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A LEMMA ON LIMITS OF ANALYTIC SETS

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Abstract

This paper is a small remark on the analyticity of a limit of analytic sets in a particular case: when the sets are complex discs.

Key words: *Analytic sets, limits of sets, complex discs.*

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1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : ||z|| < 1\}$ and $\mathbb{B} = \{z \in \mathbb{C}^{n-1} : ||z|| < 1\}$ where $n \geq 2$. Let M be a complex manifold of complex dimension n and let D be a subset of M homeomorphic to a disc. We say that D is a *singular disc* if for all $x \in D$ there exists a neighborhood $\mathcal{D}$ of x in D, and an injective holomorphic function $f : \mathbb{D} \to M$ such that $f(\mathbb{D}) = \mathcal{D}$. If $f'(0) = 0$ we say that x is a *singularity* of D, otherwise x is a *regular point* of D (this does not depend on f). The set S of singularities of D is discrete and closed in D and we have that $D \backslash S$ is a complex submanifold of M (Riemann surface).

Thus, if x is a regular point of D, there is a neighborhood U of x in M and holomorphic coordinates (w, z), $w \in \mathbb{B}$, $z \in \mathbb{D}$ on U such that $D \cap U$ is represented by $(w = 0)$. If D does not have singularities we say that it is a *regular disc*. In this case, by uniformization, there is a holomorphic map $f : E \to M$, where $E = \mathbb{D}$ or $\mathbb{C}$, such that f is a biholomorphism between E and D.

Example. Let $\mathcal{F}$ be a holomorphic foliation by curves on the complex manifold M and let $D \subset M$ be homeomorphic to a disc. If D is contained in a leaf of $\mathcal{F}$ then it is a regular disc.

We now state the principal result.

Lemma 1.1. *Let $F : \mathbb{D} \times [0,1] \to \mathbb{C}^n$ be a continuous map such that for all $t \in [0,1]$, the map $F(*, t) : \mathbb{D} \to \mathbb{C}^n$ is a homeomorphism onto its image. Thus, we have a continuous family of discs $D_t := F(\mathbb{D} \times \{t\})$. Suppose D_t is a regular disc for each $t > 0$. Then D_0 is a singular disc.*

Remark. Actually, we may only assume that $\mathcal{D}_t$ is a singular disc for all $t > 0$. Lemma 1.1 is used in [6] to prove a theorem of extension of topological equivalences of holomorphic foliations and, as a consecuence, the invariance of the algebraic multiplicity of an isolated singularity of a holomorphic foliation, by C^1 equivalences.

2 Preliminary Lemmas

We state and prove some lemmas used in the proof of Lemma 1.1.

Lemma 2.1. *Let* U *be a simply connected domain in the complex plane such that* ∂U *is a Jordan curve. Then any uniformization* $f : \mathbb{D} \to \mathrm{U}$ *extends as a homeomorphism between* $\overline{\mathbb{D}}$ *and* $\overline{\mathrm{U}}$.

Proof. See [1] p.310. □

Lemma 2.2. *For each* $k \in \mathbb{N}$, *let* $\phi_k : \mathbb{S}^1 \to \mathbb{S}^1$ *be a homeomorphism. Then* $\{\phi_k\}$ *has a subsequence which converges almost everywhere with respect to the Lebesgue measure in the circle.*

Proof. We give a sketch of the proof. By taking a subsequence we may assume that ϕ_k converges on a dense subset of $\mathbb{S}^1$. Let $\pi : \mathbb{R} \to \mathbb{S}^1$ be a covering. For each k, we may choose a lifting $f_k : \mathbb{R} \to \mathbb{R}$ of ϕ_k by π. Since ϕ_k is a homeomorphism, f_k is monotone and we may assume that f_k is increasing for all k. We may also assume that f_k converges on a dense subset R of $\mathbb{R}$. For all $y \in R$, we define $f(y) = \lim f_k(y)$. Observe that f is increasing, since so is f_k for all k. We extend f to $\mathbb{R}$ as

$$f(x) = \limsup_{y \in R, y < x} f(y).$$

It is not difficult to see that $f : \mathbb{R} \to \mathbb{R}$ is increasing. Then f is continuous on a set A of total measure. Now, it is not difficult to prove that for all $x \in A$, the sequence $f_k(x)$ converges to $f(x)$ and the lemma follows. □

Lemma 2.3. *Let* $\varphi : \mathbb{S}^1 \to \mathbb{C}$ *be a bounded measurable function. Suppose that* $\int_{\mathbb{S}^1} z^n \varphi(z) dz = 0$ *for all* $n \in \mathbb{Z} \backslash \{-1\}$. *Then, with respect to the Lebesgue measure in the circle,* φ *is almost everywhere equal to the constant* $\frac{1}{2\pi i} \int_{\mathbb{S}^1} \frac{\varphi(z)}{z} dz$.

Proof. Let $c = \frac{1}{2\pi i}\int_{\mathbb{S}^1} \frac{\varphi(z)}{z} dz$ and define $\phi := \varphi - c$. Then

$$\begin{aligned}
\int_{\mathbb{S}^1} \frac{\phi(z)}{z} dz &= \int_{\mathbb{S}^1} \frac{\varphi(z) - c}{z} dz = \int_{\mathbb{S}^1} \frac{\varphi(z)}{z} dz - \int_{\mathbb{S}^1} \frac{c}{z} dz \\
&= \int_{\mathbb{S}^1} \frac{\varphi(z)}{z} dz - c\int_{\mathbb{S}^1} \frac{dz}{z} \\
&= \int_{\mathbb{S}^1} \frac{\varphi(z)}{z} dz - (\frac{1}{2\pi i}\int_{\mathbb{S}^1} \frac{\varphi(z)}{z} dz)(2\pi i) \\
&= 0.
\end{aligned}$$

On the other hand, for any $n \neq -1$ we have that

$$\begin{aligned}
\int_{\mathbb{S}^1} z^n \phi(z) dz &= \int_{\mathbb{S}^1} z^n(\varphi(z) - c) dz = \int_{\mathbb{S}^1} z^n \varphi(z) dz - \int_{\mathbb{S}^1} z^n c dz \\
&= 0 - c\int_{\mathbb{S}^1} z^n dz \\
&= 0
\end{aligned}$$

because $z^n dz$ is an exact form whenever $n \neq -1$. Thus, we have that

$$\int_{\mathbb{S}^1} z^n \phi(z) dz = 0 \quad \text{for all} \quad n \in \mathbb{Z}. \tag{1}$$

We claim that $\int_{\mathbb{S}^1} f(z)\phi(z)dz = 0$ for all continuous function $f : \mathbb{S}^1 \to \mathbb{C}$.

By the Stone -Weierstrass approximation theorem (see [3]), f can be uniformly approximated by a sequence of functions $P_k = A_k + iB_k$, $k \in \mathbb{N}$, where $A_k, B_k : \mathbb{R}^2 \to \mathbb{R}$ are real polynomials. Let $z = x + iy$ and observe that $\bar{z} = 1/z$ if $z \in \mathbb{S}^1$. Then

$$\begin{aligned}
P_k(z) &= A_k(x,y) + iB_k(x,y) = A_k(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2}) + iB_k(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2}) \\
&= A_k(\frac{z+1/z}{2}, \frac{z-1/z}{2}) + iB_k(\frac{z+1/z}{2}, \frac{z-1/z}{2}).
\end{aligned}$$

Hence $P_k(z) = \sum a_j z^j$ where j runs on the integers, the sum is finite and the coefficients a_j depend only on the coefficients of A_k and B_k. Then

$$\int_{\mathbb{S}^1} P_k(z)\phi(z)dz = \int_{\mathbb{S}^1} (\sum_j a_j z^j)\phi(z)dz = \sum_j a_j \int_{\mathbb{S}^1} z^j \phi(z)dz = 0,$$

by (1). Since P_k converges uniformly to f and ϕ is bounded we have that

$$\int_{\mathbb{S}^1} P_k(z)\phi(z)dz \to \int_{\mathbb{S}^1} f(z)\phi(z)dz$$

as $k \to \infty$ and therefore $\int_{\mathbb{S}^1} f(z)\phi(z)dz = 0$.

Now, we take an uniformly bounded sequence of continuous functions $f_k : \mathbb{S}^1 \to \mathbb{C}$ which converges a.e. to $\bar{\phi}$. Then $\{f_k\phi\}$ is uniformly bounded and converges a.e. to $\bar{\phi}\phi = |\phi|^2$. Thus by the dominated convergence theorem we have that

$$\int_{\mathbb{S}^1} f_k(z)\phi(z)dz \to \int_{\mathbb{S}^1} |\phi(z)|^2 dz$$

as $k \to \infty$. Therefore $\int_{\mathbb{S}^1} |\phi(z)|^2 dz = 0$ and it follows that $\phi = 0$ almost everywhere. □

Lemma 2.4. *Let $\mathcal{D} \subset \mathbb{C}^n$ be a bounded set homeomorphic to a disc. Let $p \in \mathcal{D}$ and suppose that $\mathcal{D}\backslash\{p\}$ has a complex structure such that the inclusion $\mathcal{D}\backslash\{p\} \to \mathbb{C}^n$ is holomorphic. Then there is a holomorphic injective map $g : \mathbb{D} \to \mathbb{C}^n$ with $g(\mathbb{D}) = \mathcal{D}$, $g(0) = p$.*

Proof. Let A_r denote the annulus $\{z \in \mathbb{C},\ r < |z| < 1\}$ where $r \geq 0$. Since $\mathcal{D} \subset \mathbb{C}^n$ is bounded and $\mathcal{D}\backslash\{p\}$ is homeomorphic to an annulus we have (see [4]) that there exist a biholomorphism

$$g : A_r \to \mathcal{D}\backslash\{p\},$$

such that $g(z) \to p$ as $|z| \to r$. Take R with $r < R < 1$ and let Γ_r and Γ_R be denote the circles $|z| = r$ and $|z| = R$ respectively. For $r < |z| < R$

we have the formula

$$g(z) = \frac{1}{2\pi i}\int_{\Gamma_R}\frac{g(w)}{w-z}dw - \frac{1}{2\pi i}\int_{\Gamma_r}\frac{p}{w-z}dw,$$

since g extends continuously to Γ_r as $g|_{\Gamma_r} = p$. But the second integral is equal to zero because $p/(w-z)$ is holomorphic on the disc $|w| < r$, then

$$g(z) = \frac{1}{2\pi i}\int_{\Gamma_R}\frac{f(w)}{w-z}dw.$$

Therefore g extends to the disc $|z| < R$. Since g is not constant and $g|_{\Gamma_r} = p$ we necessarily have $r = 0$ and the lemma follows. □

3 Proof of Lemma 1.1

Let $p = F(x_0, 0)$ be any point in D_0. Let $\mathrm{U} \subset \mathbb{D}$ be a disc centered at x_0 and such that $\overline{\mathrm{U}} \subset \mathbb{D}$. Let $t_k > 0$ be such that $t_k \to 0$ as $k \to \infty$ an define $\mathcal{D}_k = F(\mathrm{U} \times \{t_k\})$. Clearly $\overline{\mathcal{D}}_k \subset \overline{D}_{t_k}$. By uniformization, D_{t_k} is equivalent to a subset of $\mathbb{C}$ and, since $\mathcal{D}_k$ is a proper subset of D_{t_k}, we have (again by uniformization) that $\mathcal{D}_k$ is holomorphically equivalent to the unitary disc $\mathbb{D}$. Then there is a holomorphic map $f_k : \mathbb{D} \to \mathbb{C}^n$ which is a biholomorphism between $\mathbb{D}$ and $\mathcal{D}_k$. If we think that D_{t_k} is a subset of $\mathbb{C}$, by applying Lemma 2.1, we conclude that f_k extends as a homeomorphism $f_k : \overline{\mathbb{D}} \to \overline{\mathcal{D}}_k$. We may assume that $f_k(0) = F(x_0, t_k)$ for all k; otherwise we compose f_k with a suitable Moebius transformation. Observe that $f_k(\mathbb{D})$ is contained in the compact set $F(\overline{\mathrm{U}} \times [0,1])$, hence $\{f_k\}$ is uniformly bounded and, by Montel's theorem, we can assume that f_k converges uniformly on compact sets to a holomorphic function $f : \mathbb{D} \to \mathbb{C}^n$. Note that $f(0) = p$, since

$$f(0) = \lim_{k\to\infty} f_k(0) = \lim_{k\to\infty} F(x_0, t_k) = F(x_0, 0) = p.$$

Let $\mathbb{S}^1 = \partial\mathbb{D}$ and consider for each k the homeomorphism

$$\varphi_k := f_k|_{\mathbb{S}^1} : \mathbb{S}^1 \to \partial\mathcal{D}_k.$$

Define

$$\pi : F(\partial \mathrm{U} \times [0,1]) \to \mathbb{S}^1$$

$$\pi(F(\zeta, t)) = \zeta.$$

Clearly, π maps $\partial\mathcal{D}_k = F(\partial \mathrm{U} \times \{t_k\})$ homeomorphically onto $\mathbb{S}^1$. Therefore for each k, the map

$$\phi_k := \pi \circ \varphi_k : \mathbb{S}^1 \to \mathbb{S}^1$$

is a homeomorphism. By taking a subsequence, we may assume that ϕ_k converges a.e. to a function $\phi : \mathbb{S}^1 \to \mathbb{S}^1$ (Lemma 2.2). Therefore φ_k converges a.e. to

$$\varphi := \pi^{-1} \circ \phi : \mathbb{S}^1 \to \partial\mathcal{D}_0.$$

Fix $x \in \mathbb{D}$. Since $\{\varphi_k\}$ is uniformly bounded, by the dominated convergence theorem we have that

$$\frac{1}{2\pi i} \int_{\mathbb{S}^1} \frac{\varphi_k(w)}{w - x} dw \to \frac{1}{2\pi i} \int_{\mathbb{S}^1} \frac{\varphi(w)}{w - x} dw \tag{2}$$

as $k \to \infty$. By the Cauchy's integral formula, the left part of (2) is equal to $f_k(x)$ and, since $f_k(x) \to f(x)$, we conclude that

$$f(x) = \frac{1}{2\pi i} \int_{\mathbb{S}^1} \frac{\varphi(w)}{w - x} dw. \tag{3}$$

Assertion 1. *$f : \mathbb{D} \to \mathbb{C}^n$ is not constant.*

Proof. Assume by contradiction that f is a constant function. Then $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$, where $f^{(n)}$ is the *nth* derivative of f. From (3), by induction on n, it is not difficult to prove that

$$f^{(n)}(0) = \frac{1}{2\pi i} \int_{\mathbb{S}^1} \frac{\varphi(w)}{w^{n+1}} dw = 0$$

for all $n \geq 1$. Hence

$$\int_{\mathbb{S}^1} w^n \varphi(w) dw = 0 \quad \text{for all} \quad n \leq -2. \tag{4}$$

On the other hand, for each k and any $n \geq 0$ we have that $\int_{\mathbb{S}^1} w^n \varphi_k(w) dw = 0$ because $w^n \varphi_k(w)$ extends holomorphically to $\mathbb{D}$ as $w^n f_k(w)$. Then, by the dominated convergence theorem we have that

$$\int_{\mathbb{S}^1} w^n \varphi(w) dw = \lim_{k \to \infty} \int_{\mathbb{S}^1} w^n \varphi_k(w) dw = 0 \quad \text{for all} \quad n \geq 0. \tag{5}$$

Thus, from (4) and (5):

$$\int_{\mathbb{S}^1} w^n \varphi(w) dw = 0 \quad \text{for all} \quad n \in \mathbb{Z} \backslash \{-1\}.$$

Applying Lemma 2.3 to each coordinate of φ we deduce that $\varphi = p$ a.e., which is a contradiction because $\varphi(\mathbb{S}^1) \subset \partial \mathcal{D}_0$ and $p \notin \partial \mathcal{D}_0$.

Assertion 2. $f(\mathbb{D}) \subset D_0$.

Proof. Let $z \in \mathbb{C}^n$ be such that $z = f(x)$. Then $z = \lim f_k(x)$. Since $f_k(x)$ is contained in $\mathcal{D}_k = F(\mathrm{U} \times \{t_k\})$, we have that $f_k(x) = F(x_k, t_k)$ with $x_k \in \mathrm{U}$. By taking a subsequence, we may assume that $x_k \to \bar{x} \in \overline{\mathrm{U}}$. Then

$$z = \lim_{k \to \infty} f_k(x) = \lim_{k \to \infty} F(x_k, t_k) = F(\lim_{k \to \infty} x_k, 0) = F(\bar{x}, 0) \in D_0.$$

Therefore $f(\mathbb{D}) \subset D_0$.

It follows from Assertion 1 that f' does not vanish. Then, we know that the zero set of f' is discrete and closed in $\mathbb{D}$. Hence, there exists a disc $\Omega \subset \mathbb{D}$ centered at 0 such that $f' \neq 0$ on $\Omega \backslash \{0\}$. Since f is not constant, 0 is an isolated point in $f^{-1}(p)$. Thus, we assume Ω to be small enough such that $\overline{\Omega} \cap f^{-1}(p) = \{0\}$. In particular, $f(\partial \Omega)$ does not pass through p.

Assertion 3. There is a disc $\mathcal{D} \subset D_0$ with $p \in \mathcal{D}$ and such that $\mathcal{D} \subset f(\Omega)$.

Proof. Let $x \in \Omega \backslash \{0\}$. Then $f'(x) \neq 0$ and there exists a disc $\Delta \subset \Omega$ with $x \in \Delta$ and such that $f|_{\overline{\Delta}} : \overline{\Delta} \to \mathbb{C}^n$ is injective, hence a homeomorphism onto its image, since $\overline{\Delta}$ is compact. Then $f(\Delta)$ is homeomorphic to a disc and, since $f(\Delta) \subset D_0$, we have that $f(\Delta)$ is open in D_0. Then $f(x)$ is an interior point of $f(\Omega)$ as a subset of D_0. It follows that every point $z \in f(\Omega \backslash \{0\})$ is an interior point of $f(\Omega) \subset D_0$. Thus if z is a point in the boundary of $f(\Omega)$, since z is not an interior point, we have that $z \notin f(\partial\Omega \cup \{0\}) = f(\partial\Omega) \cup \{p\}$. Therefore:

$$\partial f(\Omega) \subset f(\partial\Omega) \cup \{p\}.$$

Since $f(\partial\Omega)$ does not pass through p, we may take a disc $\mathcal{D} \subset D_0$ containing p and such that $\mathcal{D}$ is disjoint of $f(\partial\Omega)$. Finally, we claim that $\mathcal{D} \subset f(\Omega)$. Let $z \in \mathcal{D}$ and suppose that $z \notin f(\Omega)$. Since $\mathcal{D}$ contains p, we may take $x \neq 0$, close enough to 0, such that $z' := f(x) \in \mathcal{D}$. We have $z' \neq p$ because $x \neq 0$, hence we may take a path γ in $\mathcal{D} \backslash \{p\}$ connecting z and z'. Since $z \notin f(\Omega)$ and $z' \in f(\Omega)$, there exists $z'' \in \gamma$ such that $z'' \in \partial f(\Omega)$. Then, since $\partial f(\Omega) \subset f(\partial\Omega) \cup \{p\}$, we have $z'' \in f(\partial\Omega) \cup \{p\}$. But this is a contradiction because $z'' \in \gamma$ is contained in $\mathcal{D} \backslash \{p\}$, which is disjoint of $f(\partial\Omega) \cup \{p\}$.

Let $z \in \mathcal{D} \backslash \{p\}$. By Assertion 3, $z = f(x)$ with $x \in \Omega$. Since $z \neq p$ we have $x \neq 0$, hence $f'(x) \neq 0$. Then, from the proof of Assertion 3, there exists a disc $\Delta \subset \Omega$, $x \in \Delta$, such that $f|_{\Delta} : \Delta \to \mathbb{C}^n$ is injective and $f(\Delta)$ is a neighborhood of z in $\mathcal{D} \backslash \{p\}$. Since f is holomorphic it follows that $\mathcal{D} \backslash \{p\}$ is a Riemann surface and the inclusion $\mathcal{D} \backslash \{p\} \to \mathbb{C}^n$ is a holomorphic map. Then, by Lemma 2.4, there is a holomorphic injective map $g : \mathbb{D} \to \mathbb{C}^n$ with $g(\mathbb{D}) = \mathcal{D} \subset D_0$. Since p was arbitrary, it follows that D_0 is a singular disc, which finishes the proof of Lemma 1.1. □

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Resumen

Este artículo es una pequeña observación sobre la analiticidad de un límite de conjuntos analíticos, en un caso muy particular: cuando los conjuntos son discos complejos.

Palabras Clave: Conjuntos analíticos, límites de conjuntos, discos complejos.

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