\mu} Z_n^{\alpha+\mu}(x;k), \tag{7.2}$$

where $Z_n^{\alpha}(x;k)$ in Konhauser's biorthogonal polynomial.

In an attempt to obtain a result analogus to (7.2) for the polynomial $L_n^{(\alpha,\beta;\gamma,\delta)}(x,y)$ we first seek a two variable analogue of (7.1).

A two variable analogue of I^{μ} may be defined as

$$I^{\lambda,\mu}[f(x,y)] = \frac{1}{\Gamma(\lambda)\Gamma(\mu)} \int_0^x \int_0^y (x-u)^{\lambda-1}(y-v)^{\mu-1} f(u,v)\,du\,dv. \tag{7.3}$$

Putting $f(x,y) = x^{\beta} y^{\delta} L_n^{(\alpha,\beta;\gamma,\delta)}(x^{\alpha}, y^{\gamma})$ in (7.3), we obtain

$$I^{\lambda,\mu}[x^{\beta} y^{\delta} L_n^{(\alpha,\beta;\gamma,\delta)}(x^{\alpha}, y^{\gamma})]$$

$$= \frac{1}{\Gamma(\lambda)\Gamma(\mu)} \int_0^x \int_0^y (x-u)^{\lambda-1}(y-v)^{\mu-1} u^{\beta} v^{\delta} L_n^{(\alpha,\beta;\gamma,\delta)}(u^{\alpha}, v^{\gamma})\,du\,dv$$

$$= \frac{x^{\alpha+\lambda} y^{\beta+\mu}}{\Gamma(\lambda)\Gamma(\mu)} \int_0^1 \int_0^1 t^{\beta} w^{\delta} (1-t)^{\lambda-1}(1-w)^{\mu-1} L_n^{(\alpha,\beta;\gamma,\delta)}(x^{\alpha}, t^{\alpha}, y^{\gamma} w^{\gamma})$$

(by putting $u = xt$ and $v = yw$)

$$= \frac{\Gamma(\alpha n+\beta+1)\Gamma(\gamma n+\delta+1) x^{\alpha+\lambda} y^{\beta+\mu}}{(n!)^2 \Gamma(\lambda)\Gamma(\mu)} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s} x^{\alpha s} y^{\gamma r}}{r!\, s!\, \Gamma(\alpha s+\beta+1)\Gamma(\gamma r+\delta+1)} \times$$

$$\times \int_0^1 t^{\alpha s+\beta}(1-t)^{\lambda-1} dt \times \int_0^1 w^{\gamma r+\delta}(1-w)^{\mu-1} dw$$

$$= \frac{\Gamma(\alpha n+\beta+1)\Gamma(\gamma n+\delta+1) x^{\alpha+\lambda} y^{\beta+\mu}}{(n!)^2} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s} (x^{\alpha})^s (y^{\gamma})^r}{r!\, s!\, \Gamma(\alpha s+\beta+\lambda+1)\Gamma(\gamma r+\delta+\mu+1)}.$$

We thus arrive at

$$I^{\lambda,\mu}[x^\beta y^\delta L_n^{(\alpha,\beta;\gamma,\delta)}(x^\alpha, y^\gamma)]$$

$$= \frac{\Gamma(\alpha n+\beta+1)\Gamma(\gamma n+\delta+1)x^{\alpha+\lambda}y^{\beta+\mu}}{\Gamma(\alpha n+\beta+\lambda+1)\Gamma(\gamma n+\delta+\mu+1)} L_n^{(\alpha,\beta+\lambda;\gamma,\delta+\mu)}(x^\alpha, y^\gamma). \tag{7.4}$$

8 Laplace Transform

In the usual notation the Laplace transform is given by

$$L\{f(t):s\} = \int_0^\infty e^{-st} f(t)dt, R\ell(s-a) > 0 \tag{8.1}$$

where $f \in L(0,R)$ for every $R > 0$ and $f(t) = 0(e^{at}), t \to \infty$.

Using (8.1) Srivastava [13] proved

$$L\{t^\beta Z_n^\alpha(xt;k):s\}$$

$$= \frac{(\alpha+1)_{kn}\Gamma(\beta+1)}{s^{\beta+1}\ n!}\ {}_{k+1}F_k \left[\begin{matrix} -n, & \Delta(k,\beta+1); & \\ & & \left(\frac{x}{s}\right)^k \\ & \Delta(k,\alpha+1); & \end{matrix} \right], \tag{8.2}$$

provided that $R\ell(s) > 0$ and $R\ell(\beta) > -1$.

In an attempt to obtain a result analogous to (8.2) for $L_n^{(\alpha,\beta;\gamma,\delta)}(x,y)$ we take a two variable analogue of (8.1) as follows:

$$L\{f(u,v):s_1,s_2\} = \int_0^\infty \int_0^\infty e^{-s_1u-s_2v} f(u,v)dudv. \tag{8.3}$$

Now, we have

$$L\{u^{\lambda}v^{\mu}L_n^{(\alpha,\beta;\gamma,\delta)}(xu^{\alpha},yv^{\gamma}):s_1,s_2\}$$

$$=\frac{\Gamma(\alpha n+\beta+1)\Gamma(\gamma n+\delta+1)}{(n!)^2}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}\; x^s y^r}{r!\,s!\Gamma(\alpha s+\beta+1)\Gamma(\gamma r+\delta+1)}\times$$

$$\times\int_0^\infty e^{-s_1 u}\,u^{\alpha s+\lambda}du\;\times\int_0^\infty e^{-s_2 v}\,v^{\gamma r+\mu}\;dv$$

$$=\frac{\Gamma(\alpha n+\beta+1)\Gamma(\gamma n+\delta+1)}{(n!)^2\,s_1^{\lambda+1}\;s_2^{\mu+1}}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{\Gamma(\alpha s+\lambda+1)\Gamma(\gamma r+\mu+1)}{\Gamma(\alpha s+\beta+1)\Gamma(\gamma r+\delta+1)}\left(\frac{x}{s_1^{\alpha}}\right)^s\left(\frac{y}{s_2^{\gamma}}\right)^r$$

We thus arrive at

$$L\{u^{\lambda}v^{\mu}L_n^{(\alpha,\beta;\gamma,\delta)}(xu^{\alpha},yv^{\gamma}):s_1,s_2\}$$

$$=\frac{\Gamma(\alpha n+\beta+1)\Gamma(\gamma n+\delta+1)}{(n!)^2\,s_1^{\lambda+1}\;s_2^{\mu+1}}\,F^{1:1;1}_{-:1;1}\left[\begin{matrix}-n:(\lambda+1,\alpha);(\mu+1,\gamma);\\ \\ -:(\beta+1,\alpha);(\delta+1,\gamma);\end{matrix}\;\frac{x}{s_1^{\alpha}},\frac{y}{s_2^{\gamma}}\right]. \tag{8.4}$$

In the special case when $\lambda=\beta$ and $\mu=\delta$, (8.4) simplifies to the elegant form

$$\begin{aligned}&L\{u^{\beta}v^{\delta}L_n^{(\alpha,\beta;\gamma,\delta)}(xu^{\alpha},yv^{\gamma}):s_1,s_2\}\\ &=\frac{\Gamma(\alpha n+\beta+1)\Gamma(\gamma n+\delta+1)}{(n!)^2\,s_1^{\alpha n+\beta+1}\,s_2^{\gamma n+\delta+1}}\left(s_1^{\alpha}s_2^{\gamma}-xs_2^{\gamma}-ys_1^{\alpha}\right)^n.\end{aligned} \tag{8.5}$$

9 A Series Relation for $L_n^{(\alpha,\beta;\gamma,\delta)}(x,y)$

We now make use of fractional differentiation operator D_z^μ defined by (see [14], p. 285)

$$D_z^\mu\{z^\lambda\} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)}\, z^{\lambda-\mu} \tag{9.1}$$

where μ is an arbitrary complex number to show that

$$\sum_{n=0}^{\infty} \frac{n!\,(\lambda)_n L_n^{(\alpha,\beta;\gamma,\delta)}(x,y)\; t^n}{(\mu)_n\,\Gamma(\alpha n+\beta+1)\Gamma(\gamma n+\delta+1)}\, {}_1F_1\left[\begin{array}{ll} \mu-\lambda; & \\ & t \\ \mu+n; & \end{array}\right]$$

$$= e^t \psi^{(2)}\left[\begin{array}{ll} (\lambda,1) : \text{———};\text{———}; & \\ & -xt,-yt \\ (\mu,1) : (\beta+1,\alpha);(\delta+1,\gamma); & \end{array}\right] \tag{9.2}$$

Proof: We can rewrite (3.2) as

$$e^{-t}\sum_{n=0}^{\infty} \frac{n!\,L_n^{(\alpha,\beta;\gamma,\delta)}(x,y)\; t^n}{\Gamma(\alpha n+\beta+1)\Gamma(\gamma n+\delta+1)}\phi(\alpha,\beta+1;-xt)\phi(\gamma,\delta+1;-yt)$$

or as

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k\, n! L_n^{(\alpha,\beta;\gamma,\delta)}\;(x,y)\; t^{n+k}}{k!\Gamma(\alpha n+\beta+1)\Gamma(\gamma n+\delta+1)}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-x)^n\;(-y)^k\; t^{n+k}}{n!\,k!\,\Gamma(\alpha n+\beta+1)\,\Gamma(\gamma k+\delta+1)}.$$

Now multiply both sides by $t^{\lambda-1}$, and apply $D_t^{\lambda-\mu}$, we obtain

$$L.H.S = \frac{\Gamma(\lambda)}{\Gamma(\mu)} t^{\mu-1} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k\, n!(\lambda)_{n+k} L_n^{(\alpha,\beta;\gamma,\delta)}(x,y) t^{n+k}}{k!\Gamma(\alpha n+\beta+1)\Gamma(\gamma n+\delta+1)(\mu)_{n+k}}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} t^{\mu-1} \sum_{n=0}^{\infty} \frac{n!(\lambda)_n L_n^{(\alpha,\beta;\gamma,\delta)}(x,y) t^n}{(\mu)_n\,\Gamma(\alpha n+\beta+1)\,\Gamma(\gamma n+\delta+1)}\, {}_1F_1\left[\begin{matrix}\lambda+n; \\ \mu+n;\end{matrix}\; -t\right]$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} t^{\mu-1} e^{-t} \sum_{n=0}^{\infty} \frac{n!\;(\lambda)_n\; L_n^{(\alpha,\beta;\gamma,\delta)}(x,y)\; t^n}{(\mu)_n\,\Gamma(\alpha n+\beta+1)\,\Gamma(\gamma n+\delta+1)}\, {}_1F_1\left[\begin{matrix}\mu-\lambda; \\ \mu+n;\end{matrix}\; t\right]$$

(using Kümmer's transformation).

Similarly,

$$R.H.S = \frac{\Gamma(\lambda)}{\Gamma(\mu)} t^{\mu-1} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{\Gamma(\lambda+n+k)(-xt)^n(-yt)^k}{n!\,k!\,\Gamma(\mu+n+k)\Gamma(\alpha n+\beta+1)\Gamma(\gamma k+\delta+1)}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} t^{\mu-1} \psi^{(2)} \left[\begin{matrix}(\lambda,1): \text{———};\text{———}; \\ (\mu,1):(\beta+1,\alpha);(\delta+1,\gamma);\end{matrix}\; -xt,-yt\right]$$

which immediately lead to the result (9.2). □

For $\gamma = 1$, $\delta = 0$ and $y = 0$ (9.2) reduces to the result due to Prabhakar and Rekha [10] for $L_n^{\alpha,\beta}(x)$.

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Resumen

El presente artículo estudia una clase general de polinomios $L_n^{(\alpha,\beta;\gamma,\delta)}$ (x,y) de dos variables que incluye como casos particulares los polinomios $L_n^{(\alpha,\beta)}(x,y)$ de Ragab y los $L_n^{\alpha,\beta}(x)$ Prabhakar y Rekha. Se obtienen ciertas funciones generatrices, una propiedad de suma finita, representaciones integrales, la integral de contorno de Schläfli, integrales fraccionarias, transformada de Laplace y una relación de series para esta clase general de polinomios.

Palabras Clave: Polinomio de dos variables, funciones generatrices, representaciones integrales, integrales fraccionarias.

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