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INVARIANT AND REVERSIBLE MEASURES FOR RANDOM WALKS ON $\mathbb{Z}$

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Abstract

In this expository paper we study the stationary measures of a stochastic process called nearest neighbor random walk on $\mathbb{Z}$, and further we describe conditions for these measures to have the stronger property of reversibility. We consider both the cases of symmetric and non-symmetric random walk.

Key words: *Random walk, invariant measure, reversible measure.*

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Introduction

Let p be a fixed number from the interval $[0,1]$, and consider a sequence of identically distributed random variables $\{B_n : n \geq 1\}$ with values in the set $\{-1,1\}$ such that for each $n \geq 1$, $\mathsf{P}(B_n = 1) = p$ and $\mathsf{P}(B_n = -1) = q := 1 - p$. Next we define

$$X_0 = 0, \qquad X_n = \sum_{k=1}^{n} B_k, \qquad \text{for } n \geq 1.$$

The family of random variables $X = \{X_n : n \geq 0\}$ is called *nearest neighbor random walk* on $\mathbb{Z}$. This description of random walk follows that presented in [4].

The stochastic process X thus defined is a time-homogeneous Markov chain with state space $\mathbb{Z}$ and transition probabilities given by

$$p_{ij} := \mathsf{P}\left(X_{n+1} = j \mid X_n = i\right) = \begin{cases} p & \text{if } j = i+1, \\ q & \text{if } j = i-1, \\ 0 & \text{otherwise.} \end{cases}$$

As with all Markov processes, it is important to study the stationary distributions of X, also called invariant measures, for many of the long-term properties of the process are linked to this notion ([2]). Although invariant measures are initially defined as *probability* measures connected in a certain way with the transition probabilities, there are situations in which we need to allow for measures of infinite mass to enter the scene. Therefore we will say that a measure $\pi = (\pi_i)_{i \in \mathbb{Z}}$ (not necessarily finite) is invariant for X if the following condition is satisfied

$$\pi_i = \sum_{j \in \mathbb{Z}} \pi_j p_{ji}, \qquad \text{for each } i \in \mathbb{Z}.$$

Because of the specific kind of transition probabilities of the random walk X, the above condition for stationarity reduces to

$$\pi_i = p\pi_{i-1} + q\pi_{i+1}, \qquad \text{for each } i \in \mathbb{Z}. \tag{1}$$

Now let us discuss the property of reversibility. Random walks (as any other kind of Markov processes) have the property that, conditional on the knowledge of its present state, its future values are independent of its past history ([1], [3]). This property is symmetrical in time, so at least from a purely theoretical point of view we may as well consider the random walk running backwards in time ([2]). In this setting, the random walk is called reversible when it admits an initial distribution with respect to which, on every time interval, the law of the process is the same when it is run backwards as when it is run forwards, see [4, p. 107].

An equivalent condition for reversibility in terms of the transition probabilities is presented in [2, Theorem 1.9.3], and we will take it as definition here. We will say that the measure $\pi = (\pi_i)_{i\in\mathbb{Z}}$ is reversible for X if the following equations hold:

$$\pi_i p_{ij} = \pi_j p_{ji}, \qquad \text{for all } i, j \in \mathbb{Z}. \tag{2}$$

In the literature these are known as the equations of *detailed balance* (see [1], [2]). Note that if $\pi = (\pi_i)_{i\in\mathbb{Z}}$ is reversible for X, then it is also invariant, since for every fixed $i \in \mathbb{Z}$, summing both sides of (2) over j we get

$$\pi_i = \sum_{j\in\mathbb{Z}} \pi_i p_{ij} = \sum_{j\in\mathbb{Z}} \pi_j p_{ji}.$$

Thus it is clear that for any given random walk X, the set of reversible measures is included in the set of invariant measures. In this paper we want to describe the invariant measures for random walk on $\mathbb{Z}$, and to study conditions for these measures to have the stronger property of reversibility.

The rest of the paper is organized as follows. In Section 1 we consider symmetric random walk on the integers, which is the case when $p = q = \frac{1}{2}$. We describe its stationary measures, and we show that in this case all such measures are also reversible. After dispatching two trivial cases, in Section 2 we consider non-symmetric random walk on the integers, *i.e.* the case $p \neq q$ with $0 < p < 1$. We describe the invariant measures, and we find conditions so that reversibility also holds.

1 Symmetric Random Walk on $\mathbb{Z}$

This is the case when $p = q = 1/2$. Note that in this case the transition probabilities can be written as

$$p_{ij} = \begin{cases} \frac{1}{2} & \text{if } |i-j| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let's start by describing the stationary measures in this case. For each $i \in \mathbb{Z}$ fixed, equation (1) gives

$$\frac{1}{2}\pi_i + \frac{1}{2}\pi_i = \frac{1}{2}\pi_{i-1} + \frac{1}{2}\pi_{i+1},$$

or equivalently

$$\pi_i - \pi_{i-1} = \pi_{i+1} - \pi_i.$$

This says that the sequence $(\pi_i)_{i\in\mathbb{Z}}$ has constant differences, which allows us to write $\pi_i = a + id$ for each $i \in \mathbb{Z}$; where a, d are constants. Since we want $\pi_i \geq 0$ for all $i \in \mathbb{Z}$, this forces $d = 0$ and $a \geq 0$. Thus the invariant measures are "uniform": $\pi_i = a$, $i \in \mathbb{Z}$ where $a \geq 0$ is a constant.

Now let us look at reversibility. Note that in this case equations (2) hold if and only if $p_{ij} = p_{ji}$ for all $i, j \in \mathbb{Z}$, which is clearly true by the symmetry in i and j of the definition of the transition probabilities. We see that symmetric random walk is reversible, and its reversible measure is a "uniform measure".

2 Non-symmetric Random Walk on $\mathbb{Z}$

Next we are going to consider non-symmetric random walk, *i.e.* the case when $p \neq q$. Let us first get rid of two trivial cases.

Case 1: $p = 1$ and $q = 0$. The transition probabilities are now

$$p_{ij} = \begin{cases} 1 & \text{if } j = i + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that in this case equations (1) hold if and only if for each $i \in \mathbb{Z}$ we have $\pi_i = \pi_{i-1}$. Thus any constant measure is invariant. Say the invariant measure is of the form $\pi_i \equiv c$, $\forall i \in \mathbb{Z}$; for some $c \geq 0$. For $c > 0$, equations (2) are valid if and only if $p_{ij} = p_{ji}$ for all $i, j \in \mathbb{Z}$; which is clearly false. Thus reversibility only holds for the zero measure when $c = 0$.

Case 2: $p = 0$ and $q = 1$. The transition probabilities are now

$$p_{ij} = \begin{cases} 1 & \text{if } j = i - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then equations (1) hold if and only if for each $i \in \mathbb{Z}$ we have $\pi_i = \pi_{i+1}$. Again in this case any constant measure $\pi_i \equiv c$ is invariant. As in the previous case, reversibility is not possible when $c > 0$ since $p_{ij} = p_{ji}$ is not true for at least one pair of indices $i, j \in \mathbb{Z}$.

Now we restrict our attention to non-symmetric random walk with $p, q \in (0, 1)$. As in the previous section, we start by describing the stationary measures. For each fixed $i \in \mathbb{Z}$, equation (1) gives

$$p\pi_i + q\pi_i = p\pi_{i-1} + q\pi_{i+1},$$

and rearranging the terms in this equality we get

$$p(\pi_i - \pi_{i-1}) = q(\pi_{i+1} - \pi_i), \qquad \text{or} \qquad \pi_{i+1} - \pi_i = \frac{p}{q}(\pi_i - \pi_{i-1}).$$

Let us designate $r = p/q$ and $y_i = \pi_i - \pi_{i-1}$. Thus for each $i \in \mathbb{Z}$ we have $y_{i+1} = ry_i$, so we get $y_i = y_0 r^i$. Then it is not difficult to see that the sequence π is given by

$$\pi_i = \pi_0 + y_0 \left(\frac{r - r^{i+1}}{1-r} \right), \qquad \forall i \in \mathbb{Z};$$

or equivalently

$$\pi_i = \left(\pi_0 + \frac{y_0 r}{1-r} \right) + \left(-\frac{y_0 r}{1-r} \right) r^i, \qquad \forall i \in \mathbb{Z}. \tag{3}$$

Note that $p \neq q$ implies $r \neq 1$, so the above expression makes sense. Also, since the value of r is completely determined by that of p; in order to determine the whole sequence π we need to fix two of its values: π_0 and π_{-1} (recall $y_0 = \pi_0 - \pi_{-1}$).

So far we have described the stationary measures as being of the form $\pi_i = a + br^i$, $\forall i \in \mathbb{Z}$; where

$$a = \pi_0 + \frac{y_0 r}{1-r} \qquad \text{and} \qquad b = -\frac{y_0 r}{1-r}.$$

A natural question arises: Under what circumstances is π a positive measure? Note that

$$\inf_{i \in \mathbb{Z}} \{a + br^i\} = \begin{cases} a & \text{if } b \geq 0, \\ -\infty & \text{if } b < 0. \end{cases}$$

Thus in order for π to be positive we must require that $a \geq 0$ and $b \geq 0$. To recap, all measures of the form $a + br^i, i \in Z$ with $a, b \geq 0$ are stationary measures for the non-symmetric random walk.

Next we proceed to find conditions for reversibility. Fix an arbitrary $i \in \mathbb{Z}$, we want conditions for (2) to hold for all $j \in \mathbb{Z}$. We will do this by cases.

First note that if $j \notin \{i-1, i, i+1\}$ then $p_{ij} = p_{ji} = 0$, and (2) holds trivially. Also, if $j = i$ then $p_{ij} = p_{ji} = 0$, and again (2) clearly holds. It remains only to consider the cases $j = i - 1$ and $j = i + 1$.

If $j = i - 1$, then $p_{ij} = q$ and $p_{ji} = p$; so that

$$\begin{aligned} \text{(2) holds} \quad &\Leftrightarrow \quad (a + br^i)q = (a + br^{i-1})p \\ &\Leftrightarrow \quad a + br^i = (a + br^{i-1})r \\ &\Leftrightarrow \quad a(1-r) = 0. \end{aligned}$$

Since we know $r \neq 1$, we see that (2) holds if and only if $a = 0$.

If $j = i + 1$, then $p_{ij} = p$ and $p_{ji} = q$; so that

$$\begin{aligned} \text{(2) holds} \quad &\Leftrightarrow \quad (a + br^i)p = (a + br^{i+1})q \\ &\Leftrightarrow \quad (a + br^i)r = a + br^{i+1} \\ &\Leftrightarrow \quad a(r-1) = 0. \end{aligned}$$

Again we see that (2) holds if and only if $a = 0$.

Thus a necessary and sufficient condition for this process to be reversible is that $a = 0$, or equivalently

$$\pi_0 = -\frac{y_0 r}{1-r}. \tag{4}$$

Recalling that $y_0 = \pi_0 - \pi_{-1}$, we can see that the above equation holds if and only if $\pi_0 = r\pi_{-1}$. This says that to get the whole measure we only need to fix a value for π_0. Also note that in this case $\pi_0 = -\frac{y_0 r}{1-r} = b$, so in the reversible case we have

$$\pi_i = \pi_0 r^i, \qquad \forall i \in \mathbb{Z}.$$

Unlike the symmetric case, we see that for non-symmetric random walks the reversible measures form a proper subset of the invariant measures.

References

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Resumen

En este artículo estudiamos las medidas estacionarias de un proceso estocástico llamado camino aleatorio en $\mathbb{Z}$, y además describimos condiciones para que dichas medidas tengan la propiedad adicional de reversibilidad. Consideramos ambos casos: caminos aleatorios simétricos y no simétricos.

Palabras Clave: camino aleatorio, medida invariante, medida reversible.

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