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ON A NEW CLASS OF CONTINUITY VIA RARE SETS

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Abstract

The notion of rare continuity was introduced by Popa [15] as a new generalization of weak continuity [7]. In this paper, we introduce a new class of functions called rarely pre-θ-continuous functions as a new generalization of the class of strongly θ-precontinuous functions and investigate some of its fundamental properties.

Key words: *Rare set, pre-θ-open, rarely continuous, rarely almost compact.*

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Introducción

In 1979, Popa [15] introduced the useful notion of rare continuity as a generalization of weak continuity [7]. The class of rarely continuous functions has been further investigated by Long and Herrington [8] and Jafari [5] and [6]. In 2001, Noiri [12] introduced the notion of pre-θ-open sets which are stronger than preopen sets.

The purpose of the present paper is to introduce the concept of rare pre-θ-continuity in topological spaces as a new type of rare continuity. We investigate several properties of rarely pre-θ-continuous functions. The notion of I.pre-θ-continuity is also introduced which is weaker than strongly θ-precontinuity and stronger than rare pre-θ-continuity. It is shown that when the codomain of a function is pre-regular, then the notions of rare pre-θ-continuity and I.pre-θ-continuity are equivalent.

1 Preliminaries

Throughout this paper, (X,τ) and (Y,σ) (or simply, X and Y) denote topological spaces on which no separation axioms are assumed unless explicitly stated. If S is any subset of a space X, then $Cl(S)$ and $Int(S)$ denote the closure and the interior of S, respectively. Recall that a set S is called regular open (resp. regular closed) if $S = Int(Cl(S))$ (resp. $S = Cl(Int(S))$.

The subset $S \subset X$ is called preopen [9] (resp. α-open [11] if $S \subset Int(Cl(S))$ (resp. $S \subset Int(Cl(Int(S))))$. The complement of a preopen set is called preclosed. The intersection of all preclosed sets containing S is called the preclosure [4] of S and is denoted by $pCl(S)$. The preinterior of S is defined by the union of all preopen sets contained in S and is denoted by $pInt(S)$. The family of all preopen sets of X is denoted by $PO(X)$.

A point $x \in X$ is called a pre-θ-cluster [12] point of S if $S \cap pCl(U) \neq \emptyset$ for each preopen set U containing x. The set of all pre-θ-cluster points of S is called the pre-θ-closure of S and is denoted by $pCl_\theta(S)$. A subset S is called pre-θ-closed if $pCl_\theta(S) = S$. The complement of a pre-θ-closed set is called pre-θ-open. The family of all pre-θ-open (resp. pre-θ-closed) sets of a space X is denoted by $P\theta O(X)$ (resp. $P\theta C(X)$). We set $P\theta O(X, x) = \{U \mid x \in U \in P\theta O(X)\}$, similarly for $P\theta C(X, x)$.

The pre-θ-interior of S denoted by $pInt_\theta(S)$ is defined as follows:

$$pInt_\theta(S) = \{x \in X : \text{for some preopen subset } U \text{ of } X, x \in U \subset pCl(U) \subset S\}$$

Recall that a space X is said to be pre-regular [12] if for each preclosed set F and each point $x \in X - F$, there exists disjoint preopen sets U and V such that $x \in U$ and $F \subset V$, equivalently for each $U \in PO(X)$ and each point $x \in U$, there exists $V \in PO(X, x)$ such that $x \in V \subset pCl(V) \subset U$.

Lemma 1.1 *Let S be a subset of X, then:*

(1) S is a pre-θ-preopen set if and only if $S = pInt_\theta(S)$.

(2) $X - pInt_\theta(S) = pCl_\theta(X - S)$ and $pInt_\theta(X - S) = X - pCl_\theta(S)$.

Recall that a rare set is a set with no interior points.

Lemma 1.2 *([13], [3]).*

(1) $pCl(S) \subset pCl_\theta(S)$ (resp. $pInt_\theta(S) \subset pInt(S)$) for any subset S of X.

(2) For a preopen (resp. preclosed) subset S of X, $pCl(S) = pCl_\theta(S)$ (resp. $pInt_\theta(S) = pInt(S)$).

Lemma 1.3 *[1]. A space X is pre-regular if and only if $pCl(S) = pCl_\theta(S)$ for any subset S of X.*

Lemma 1.4 *If X is a pre-regular space, then:*

(1) Every preclosed subset S of X is pre-θ-closed (i.e., preclosed sets and pre-θ-closed sets coincide).

(2) $pCl_\theta(S)$ (resp. $pInt_\theta(S)$) is a pre-θ-closed set (resp. pre-θ-open set).

Definition 1 *A function $f : X \to Y$ is called:*

1) Weakly continuous [7] (resp. weakly pre-θ-continuous [2]) if for each $x \in X$ and each open set G containing $f(x)$, there exists $U \in O(X,x)$ (resp. $U \in P\theta O(X,x)$) such that $f(U) \subset Cl(G)$.

2) Rarely continuous [15] if for each $x \in X$ and each $G \in O(Y, f(x))$, there exists a rare set R_G with $G \cap Cl(R_G) = \emptyset$ and $U \in O(X,x)$ such that $f(U) \subset G \cup R_G$.

2 Rare Pre-θ-Continuity

Definition 2 *A function $f : X \to Y$ is called rarely pre-θ-continuous if for each $x \in X$ and each $G \in O(Y, f(x))$, there exists a rare set R_G with $G \cap Cl(R_G) = \emptyset$ and $U \in P\theta O(X,x)$ such that $f(U) \subset G \cup R_G$.*

Example 2.1 *Let X and Y be the real line with indiscrete and discrete topologies, respectively. The identity function is rarely pre-θ-continuous.*

We must admit that we do not have examples to convince the reader apropos our claim, i.e., example showing that a rarely pre-θ-continuous function is independent of rarely continuous. Therefore we ask the reader to find examples to prove or disprove our claim.

Theorem 2.2 *The following statements are equivalent for a function $f : X \to Y$:*

(1) The function f is rarely pre-θ-continuous at $x \in X$.

(2) For each set $G \in O(Y, f(x))$, there exists $U \in P\theta O(X, x)$ such that $Int[f(U) \cap (Y \setminus G)] = \emptyset$.

(3) For each set $G \in O(Y, f(x))$, there exists $U \in P\theta O(X, x)$ such that $Int[f(U)] \subset Cl(G)$.

Proof.

(1)→ (2): Let $G \in O(Y, f(x))$. By $f(x) \in G \subset Int(Cl(G))$ and the fact that $Int(Cl(G)) \in O(Y, f(x))$, there exists a rare set R_G with $Int(Cl(G)) \cap Cl(R_G) = \emptyset$ and a pre-θ-open set $U \subset X$ containing x such that $f(U) \subset Int(Cl(G)) \cup R_G$. We have $Int[f(U) \cap (Y - G)] = Int[f(U)] \cap Int(Y - G) \subset Int[Cl(G) \cup R_G] \cap (Y - Cl(G)) \subset (Cl(G) \cup Int(R_G)) \cap (Y - Cl(G)) = \emptyset$.

(2) → (3): It is straightforward.

(3) → (1): Let $G \in O(Y, f(x))$. Then by (3), there exists $U \in P\theta O(X, x)$ such that $Int[f(U)] \subset Cl(G)$. We have $f(U) = [f(U) - Int(f(U))] \cup Int(f(U)) \subset [f(U) - Int(f(U))] \cup Cl(G) = [f(U) - Int(f(U))] \cup G \cup (Cl(G) - G) = [(f(U) - Int(f(U))) \cap (Y - G)] \cup G \cup (Cl(G) - G)$. Set $R^* = [f(U) - Int(f(U))] \cap (Y - G)$ and $R^{**} = (Cl(G) - G)$. Then R^* and R^{**} are rare sets. Moreover $R_G = R^* \cup R^{**}$ is a rare set such that $Cl(R_G) \cap G = \emptyset$ and $f(U) \subset G \cup R_G$. This shows that f is rarely pre-θ-continuous.

Theorem 2.3 *Let X be a pre-regular space. Then the following statements are equivalent for a function $f : X \to Y$:*

1) The function f is rarely pre-θ-continuous at $x \in X$.

2) For each $G \in O(Y, f(x))$, there exists a rare set R_G with $G \cap Cl(R_G) = \emptyset$ such that $x \in pInt_\theta(f^{-1}(G \cup R_G))$.

3) For each $G \in O(Y, f(x))$, there exists a rare set R_G with $Cl(G) \cap R_G = \emptyset$ such that $x \in pInt_\theta(f^{-1}(Cl(G) \cup R_G))$.

4) *For each* $G \in RO(Y, f(x))$, *there exists a rare set* R_G *with* $G \cap Cl(R_G) = \emptyset$ *such that* $x \in pInt_\theta(f^{-1}(G \cup R_G))$.

Proof.

1) → 2): Suppose that $G \in O(Y, f(x))$. Then there exists a rare set R_G with $G \cap Cl(R_G) = \emptyset$ and $U \in P\theta O(X, x)$ such that $f(U) \subset G \cup R_G$. It follows that $x \in U \subset f^{-1}(G \cup R_G)$. This implies that $x \in pInt_\theta(f^{-1}(G \cup R_G))$.

2)→ 3): Suppose that $G \in O(Y, f(x))$. Then there exists a rare set R_G with $G \cap Cl(R_G) = \emptyset$ such that $x \in pInt_\theta(f^{-1}(G \cup R_G))$. Since $G \cap Cl(R_G) = \emptyset, R_G \subset Y - G$, where $Y - G = (Y - Cl(G)) \cup (Cl(G) - G)$. Now, we have $R_G \subset (R_G \cup (Y - Cl(G)) \cup (Cl(G) - G))$. Set $R^* = R_G \cap (Y - Cl(G))$. It follows that R^* is a rare set with $Cl(G) \cap R^* = \emptyset$. Therefore $x \in PInt_\theta[f^{-1}(G \cup R_G)] \subset pInt_\theta[f^{-1}(Cl(G) \cup R^*)]$.

3) → 4): Assume that $G \in RO(Y, f(x))$. Then there exists a rare set R_G with $Cl(G) \cap R_G = \emptyset$ such that $x \in pInt_\theta[f^{-1}(Cl(G) \cup R_G)]$. Set $R^* = R_G \cup (Cl(G) - G)$. It follows that R^* is a rare set and $G \cap Cl(R^*) = \emptyset$. Hence
$x \in pInt_\theta[f^{-1}(Cl(G) \cup R_G)] = pInt_\theta[f^{-1}(G \cup (Cl(G) - G) \cup R_G)] = pInt_\theta[f^{-1}(G \cup R^*)]$.

4) → 1): Let $G \in O(Y, f(x))$. By $f(x) \in G \subset Int(Cl(G))$ and the fact that $Int(Cl(G)) \in RO(Y)$, there exists a rare set R_G and $Int(Cl(G)) \cap Cl(R_G) = \emptyset$ such that $x \in PInt_\theta[f^{-1}(Int(Cl(G)) \cup R_G)]$. Let $U = pInt_\theta[f^{-1}(Int(Cl(G)) \cup R_G]$. Since X is a pre-regular space, $U \in P\theta O(X, x)$ (Lemma 1.4) and, therefore $f(U) \subset Int(Cl(G)) \cup R_G$. Hence, we have $Int[f(U) \cap (Y - G)] = \emptyset$, and by Theorem 2.2, f is rarely pre-θ-continuous.

Remark 2.4 *Note that in Theorem 2.3, the fact that* X *is a pre-regular space was only used in* 4) → 1). *Therefore the following implications are always true:* 1) → 2 → 3) → 4).

Recall that a function $f : X \to Y$ is strongly θ-precontinuous [12] if for each set $G \in O(Y, f(x))$, there exists $U \in PO(X, x)$ such that $f(pCl(U)) \subset G$ equivalently [3] if for each set $G \in O(Y, f(x))$, there exists $U \in P\theta O(X, x)$ such that $f(U) \subset G$.

With this characterization in mind, we define the following notion which is a new generalization of strongly θ-precontinuous.

Definition 3 *A function $f : X \to Y$ is I.pre θ-continuous at $x \in X$ if for each set $G \in O(Y, f(x))$, there exists $U \in P\theta O(X, x)$ such that $Int[f(U)] \subset G$.*

If f has this property at each point $x \in X$, then we say that f is I.pre-θ-continuous on X.

Example 2.5 *Let $X = \{a, b\}$ have the Sierpinski topology $\tau = \{X, \emptyset, \{a\}\}$. Then a function $f : X \to X$ defined by $f(a) = b$ and $f(b) = a$ is I.pre-θ-continuous at $a \in X$ but it is not strongly θ-precontinuous.*

Remark 2.6 *It should be noted that I.pre-θ-continuity is weaker than strongly θ-precontinuous and stronger than rare pre-θ-continuity.*

Question 2.7 *Are there examples showing that a function is rarely pre-θ-continuous but not I.pre-θ-continuous?*

Theorem 2.8 *Let Y be a regular space. Then the function $f : X \to Y$ is I.pre-θ-continuous on X if and only if f is rarely pre-θ-continuous on X.*

Proof. We prove only the sufficient condition since the necessity condition is evident (Remark 2.6).

Let f be rarely pre-θ-continuous on X and $x \in X$. Suppose that $f(x) \in G$, where G is an open set in Y. By the regularity of Y, there exists an open set $G_1 \in O(Y, f(x))$ such that $Cl(G_1) \subset G$. Since f is rarely pre-θ-continuous, then there exists $U \in P\theta O(X, x)$ such that

$Int[f(U)] \subset Cl(G_1)$ (Theorem 2.2). This implies that $Int[f(U)] \subset G$ and therefore f is I.pre-θ-continuous on X.

We say that a function $f : X \to Y$ is almost pre- θ-open, if the image of a pre-θ-open set is open.

Theorem 2.9 *If $f : X \to Y$ is an almost pre-θ-open rarely pre-θ-continuous function, then f is weakly pre-θ-continuous.*

Proof. Suppose that $x \in X$ and $G \in O(Y, f(x))$. Since f is rarely pre-θ-continuous, there exists a rare set R_G with $Cl(R_G) \cap U = \emptyset$ and $U \in P\theta O(X, x)$ such that $f(U) \subset G \cup R_G$. This means that $(f(U) \cap (Y \setminus Cl(G)) \subset R_G$. Since the function f is almost pre-θ-open, then $f(U) \cap (Y \setminus Cl(G))$ is open. But the rare set R_G has no interior points. Then $f(U) \cap (Y \setminus Cl(G)) = \emptyset$. This implies that $f(U) \subset Cl(G)$ and thus f is weakly pre-θ-continuous.

Definition 4 *Let $A = \{G_i\}$ be a class of subsets of X. By rarely union sets [5] of A we mean $\{G_i \cup R_{G_i}\}$, where each R_{G_i} is a rare set such that each of $\{G_i \cap Cl(R_{G_i})\}$ is empty.*

Recall that, a subset B of X is said to be rarely almost compact relative to X [5] if for every open cover of E by open sets of X, there exists a finite subfamily whose rarely union sets cover B.
A topological space X is said to be rarely almost compact [5] if the set X is rarely almost compact relative to X.
A subset K of a space X is said to be $P\theta O$-compact relative to X if for every cover of K by pre-θ-open sets in X has a finite subcover. A space X is said to be $P\theta O$-compact if the set X is $P\theta O$-compact relative to X.

Theorem 2.10 *Let $f : X \to Y$ be rarely pre-θ-continuous and K be a $P\theta O$-compact set relative to X. Then $f(K)$ is rarely almost compact subset relative to Y.*

Proof. Suppose that Ω is a open cover of $f(K)$. Let B be the set of all V in Ω such that $V \cap f(K) \neq \emptyset$. Then B is a open cover of $f(K)$. Hence for each $k \in K$, there is some $V_k \in B$ such that $f(k) \in V_k$. Since f is rarely pre-θ-continuous there exists a rare set R_{V_k} with $V_k \cap Cl(R_{V_k}) = \emptyset$ and a pre-θ-open set U_k containing k such that $f(U_k) \subset V_k \cup R_{V_k}$. Hence there is a finite subfamily $\{U_k\}_{k \in \triangle}$ which covers K, where $\triangle$ is a finite subset of K. The subfamily $\{V_k \cup R_{V_k}\}_{k \in \triangle}$ also covers $f(K)$.

Theorem 2.11 *Let $f : X \to Y$ be rarely continuous and X be a pre-regular space. Then f is rarely pre-θ-continuous.*

Proof. It follows of ([1], Lemma 4.1) or Lemma 1.3.

Lemma 2.12 *(Long and Herrington [8]). If $g : Y \to Z$ is continuous and one-to-one, then g preserves rare sets.*

Theorem 2.13 *If $f : X \to Y$ is rarely pre-θ-continuous and $g : Y \to Z$ is continuous and one-to-one, the $g \circ f : X \to Z$ is rarely pre-θ-continuous.*

Proof. Suppose that $x \in X$ and $(g \circ f)(x) \in V$, where V is an open set in Z. By hypothesis, g is continuous, therefore there exists an open set $G \subset Y$ containing $f(x)$ such that $g(G) \subset V$. Since f is rarely pre-θ-continuous, there exists a rare set R_G with $G \cap Cl(R_G) = \emptyset$ and a pre-θ-open set U containing x such that $f(U) \subset G \cup R_G$. It follows from Lemma 2.14 that $g(R_G)$ is a rare set in Z. Since R_G is a subset of $Y \setminus G$ and g is injective, we have $Cl(g(R_G)) \cap V = \emptyset$. This implies that $(g \circ f)(U) \subset V \cup g(R_G)$. Hence the result.

Recall that, a function $f : X \to Y$ is called pre-θ-open if $f(U)$ is pre-θ-open in Y for every pre-θ-open set U of X.

Theorem 2.14 *Let $f : X \to Y$ be pre-θ-open and $g : Y \to Z$ a function such that $g \circ f : X \to Z$ is rarely pre-θ-continuous. Then g is rarely pre-θ-continuous.*

Proof. Let $y \in Y$ and $x \in X$ such that $f(x) = y$. Let $G \in O(Z, (g \circ f)(x))$. Since $g \circ f$ is rarely pre-θ-continuous, there exists a rare set R_G with $G \cap Cl(R_G) = \emptyset$ and $U \in P\theta O(X, x)$ such that $(g \circ f)(U) \subset G \cup R_G$. But $f(U)$ (say V) is a pre-θ-open set containing $f(x)$. Therefore, there exists a rare set R_G with $G \cap Cl(R_G) = \emptyset$ and $V \in P\theta O(Y, y)$ such that $g(V) \subset G \cup R_G$, i.e., g is rarely pre-θ-continuous.

Definition 5 *A space X is called r-separate [6] if for every pair of distinct points x and y in X, there exists rare sets R_{U_x}, R_{U_y} and open sets U_x and U_y with $U_x \cap Cl(R_{U_x}) = \emptyset$ and $U_y \cap Cl(R_{U_y}) = \emptyset$ such that $(U_x \cup R_{U_x}) \cap (U_y \cup R_{U_y}) = \emptyset$.*

In [15], Popa obtained the following result.

Theorem 2.15 *The function $f : X \to Y$ is rarely continuous if and only if for each open set $G \subset Y$, there exists a rare set R_G with $G \cap Cl(R_G) = \emptyset$ such that $f^{-1}(G) \subset Int[f^{-1}(G \cup R_G)]$.*

Theorem 2.16 *Let Y is r-separate and every preopen subset of X is α-open. If $f : X \to Y$ is rarely pre-θ-continuous injection, then X is Hausdorff.*

Proof. Since f is injective, then $f(x) \neq f(y)$ for any distinct points x and y in X. Since Y is r-separate, there exists open sets G_1 and G_2 in Y containing $f(x)$ and $f(y)$, respectively, and rare sets $R_{G_1}$and R_{G_2} with $G_1 \cap Cl(R_{G_1}) = \emptyset$ and $G_2 \cap Cl(R_{G_2}) = \emptyset$ such that $(G_1 \cup R_{G_1}) \cap (G_2 \cup R_{G_2}) = \emptyset$. Therefore $Int[f^{-1}(G_1 \cup R_{G_1})] \cap Int[f^{-1}(G_2 \cup R_{G_2})] = \emptyset$. Since every preopen subset of X is α-open, and using ([1], Lemma 4.3) we obtain of that a rarely pre-θ-continuous is rarely continuous and by Theorem 2.15, we have $x \in f^{-1}(G_1) \subset Int[f^{-1}(G_1 \cup R_{G_1})]$ and $y \in f^{-1}(G_2) \subset Int[f^{-1}(G_2 \cup R_{G_2})]$. This shows that X is Hausdorff.

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Resumen

Popa [15] introdujo la noción de continuidad rara como una generalización de continuidad débil [7]. En este trabajo, introducimos una nueva clase de funciones llamadas funciones raramente pre-θ- continuas como una nueva generalización de la clase de funciones fuertemente θ-precontinuas e investigamos algunas de sus propiedades fundamentales.

Palabras Clave: Conjunto raro, pre-θ-abierto, raramente continuas, raramente casi compacto.

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