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# ON THE KIRCHHOFF EQUATION IN NONCYLINDRICAL DOMAINS OF $\mathbb{R}^n$

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***Abstract***

*In this paper we investigate a model of Kirchhoff type for vibrations of elastic bodies represented by bounded open sets $\Omega_t$ of $\mathbb{R}^n$ when the boundaries $\Gamma_t$ are moving with the time $t$. With restrictions on the rest position and the initial velocity we prove global existence and uniqueness of solutions, in the Sobolev class, for a certain mixed problem with zero Dirichlet boundary conditions.*

***Key words:*** *Kirchhoff equation; vibration, moving boundary, Sobolev spaces.*

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# 1 Basic Notes

Let $\Omega$ be a bounded open set of $\mathbb{R}^n$ with boundary $\Gamma$ of class $C^2$ and $K = K(t)$ a mapping from $[0, \infty)$ to $[0, \infty)$. We consider the family of deformations $\{\Omega_t\}_{t \geq 0}$ of $\Omega$ defined by

$$\Omega_t = \{x \in \mathbb{R}^n;\ x = K(t)y,\ \text{for all } y \in \Omega\},$$

that is, $\Omega_t = K(t)\Omega$. We set $\Omega_0 = \Omega$ and denote by $V_0$ the Lebesgue measure of $\Omega$. Let $u = u(x, t)$ be a real function defined for $x \in \Omega_t$ and all $t \geq 0$. We consider the class of partial differential equations, of the Kirchhoff type, defined by

$$\frac{\partial^2 u}{\partial t^2} - \left(a(t) + \hat{b}(t) \int_{\Omega_t} |\nabla u(x,t)|^2 \, dx\right) \Delta u = 0, \tag{1.1}$$

where

$$a(t) = \frac{\tau_0 - k}{m} + \frac{k}{m} K(t)^n; \quad \hat{b}(t) = \frac{k}{2mV_0 K(t)^n},$$

$\tau_0$ , $k$ $m$ positive constants and $\tau_0 > k$.

**Remark 1.1**. We could consider a general model of Kirchhoff type as

$$\frac{\partial^2 u}{\partial t^2} - M\left(t, x, \int_{\Omega_t} |\nabla u(x,t)|^2 \, dx\right) \Delta u = 0.$$

In the present investigation we are restricted to the case
$M(t, \lambda) = a(t) + \hat{b}(t)\lambda$ given by (1.1).

**Examples**

- Suppose $K(t) = 1$ in (1.1) for all $t \geq 0$. We obtain

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\tau_0}{m} + \frac{k}{2mV_0} \int_{\Omega} |\nabla u(x,t)|^2 \, dx\right) \Delta u = 0. \tag{1.2}$$

In this case $K(t) = 1$ when $n = 1$ and $\Omega = (0, L)$, $L > 0$, we have

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\tau_0}{m} + \frac{k}{2m\,L} \int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx\right) \frac{\partial^2 u}{\partial x^2} = 0. \tag{1.3}$$

The model (1.3) was proposed by Kirchhoff [9], see also Carrier [5]. It represents the small vertical vibrations of an stretched elastic string when the tension is variable but the ends of the string are fixed at 0 and $L$. In (1.3) $\tau_0$ is the initial tension, $m$ the mass of the string and $k$ the Young's modulus of the material of the string. For mathematical aspects of (1.3) see Bernstein [3], Dickey [6].

In the case of constant tension $\tau_0$ , the model (1.3) reduces to

$$\frac{\partial^2 u}{\partial t^2} - \frac{\tau_0}{m}\frac{\partial^2 u}{\partial x^2} = 0,$$

obtained by d'Alembert (1714-1793) and Euler (1707-1783).

Returning to the particular case (1.2) it can be written, in general, as

$$\frac{\partial^2 u}{\partial t^2} - M\left(\int_\Omega |\nabla u(x,t)|^2\, dx\right)\Delta u = 0 \tag{1.4}$$

or

$$u''(t) + M\big(||u(t)||^2\big)Au = 0 \tag{1.5}$$

in the operator notation. In (1.5) we consider the Hilbert spaces $V \subset H \subset V'$, where $V'$ is the dual of $V$ and the embeddings are continuous and dense. By $||\cdot||$ we denote the norm in $V$ and $A\colon V \to V'$ is a bounded linear operator. For (1.4) see Pohozhaev [18] and for (1.5) see Lions [12].

When we suppose $M(\lambda) \geq m_0 > 0$ and $C^1(0,\infty)$, Pohozhaev [18] proved that the mixed problem for (1.4) has global solution in $t$ when the initial data $u(x,0)$, $u_t(x,0)$ are restricted to belong to a class of functions called Pohozhaev's Class. This result can also be seen in Lions [12] for the operator model (1.5). It is interesting call the attention to the reader that Pohozaev, [17], [19] proved that if $M(\lambda) = (c_0 + c_1\lambda)^{-2}$, $c_0 > 0$, $c_1 \geq 0$, constants, the mixed problem for (1.4) has global solution without restrictions on the initial data. The model (1.5) was also analysed by Arosio-Spagnolo [1] and Hazoya-Yamada [8] when $M(\lambda) \geq 0$. Milla Miranda and San Gil Jutuca [16] investigated the mixed problem (1.4) for a partition of $\Gamma$, boundary of $\Omega$, in $\Gamma_1$ and $\Gamma_2$. They considered Dirichlet-Newmann condition of the type $u = 0$ on $\Gamma_0$ and $\dfrac{\partial u}{\partial v} + \delta(x)u' = 0$ on $\Gamma_1$, $\delta \geq \delta_0 > 0$. They proved the existence of global solutions for $t \geq 0$. A tentative to obtain explicit solution for (1.3) can be seen in Ebihara-Tanaka-Nakashina [7] and blow up for a perturbation of (1.4) in Bainov-Minchev [2].

• Let us consider (1.1) when $n = 1$. Thus $\Omega_t = (\alpha(t), \beta(t))$ are the deformations of a fixed interval $\Omega = (\alpha_0, \beta_0)$ by the function $K(t) = \dfrac{\gamma(t)}{\gamma_0}$, where $\gamma(t) = \beta(t) - \alpha(t)$ and $\gamma_0 = \beta_0 - \alpha_0$. In this case we obtain from (1.1)

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\tau_0}{m} + \frac{k}{m}\,\frac{\gamma(t) - \gamma_0}{\gamma_0} + \frac{k}{2m\gamma(t)} \int_{\alpha(t)}^{\beta(t)} \left(\frac{\partial u}{\partial x}\right)^2 dx\right) \frac{\partial^2 u}{\partial x^2} = 0. \quad (1.6)$$

The equation (1.6) is the model for small vertical vibrations of an stretched elastic string when the ends $\alpha_0 < \beta_0$ move to $\alpha(t) < \beta(t)$ after the time $t$, that is, we suppose

$$\alpha(t) \leq \alpha_0 < \beta_0 \leq \beta(t), \text{ for } t \geq 0.$$

This model can be seen in Medeiros-Limaco-Menezes [14].

• When $\{\Omega_t\}_{t\geq 0}$ are the deformations of a circle $\Omega$ of the plane $\mathbb{R}^2$ the model (1.1) reduces to

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\tau_0 - k}{m} + \frac{k}{m} K(t)^2 + \frac{k}{2mV_0\, K(t)^2} \int_{\Omega_t} |\nabla u(x,t)|^2\, dx\right) \Delta u = 0. \quad (1.7)$$

The mixed problem for (1.7) was investigated by Limaco-Medeiros [10] and Medeiros-Limaco [15], applying a technique which takes in consideration the geometry of $\Omega$.

In the present article we investigate a mixed problem for (1.1) in the general case when $\Omega_t = K(t)\Omega$ is a family of bounded open set of $\mathbb{R}^n$ with moving boundary $\Gamma_t$. $\square$

# 2 Mathematical Analysis

## 2.1 Preliminaries and Main Results

Let $\Omega$ be a bounded open set of $\mathbb{R}^n$, with $C^2$ boundary, and $K = K(t)$, a function $K\colon [0,\infty) \to [0,\infty)$. Represent by $\{\Omega_t\}$ the family of deformations of $\Omega$ by $K = K(t)$ defined by:

$$\Omega_t = \{x \in \mathbb{R}^n;\ x = K(t)y,\ \forall y \in \Omega,\ 0 \le t < \infty\}.$$

We consider the noncylindrical domain $\widehat{Q}$ of $\mathbb{R}^{n+1}$ defined by

$$\widehat{Q} = \bigcup_{t>0} (\Omega_t \times \{t\})$$

whose lateral boundary is denoted by $\widehat{\Sigma}$ defined by

$$\widehat{\Sigma} = \bigcup_{t>0} (\Gamma_t \times \{t\}),$$

where $\Gamma_t$ is the boundary of $\Omega_t$. We suppose $\Gamma_t$ and $\widehat{\Sigma}$ regulars.

HYPOTHESIS

- We suppose $\tau_0 > k$.
- $K \in W^{3,\infty}(0,\infty)$, $\quad K(t) \ge K_0 > 0$, $\quad K'(t) \ge 0$ and $|K''(t)| \le C_0|K'(t)|$, $\quad |K'''(t)| \le C_1\, K'(t)$.
  By $K'$ we represent $\dfrac{dK}{dt}\cdot$
  $C_0$, $C_1$ positive constants

- Represent by

$$S_0 = \underset{t>0}{\to} \text{Sup}\,\{K(t), K'(t), |K''(t)|, |K'''(t)|\},$$

finite number by hypothesis.

**Remark 2.1**. We have by embedding theorems

- $||v||_{H_0^1(\Omega)} \leq \tilde{C}_1 |\Delta v|_{L^2(\Omega)}$ for $v \in H_0^1(\Omega) \cap H^2(\Omega)$
- $||v||_{H^2(\Omega)} \leq \tilde{C}_2 |\Delta v|_{L^2(\Omega)}$
- $|v|_{L^2(\Omega)} \leq \tilde{C}_3 ||v||_{H_0^1(\Omega)}$

The constants $\tilde{C}_1$, $\tilde{C}_2$, $\tilde{C}_3$ are positives and we set $d = d(\Omega)$ to denote the diameter of $\Omega$.

We consider the operator

$$\widehat{L}u(x,t) = \frac{\partial^2 u}{\partial t^2} - \left(a(t) + \hat{b}(t) \int_{\Omega_t} |\nabla u(x,t)|^2 \, dx\right) \Delta u(x,t), \tag{2.1}$$

defined for real functions $u(x,t)$ for $(x,t) \in \widehat{Q}$.

Let us consider the change of variables $x = K(t)y$, $x \in \Omega_t$ and $y \in \Omega$ and let us define the function $v = v(y,t)$ by

$$v(y,t) = u(K(t)y, t). \tag{2.2}$$

We modify $\widehat{L}u(x,t)$ and we obtain

$$\begin{aligned} \check{L}v(y,t) = \frac{\partial^2 v}{\partial t^2} - \frac{1}{K(t)^2} \left(a(t) + b(t) \int_{\Omega} |\nabla v(y,t)|^2 \, dy\right) \Delta v(y,t) - \\ - \frac{\partial}{\partial y_i} \left(a_{ij}(y,t) \frac{\partial v}{\partial y_j}\right) + b_i(y,t) \frac{\partial v'}{\partial y_i} + c_i(y,t) \frac{\partial v}{\partial y_i}, \end{aligned} \tag{2.3}$$

where

- $a_{ij}(y,t) = -\left(\dfrac{K'}{K}\right)^2 y_i y_j$
- $b_i(y,t) = -2\left(\dfrac{K'}{K}\right) y_i$
- $\check{c}_i(y,t) = -\left[\dfrac{(n-1)(K')^2 + KK''}{K^2}\right] y_i$
- $a(t) = \dfrac{\tau_0 - k}{m} + \dfrac{k}{m} K(t)^n$
- $b(t) = \dfrac{k}{2mV_0 K(t)^2}$

For the computation see Appendix 1.

VISCOSITY

Our method consists in add a viscosity of the type $\delta u'(x,t)$, $\delta > 0$, to the operator $Lu(x,t)$ and restrict the initial data of the mixed problem to be proposed to $\widehat{L}u(x,y)$ in $\widehat{Q}$.

In fact, we consider in $\widehat{Q}$ the operator, for $\delta > 0$,

$$\widehat{L}u(x,t) + \delta u'(x,t), \quad \text{for} \quad \delta > 0, \tag{2.4}$$

where $u'(x,t)$ is the derivative with respect to $t$.

The change of variables gives

$$\delta u'(x,t) = \left(-\delta \frac{K'}{K}\right) y_i \frac{\partial v}{\partial y_i} + \delta v'(y,t),$$

and we modify the coefficient $\check{c}_i(y,t)$ obtaining

$$c_i(y,t) = \check{c}_i(y,t) - \delta\left(\frac{K'}{K}\right) y_i$$

or

$$c_i(y,t) = -\left[\frac{(n-1)(K')^2 + KK'' + \delta KK'}{K^2}\right] y_i\,.$$

Thus the change of variables and the viscosity $\delta u'(x,t)$ modify the operator $\check{L}v(y,t)$ and we obtain

$$Lv(y,t) = \frac{\partial^2 v}{\partial t^2} - \frac{1}{K(t)^2}\left(a(t) + b(t)\int_\Omega |\nabla v(y,t)|^2\, dy\right)\Delta v(y,t) - \quad (2.5)$$
$$- \frac{\partial}{\partial y_i}\left(a_{ij}(y,t)\,\frac{\partial v}{\partial y_j}\right) + b_i(y,t)\,\frac{\partial v'}{\partial y_i} + c_i(y,t)\,\frac{\partial v}{\partial y_i}\cdot$$

Consequently we have the following equivalent mixed problem (2.6) noncylindrical and (2.7) cylindrical

$$\left|\begin{array}{l} \widehat{L}u(x,t) + \delta u'(x,t) = 0 \quad \text{in} \quad \widehat{Q}, \quad \delta > 0 \\ u(x,t) = 0 \quad \text{on} \quad \widehat{\Sigma} \\ u(x,0) = u_0(x), \;\; u'(x,0) = u_1(x) \quad \text{in} \quad \Omega \end{array}\right. \quad (2.6)$$

$$\left|\begin{array}{l} Lv(y,t) + \delta v'(y,t) = 0 \quad \text{in} \quad Q = \Omega \times (0,\infty) \\ v(y,t) = 0 \quad \text{on} \quad \Sigma = \Gamma \times (0,\infty), \;\; \Gamma \text{ the boundary of } \Omega \\ v(y,0) = v_0(y), \;\; v'(y,0) = v_1(y) \quad \text{in} \quad \Omega \end{array}\right. \quad (2.7)$$

We investigate (2.7) by Galerkin method. First we fixe some notation.

In fact, we will find later, calculus of the Estimate IV, the function $G(t) = G(v(t))$, where $v = v(t)$ is the Galerkin approximation $v_\nu(t)$ for the solution of (2.7). This function is given by:

$$G(t) = \frac{|v''|^2}{2} + \frac{\delta^2}{16}\,|v'|^2 + \frac{\delta}{8}\,(v'',v') + \quad (2.8)$$
$$+ \left(\frac{\tau_0 - k}{m}\right)\frac{||v'||^2}{K^2} + \frac{\delta}{72\,S_0^2}\,(\nabla v', \nabla v) + \frac{\delta^2}{144\,S_0^2}\left(\frac{\tau_0 - k}{m}\right)||v||^2 +$$
$$+ \frac{k}{m}\,K^n\,\frac{||v'||^2}{K^2} + \frac{b(t)}{K^2}\,||v||^2\,||v'||^2 + a'(t,v,v) + H(t),$$

where $H(t) = H(v(t))$ is given by

$$H(t) = |v'|^2 + \frac{a(t)}{K^2}\,||v||^2 + \frac{b(t)}{2K^2}\,||v||^4 + a(t,v,v).$$

We will prove in Appendix 2 that $G(t) \geq \widetilde{G}_0(t)$ where

$$\widetilde{G}_0(t) = \frac{|v''|^2}{4} + \left(\frac{3\delta^2}{64} + 1\right)|v'|^2 + \left(\frac{8}{9}\,\frac{\delta^2}{288m\,S_0^2}\,\frac{\tau_0 - k}{2mS_0^2}\right)||v||^2 +$$

$$+\frac{1}{2}\,\frac{k}{m}\,K^{n-2}\,||v||^2 + \frac{b(t)}{K^2}\,||v||^2\,||v'||^2 + \frac{b(t)}{K^2}\,||v||^2.$$

Thus

$$\widetilde{G}_0(t) \leq G(t) \leq G_0(t) \tag{2.9}$$

where $G_0(t) = G_0(v(t))$ is given by

$$G_0(t) = \frac{9}{16}\,|v''|^2 + \left(\frac{\delta^2\tilde{C}_3^2}{16} + \frac{\delta}{16}\,\tilde{C}_3^2 + \frac{\tau_0 - k}{K_0^2} +\right.$$

$$\left.+\frac{\delta}{144\,S_0^2} + 1 + \frac{k}{m}\,S_0^{n-2} + \tilde{C}_3^2\right)||v'||^2 + \left(\frac{\delta}{144\,S_0^2} + \frac{\delta^2}{144\,S_0^2}\left(\frac{\tau_0 - k}{m}\right) +\right.$$

$$\left.+\left(\frac{(C_0+1)S_0^3}{K_0^2}\,d^3\right)^2 + \frac{a(t)}{K^2} + \frac{S_0^2}{K_0^2}\,d^2\right)||v||^2 + \frac{1}{2}\,\frac{b(t)}{K^2}\,||v||^4 +$$

$$+\frac{b(t)}{K^2}\,||v||^2\,||v'||^2 \tag{2.10}$$

The inequality (2.9) is fundamental for the conclusions contained in this work. In fact, when we obtain the estimate $G(t) \leq$ cst for all $t \geq 0$ then (2.9) implies $\widetilde{G}_0(t) \leq$ cst for all $t \geq 0$ what implies uniform estimates for $|v''(t)|$, $||v'(t)||$ and $||v(t)||$. The left hand side of (2.9) helps to obtain the exponential decay, cf. Section 3.

We need evaluate $G_0(t)$ at $t = 0$. It follows that we need evaluate $|v''(0)|^2$ with $v''(0) = v''_\nu(0)$ in the approximate equation (2.26). We obtain from the approximate equation (2.26)

$$|v''(0)|^2 \leq \frac{1}{K_0^2}\,(a(0) + b(0)||v_0||^2)\,|\Delta v_0|\,|v''(0)| + |a(0)v_0, v''(0))| +$$

$$+ \left|b_i(0)\,\frac{\partial v_1}{\partial y_i}\right|\,|v''(0)| + \left|c_i(0)\,\frac{\partial v_0}{\partial y_i}\right|\,|v''(0)|.$$

By the inequality $2\alpha\beta \leq \alpha^2 + \beta^2$, for positive numbers, we obtain:

$$|v''(0)|^2 \leq \left\{\left[\frac{\sqrt{5}}{K_0^2}\,(a(0) + b(0)||v_0||^2)\right]^2 + \left[\sqrt{5}\left(d^2\tilde{C}_3^2\right)^2\right]^2\right\}|\Delta v_0|^2 +$$

$$+ \left[\left(\sqrt{5}(n+1)d\right)^2 + \left(\frac{\sqrt{5}}{K_0^2}\,(n - 1 + \delta)S_0^2\right]\,||v_0||^2 + \left(\frac{\sqrt{5}\,S_0\,d}{K_0}\right)^2\,||v_1||^2 \tag{2.11}$$

For this method see Lions [12]. Then we have by (2.9) and (2.11) the inequality

$$0 \le G(0) \le \frac{9}{16}\left\{\left[\frac{\sqrt{5}}{K_0^2}\big(a(0)+b(0)||v_0||^2\big)\right]^2 + \right.$$

$$\left. + \big(\sqrt{5}(d^2\tilde{C}_2)^2\big)^2\right\}|\Delta v_0|^2 + \frac{9}{16}\left\{\sqrt{5}(n+1+d)\right)^2 +$$

$$+ \left(\frac{\sqrt{5}}{K_0^2}(n-1+d)S_0^2\right)^2\Bigg\}||v_0||^2 + \frac{9}{16}\left(\frac{\sqrt{5}\,S_0 d}{K_0}\right)^2||v_1||^2 +$$

$$+ \left(\frac{\delta^2\tilde{C}_3^2}{16} + \frac{\delta}{16}\tilde{C}_3^2 + \frac{\tau_0 - k}{m K_0^2} + \frac{\delta}{144\,S_0^2} + 1 + \frac{k}{m}\,S_0^{n-2} + \tilde{C}_3^2\right)||v_1||^2 +$$

$$+ \left[\frac{\delta}{144\,S_0^2} + \frac{\delta}{144\,S_0^2}\left(\frac{\tau_0 - k}{m}\right) + \left(\frac{(C_0+1)S_0^3 d^3}{K_0^3}\right)^2 + \frac{a(0)}{K_0^2} + \right.$$

$$\left. + \frac{S_0^2}{K_0^2}\,d^2\right]||v_0||^2 + \frac{b(0)}{K_0^2}\,||v_0||^2\,||v_1||^2 + \frac{3}{2}\,\frac{b(0)}{K_0^2}\,||v_0||^4 =$$

$$= A_0|\Delta v_0|^2 + B_0||v_0||^2 + C_0||v_1||^2 + \frac{b(0)}{K_0^2}\,||v_0||^2\,||v_1||^2 + \frac{3}{2}\,\frac{b(0)}{K_0^2}\,||v_0||^4. \tag{2.12}$$

By the change of variables $x = K(t)y$ from $\Omega$ to $\Omega_t$, we obtain

$$\begin{aligned} & A_0|\Delta v_0|^2 + B_0||v_0||^2 + C_0||v_1||^2 + \frac{b(0)}{2}\,||v_0||^2\,||v_1||^2 + \\ & + \frac{3}{2}\,\frac{b(0)}{K_0^2}\,||v_0||^4 \le \widehat{A}_0|\Delta u_0|^2 + \widehat{B}_0||u_0||^2 + \widehat{C}_0||u_1||^2 + \\ & + \widehat{D}_0||u_0||^2\,||u_1||^2 + \widehat{E}_0||u_0||^4 \end{aligned} \tag{2.13}$$

with $A_0$, $B_0$, $C_0$, $\widehat{A}_0$, $\widehat{B}_0$, $\widehat{C}_0$, $\widehat{D}_0$, $\widehat{E}_0$ constants.

We have the results

**Theorem 2.1.** Given $u_0 \in H_0^1(\Omega_0) \cap H^2(\Omega_0)$ and $u_1 \in H_0^1(\Omega_0)$ such that

$$\left(\widehat{A}_0|\Delta u_0|^2 + \widehat{B}_0||u_0||^2 + \widehat{C}_0||u_1||^2 + \widehat{D}_0||u_0||^2\,||u_1||^2 + \widehat{E}_0||u_0||^2\right)^2 \leq$$
$$\leq \frac{1}{(nS_0^2)^2\left[(144\sqrt{a})^2 + (\delta^2\sqrt{b})^2\right]} \cdot \frac{1}{288\,m^2}\,\delta^5(\tau_0 - k)^4, \tag{2.14}$$

there exists a unique numerical function $u\colon \widehat{Q} \to \mathbb{R}$ satisfying, for all $T > 0$:

$$u \in L^\infty(0,T; H_0^1(\Omega_t)) \cap L^2(0,T; H^2(\Omega_t)) \tag{2.15}$$

$$u' \in L^\infty(0,T; H_0^1(\Omega_t)) \tag{2.16}$$

$$u'' \in L^\infty(0,T; L^2(\Omega_t)), \tag{2.17}$$

which is solution of (2.7).

**Theorem 2.2.** Given $v_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ and $v_1 \in H_0^1(\Omega)$ such that

$$\left(A_0|\Delta v_0|^2 + B_0||v_0||^2 + C_0||v_1||^2 + \frac{b(0)}{2}\,||v_0||^2\,||v_1^2|| + \frac{3}{2}\,\frac{b(0)}{K_0^2}\,||v_0||^4\right)^2 \leq$$

$$\leq \frac{1}{(nS_0^2)^2[(144\sqrt{a})^2 + (\delta^2\sqrt{b})^2]}\,\frac{1}{288\,m^2}\,\delta^5(\tau_0 - k)^4, \tag{2.18}$$

there exists a unique real function $v\colon Q \to \mathbb{R}$ such that, for all $T > 0$

$$v \in L^\infty(0,T;H_0^1(\Omega)) \cap L^2(0,T;H^2(\Omega)) \tag{2.19}$$

$$v' \in L^\infty(0,T;H_0^1(\Omega)) \tag{2.20}$$

$$v'' \in L^\infty(0,T;L^2(\Omega)), \tag{2.21}$$

which is solution of (2.7).

## 2.2 Proof of the Theorems

We prove the Theorem 2.2, which is the cylindrical case. We employ Galerkin's method making use of a Hilbertian basis, cf. Brezis [4], of the spectral objects of the problem $((w_i, w)) = \lambda_i(w_i, w)$, $i = 1, 2, \ldots,$ for all $w \in H_0^1(\Omega)$. By $((\ ,\ ))$ we represent the scalar product in $H_0^1(\Omega)$ and $(\ ,\ )$ the scalar product in $L^2(\Omega)$. The correspondents norms are, respectively $\|\cdot\|$ and $|\cdot|$. We know that $w_i \in H_0^1(\Omega) \cap H^2(\Omega)$ for all $i = 1, 2, \ldots$ We represent by $V_\nu$ the $\nu$ dimensional subspace generated by the first $\nu$ vectors $w_i$.

The details will appear in another paper.

# 3 Asymptotic Behavior

We prove that

$$\widetilde{G}_0(t) \le G(0)\, e^{-ct}, \quad \text{for all} \quad t \ge 0$$

and $c$ positive constant.

# References

[1] A. Arosio and S. Spagnolo: *Global solutions to the Cauchy problem for a nonlinear hyperbolic equations.* Nonlinear Partial Differential Equations and their Applications. Collège de France Seminar, Vol. 6, edited by H. Brezis and J.L. Lions, Pitman, London, 1984.

[2] D. Bainov and E. Minchev: *Above estimates of the interval of existence of solutions of the nonlinear Kirchhoff equation.* Comptes Rendus de l'Academi Bulgare des Sciences, Tome 48, nº 5 (1995) 13-14.

[3] S. Bernstein: *Sur une classe d'équations fonctionelles aux derivées partielles.* Isv. SSSR Math. 4 (1940) 17-20.

[4] H. Brezis: *Analyse Fonctionelle (Théorie et Applications).* Masson, Paris, 1983.

[5] C.E. Carrier: *On the vibrations problem of elastic strings.* Q.J. Appl. Math. (1953) 151-165.

[6] R.W. Dickey: *Infinity system of semilinear oscillations equations related to the string.* Proc. of the A.M.S. Vol. 23 (1969) 459-468.

[7] Y. Ebihara, Y. Tanaka and Y. Nakashina: *On exact solutions of Kirchhoff quasilinear hyperbolic equations with absorving term.* Fukuoka University Science Repports. Vol. 22, Nº 2 (1992) 97-106.

[8] M. Hazoya and Y. Yamada: *On some nonlinear wave equations II - Global existence and energy decay of solutions.* J. Fac. Sci. (Univ. Tokyo), Vol. 38, Nº 2 (1991) 230-250.

[9] G. Kirchhoff: *Vorlesungen der Mechanik.* Tauber Leipzig, 7 (1883) 444.

[10] J. Limaco and L.A. Medeiros: *Vibrations of elastic membranes with moving boundaries.* (To appear in Nonlinear Analysis TMA).

[11] J. LIMACO and L.A. MEDEIROS: *Kirchhoff Carrier elastic strings in noncylindrical domains.* Portugaliae Mathematica, Vol. 56, Fasc. 4 (1999) 495-499.

[12] J.L. LIONS: *On some question in boundary value problems of mathematical physics.* Contemporary Development in Continuum Mechanics and Partial Differential Equations - Edit by G.M. de La Penha and L.A. Medeiros, North-Holland, Amsterdam, (1978) 285-346.

[13] J.L. LIONS: *Quelques méthodes de résolution des problèmes aux limites non-linéaires.* Dunod, Paris, 1969.

[14] L.A. MEDEIROS: J. LIMACO and S.B. MENEZES: *Vibrations of elastic strings* (Mathematical Aspects) - Part One and Part Two. (To appear in Journal of Computational Analysis and Applications).

[15] L.A. MEDEIROS and J. LIMACO: *On a model for vibrations of elastic membranes.* International Journal of Differential Equations and Applications, Vol. 2, Nº 2 (2001) 209-236.

[16] M. MILLA MIRANDA and L.P. SAN GIL JUTUCA: *Existence and boundary stabilization of solutions for Kirchhoff equation.* Comm. Partial Differential Equations, 24 (9 & 10) (1999) 1750-1800.

[17] S.I. POHOZHAEV: *Quasilinear hyperbolic equation of Kirchhoff type and conservation law.* Tr. Mosk. Energ. Inst. Moscow, Nº 201 (1974) 118-126.

[18] S.I. POHOZHAEV: *On a class of quasilinear hyperbolic equations.* Math. SSSR Sbornik, 25 (1975) 145-158.

[19] S.I. POHOZHAEV: *The Kirchhoff quasilinear hyperbolic equation.* Differential Equations, Vol. 21, Nº 1 (1985) 101-108 (in Russian).

**Resumen**

En este trabajo nosotros investigaremos un modelo del tipo Kirchhoff para vibraciones de un cuerpo elástico representado por un conjunto abierto acotado de $\mathbb{R}^n$ representado por $\Omega_t$ con la frontera $\Gamma_t$ que se mueve con el tiempo t. Con restricciones sobre la posición en reposo y la velocidad inicial probaremos existencia global y unicidad de las soluciones en una clase de espacios de Sobolev, para un problema mixto con condiciones nulas de Dirichlet en la frontera.

**Palabras Clave:** Ecuación de Kirchhoff, Vibraciones, frontera móvil, espacios de Sobolev.

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