THE PENALTY METHOD AND BEAM EVOLUTION EQUATIONS

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Abstract

In this article, we present results concerning the existence of solutions for a beam evolution equation with variable coefficients in increasing noncylindrical domains.

Key words: *Beam evolution equation, existence, penalty method, noncylindrical domains.*

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1 Introduction

The main purpose of this paper will be to present results concerning the existence of solutions for the equation

$$a(x)u'' + \Delta(b(x)\Delta u) - \widehat{M}(x,t,\int_{\Omega} |\nabla u(x,t)|^2 dx)\Delta u + \delta u' = 0 \quad (1.1)$$

in a noncylindrical domain $\hat{Q}$.

The system (1.1) describes the problem of vertical flexion of fully clamped beams. For the cylindrical domains several mathematical aspects related with (1.1) were researched during the last years. We can mention the results of R.W.Dickey [4], J. Ball [1], for $n = 1$, they consider $a(x) = b(x) = 1$ and $M(x,t,\lambda) = P_0 + P_1\lambda$; J.G. Eiesley[5] studied the problem (1.1) for $n = 2$, $a(x) = b(x) = 1$ and $M(x,t,\lambda) = P_0 + P_1\lambda$. The equation (1.1) in an abstract framework, namely $u'' + A^2u - (P_0 + P_1M(|A^{\frac{1}{2}}u|^2))Au = 0$ was studied by L.A. Medeiros [10], D.C. Pereira [12], P. Biler [2] and E.H. Brito [3].

In noncylindrical domains the equation (1.1) was studied by J. Límaco, H.R. Clark and L.A. Medeiros [7], where they considered the function $M(x,t,\lambda)$ with the following properties:

- $M(x,t,\lambda) \geqslant 0$
- $|\nabla_x M| \leq C_1|\lambda|^p$
- $|\frac{\partial M}{\partial t}| \leq C_2|\lambda|^p$
- $|\frac{\partial M}{\partial \lambda}| \leq C_3|\lambda|^{p-1}$ for $p \geqslant 1$.

In this paper we generalize the result of J. Límaco et al. [7] considering the C^1 function M of three variables $x \in \Omega$, $t \geqslant 0$ and $\lambda \geqslant 0$, such that

$$0 \leq M(x,t,\lambda) \leq \phi(\lambda), \ | \nabla_x M| \leq C_1\sqrt{\lambda}\, f(\lambda),$$
$$|\tfrac{\partial M}{\partial t}| \leq C_2\, g(\lambda), \ |\tfrac{\partial M}{\partial \lambda}| \leq C_3\, h(\lambda), \tag{1.2}$$

where f, g, and h are increasing C^1 functions in $[0,\infty[$ and ϕ is an increasing and continuous function in $[0,\infty[$ with

$$\lim_{\lambda\to 0} g(\lambda) = \lim_{\lambda \to 0} \phi(\lambda) = 0.$$

We study the problem on an increasing noncylindrical domain $\widehat{Q}$, where Ω_s represents the section of $\widehat{Q} \cap \{t = s\}$, for $0 \leq s \leq T$, Γ_s is the boundary of Ω_s and the lateral boundary of $\widehat{Q}$ is given by $\sum_s = \bigcup\limits_{0<s<T} \Gamma_s$. The boundary of $\widehat{Q}$ is given by $\widehat{\partial Q} = \Omega_0 \bigcup \sum_+ \bigcup \Omega_T$. We suppose that $\Omega_0 \subset \Omega$ and $\widehat{Q}$ is contained in a cylinder $Q = \Omega \times [0,T]$. The function $\widehat{M}(x,t,\lambda)$ is a restriction of $M(x,t,\lambda)$ to $(x,t) \in \widehat{Q}$ and $\lambda > 0$, where $M(x,t,\lambda)$ is a function defined in the cylinder Q. We say that $\widehat{Q}$ is increasing, if Ω_s increase with s. The method we will employ to solve the problem is the penalty method idealized by J.L. Lions [8], [9].

2 Main Result

We consider the boundary value problem

$$\begin{aligned}
&a(x)u''(x,t) + \Delta(b(x)\Delta u(x,t)) \\
&-\widehat{M}\Big(x,t,\int_\Omega |\nabla u(x,t)|^2\,dx\Big)\Delta u(x,t) + \delta u'(x,t) = 0 && \text{in} \quad \widehat{Q}, \\
&u(x,t) = \frac{\partial u(x,t)}{\partial \nu} = 0 && \text{on} \quad \widehat{\Sigma}, \\
&u(x,0) = u_0(x), \ \ u'(x,0) = u_1(x) && \text{in} \quad \Omega_0.
\end{aligned} \tag{2.1}$$

Remark 2.1. *The functions $a(x)$ and $b(x)$ are defined in $\Omega \times [0,T]$. Let $\widehat{a}$ and $\widehat{b}$ restrictions of a and b to $\widehat{Q}$. In (2.1) we are considering $a = \widehat{a}$ and $b = \widehat{b}$.*

On the functions of system (2.1) we consider the following hypotheses:

$$\begin{aligned} & a(x),\ b(x) \in L^\infty(\Omega), \text{ such that} \\ & 0 < a_0 < a(x) < 1,\ 0 < b_0 < b(x) < b_1 \end{aligned} \tag{2.2}$$

and the function $M(x,t,\lambda)$ satisfying (1.2)

Definition 2.1. *The function $u : \widehat{Q} \to \mathbb{R}$ is a weak solution of (2.1), if $u \in L^2(0,T; H_0^2(\Omega_t))$, $u' \in L^2(0,T; L^2(\Omega_t))$ and*

$$-\int_0^T \int_{\Omega_t} a(x)\ u'(x,t)\phi'(x,t)\,dx\,dt + \int_0^T \int_{\Omega_t} b(x)\,\Delta u(x,t)\,\Delta\phi(x,t)\,dx\,dt$$

$$+ \int_0^T \int_{\Omega_t} \widehat{M}(x,t,\int_{\Omega_t} |\nabla u(x,t)|^2 dx)\ \nabla u(x,t)\cdot\nabla\phi(x,t)\,dx\,dt$$

$$+ \int_0^T \int_{\Omega_t} [\nabla_x \widehat{M}(x,t,\int_{\Omega_t} |\nabla u(x,t)|^2 dx)\cdot\nabla u(x,t)]\,\phi(x,t)\,dx\,dt$$

$$+ \delta \int_0^T \int_{\Omega_t} u'(x,t)\,\phi(x,t)\,dx\,dt = 0$$

for all $\phi \in L^2(0,T; H_0^2(\Omega_t))$, $\phi' \in L^2(0,T; L^2(\Omega_t))$, with $\phi(x,0) = \phi(x,T) = 0, \quad u(x,0) = u_0(x), \quad u'(x,0) = u_1(x)$ for all $x \in \Omega_0$.

Theorem 2.1. *Given $u_0 \in H_0^2(\Omega_0)$, $u_1 \in L^2(\Omega_0)$, and*

$$F(H_0) < \frac{\delta b_0}{5} \tag{2.3}$$

where F and H_0 are defined in (2.32) and (2.35) respectively. Then there exists at least one real function $u : \widehat{Q} \to \mathbb{R}$, weak solution of (2.1).

Proof. We will use the penalty method given by J.L. Lions [9] to transform the noncylindrical problem from $\widehat{Q}$ into a cylindrical problem in Q and then we will use the Faedo-Galerkin method.

We define the function

$$\chi \text{ by } \chi(x,t) = \begin{cases} 1 \text{ in } \Omega\times]0,T[\setminus \{\widehat{Q} \cup (\Omega_0 \times \{0\})\} \\ \\ 0 \text{ in } \widehat{Q} \cup (\Omega_0 \times \{0\}) \end{cases}$$

Let $\widetilde{u}_0$, $\widetilde{u}_1$ be the extension of $u_0, u_1$to Ω defined zero outside of $\Omega - \Omega_0$. Then $\widetilde{u}_0 \in H_0^2(\Omega)$ and $\widetilde{u}_1 \in L^2(\Omega)$. Thus, we have the following penalized problem:

Given $\varepsilon > 0$, we look for $u_\varepsilon : Q \to \mathbb{R}$, such that

$$\begin{aligned} &-\int_Q a(x)\ u_\varepsilon'(x,t)\phi'(x,t)\,dx\,dt + \int_Q b(x)\,\Delta u_\varepsilon(x,t)\,\Delta\phi(x,t)\,dx\,dt \\ &+\int_Q M(x,t,||u_\varepsilon(t)||^2)\ \nabla u_\varepsilon(x,t)\cdot\nabla\phi(x,t)\,dx\,dt \\ &+\int_Q [\nabla_x M(x,t,||u_\varepsilon(t)||^2)\cdot\nabla u_\varepsilon(x,t)]\,\phi(x,t)\,dx\,dt \\ &+\delta\int_Q u_\varepsilon'(x,t)\,\phi(x,t)\,dx\,dt + \frac{1}{\varepsilon}\int_Q \chi(x,t)\ u_\varepsilon'(x,t)\phi(x,t)\,dx\,dt \\ &+\frac{1}{\varepsilon}\int_Q \chi(x,t)\ \nabla u_\varepsilon(x,t)\cdot\nabla\phi(x,t)\,dx\,dt = 0, \end{aligned} \tag{2.4}$$

for all $\phi \in L^2(0,T;H_0^2(\Omega))$, $\phi' \in L^2(0,T;L^2(\Omega))$, with $\phi(x,0) = \phi(x,T) = 0$, and the initial conditions $u_\varepsilon(x,0) = \widetilde{u}_0(x)$, $u_\varepsilon'(x,0) = \widetilde{u}_1(x)$ for all $x \in \Omega$.

Note that the penalized problem is cylindrical, then we can employ Faedo-Galerkin approximate method with a hilbertian basis $(w_i)_{i\in N}$ of $H_0^2(\Omega)$, such that $w_1 = \widetilde{u}_0$. We denote by $V_m = [w_1, ..., w_m]$ the subspace of $H_0^2(\Omega)$, generated by $\{\widetilde{u}_0, w_2, ..., w_m\}$. We want to find $u_{\varepsilon m}(t) = \sum\limits_{i=1}^{m} g_{im}(t)\, w_i \in V_m$, solution of the following initial value problem:

$$
\begin{aligned}
&(a\ u''_{\varepsilon m}(t), w) + (b\,\Delta u_{\varepsilon m}(t), \Delta w)\\
&+(M(t, ||u_{\varepsilon m}(t)||^2)\ \nabla u_{\varepsilon m}(t), \nabla w)\\
&+(\nabla_x M(t, ||u_{\varepsilon m}(t)||^2) \cdot \nabla u_{\varepsilon m}(t), w)\\
&+\delta(\ u'_{\varepsilon m}(t), w)+\frac{1}{\varepsilon}(\chi(t)u'_{\varepsilon m}(t), w)\\
&+\frac{1}{\varepsilon}(\chi(t)\nabla u_{\varepsilon m}(t), \nabla w) = 0, \text{ for every } w \in V_m\\
&u_{\varepsilon m}(x,0) = u_{om} \to \widetilde{u}_0 \text{ in } H_0^2(\Omega)\\
&\text{and } u'_{\varepsilon m}(x,0) = u_{1m} \to \widetilde{u}_1 \text{ in } L^2(\Omega).
\end{aligned}
\tag{2.5}
$$

By the Caratheodory theorem, the problem (2.5) has local solution $u_{\varepsilon m}$ on some interval $[0, t_m[$ for each $\varepsilon > 0$ fixed. To extend the solutions to the interval $[0, T[$ for every $T > 0$ and to take to the limit as $m \to \infty$, we need a priori estimates.

First Estimate. Taking $w = u'_{\varepsilon m}(t)$ in (2.5), we have

$$
\begin{aligned}
&(a\ u''_{\varepsilon m}(t), u'_{\varepsilon m}(t)) + (b\,\Delta u_{\varepsilon m}(t), \Delta u'_{\varepsilon m}(t))\\
&+(M(t, ||u_{\varepsilon m}(t)||^2)\ \nabla u_{\varepsilon m}(t), \nabla u'_{\varepsilon m}(t))\\
&+(\nabla_x M(t, ||u_{\varepsilon m}(t)||^2) \cdot \nabla u_{\varepsilon m}(t), u'_{\varepsilon m}(t)) + \delta(\ u'_{\varepsilon m}(t), u'_{\varepsilon m}(t))\\
&+\frac{1}{\varepsilon}(\chi(t)u'_{\varepsilon m}(t), u'_{\varepsilon m}(t)) + \frac{1}{\varepsilon}(\chi(t)\nabla u_{\varepsilon m}(t), \nabla u'_{\varepsilon m}(t)) = 0
\end{aligned}
\tag{2.6}
$$

On the other hand, we have the following identities

$$(a\ u''_{\varepsilon m}(t), u'_{\varepsilon m}(t)) = \frac{1}{2}\frac{d}{dt}\int_{\Omega} a(x)\,|u'_{\varepsilon}(x,t)|^2\,dx \tag{2.7}$$

$$(b\ \Delta u_{\varepsilon m}(t), \Delta u'_{\varepsilon m}(t)) = \frac{1}{2}\frac{d}{dt}\int_{\Omega} b(x)\,|\Delta u_{\varepsilon m}(t)|^2\,dx \tag{2.8}$$

$$\begin{aligned}
&(M(t, \|u_{\varepsilon m}(t)\|^2)\ \nabla u_{\varepsilon m}(t), \nabla u'_{\varepsilon m}(t)) \\
&= \frac{1}{2}\frac{d}{dt}\int_{\Omega} M(x,t,\|u_{\varepsilon m}(t)\|^2)\,|\nabla u_{\varepsilon m}(t)|^2\,dx \\
&-\frac{1}{2}\int_{\Omega} \frac{\partial}{\partial t}M(x,t,\|u_{\varepsilon m}(t)\|^2)\,|\nabla u_{\varepsilon m}(t)|^2\,dx \\
&-\left[\int_{\Omega} \frac{\partial}{\partial \lambda}M(x,t,\|u_{\varepsilon m}(t)\|^2)\,|\nabla u_{\varepsilon m}(t)|^2\,dx\right](\nabla u_{\varepsilon m}(t), \nabla u'_{\varepsilon m}(t))
\end{aligned} \tag{2.9}$$

Using (1.2) and Cauchy-Schwarz inequality we have the following inequalities

$$\begin{aligned}
&\left|\int_{\Omega} [\nabla_x M(x,t,\|u_{\varepsilon m}(t)\|^2)\cdot \nabla u_{\varepsilon m}(t)]\ u'_{\varepsilon m}(x,t)\,dx\right| \\
&\le \int_{\Omega} |\nabla_x M(x,t,\|u_{\varepsilon m}(t)\|^2)|\ |\nabla u_{\varepsilon m}(t)|\ |\,u'_{\varepsilon m}(x,t)\,|dx \\
&\le C_1\|u_{\varepsilon m}(t)\|\,f(\|u_{\varepsilon m}(t)\|^2)\int_{\Omega} |\nabla u_{\varepsilon m}(t)|\ |\,u'_{\varepsilon m}(x,t)\,|dx \\
&\quad \le C_1 f(\|u_{\varepsilon m}(t)\|^2)\,\|u_{\varepsilon m}(t)\|^2|\,u'_{\varepsilon m}(x,t)\,|
\end{aligned} \tag{2.10}$$

Again, from (1.2) we obtain

$$\left|\frac{1}{2}\int_\Omega \frac{\partial}{\partial t}M(x,t,||u_{\varepsilon m}(t)||^2)\,|\nabla u_{\varepsilon m}(t)|^2\,dx\right| \le \frac{1}{2}C_2\,g(||u_{\varepsilon m}(t)||^2)\;||u_{\varepsilon m}(t)||^2 \tag{2.11}$$

$$\left|\int_\Omega \frac{\partial}{\partial \lambda}M(x,t,||u_{\varepsilon m}(t)||^2)\,|\nabla u_{\varepsilon m}(x,t)|^2\,dx\,(\nabla u_{\varepsilon m}(t),\nabla u'_{\varepsilon m}(t))\right| \le C_3\,h(||u_{\varepsilon m}(t)||^2)\;||u_{\varepsilon m}(t)||^2|\Delta u_{\varepsilon m}(t)|\,|u'_{\varepsilon m}(t)| \tag{2.12}$$

From (2.6) - (2.12), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \{a(x)\;|u'_{\varepsilon m}(t)|^2 + b(x)\,|\Delta u_{\varepsilon m}(x,t)|^2\\
&+M(x,t,||u_{\varepsilon m}(t)||^2)|\nabla u_{\varepsilon m}(t)|^2\}dx\\
&+\delta\,|u'_{\varepsilon m}(t)|^2 + \frac{1}{\varepsilon}|\chi(t)u'_{\varepsilon m}(t)|^2 + \frac{1}{\varepsilon}(\chi(t)\nabla u_{\varepsilon m}(t),\nabla u'_{\varepsilon m}(t))\\
&\le C_1 f(||u_{\varepsilon m}(t)||^2)\,||u_{\varepsilon m}(t)||^2|\,u'_{\varepsilon m}(x,t)\,|\\
&+\tfrac{1}{2}C_2\,g(||u_{\varepsilon m}(t)||^2)\;||u_{\varepsilon m}(t)||^2\\
&+C_3\,h(||u_{\varepsilon m}(t)||^2)\;||u_{\varepsilon m}(t)||^2|\Delta u_{\varepsilon m}(t)|\,|u'_{\varepsilon m}(t)|.
\end{aligned}$$

And using the inequality

$$||v|| \le C_0|\Delta v| \quad \text{for every } v\in H_0^2(\Omega) \tag{2.13}$$

we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \{a(x)\ |u'_{\varepsilon m}(t)|^2 + b(x)\,|\Delta u_{\varepsilon m}(x,t)|^2 \\
&+M(x,t,||u_{\varepsilon m}(t)||^2)\ |\nabla u_{\varepsilon m}(t)|^2\}dx \\
&+\delta\,|u'_{\varepsilon m}(t)|^2 + \frac{1}{\varepsilon}|\chi(t)\,u'_{\varepsilon m}(t)|^2 + \frac{1}{\varepsilon}(\chi(t)\nabla u_{\varepsilon m}(t), \nabla u'_{\varepsilon m}(t)) \\
&\leq C_0^2 C_1 f(||u_{\varepsilon m}(t)||^2)\ |\Delta u_{\varepsilon m}(t)|^2\,|u'_{\varepsilon m}(t)|^2 \\
&+\frac{C_0^2 C_2}{2}\,g(||u_{\varepsilon m}(t)||^2)\ |\Delta u_{\varepsilon m}(t)|^2 \\
&+C_0^2 C_3\,h(||u_{\varepsilon m}(t)||^2)\ |\Delta u_{\varepsilon m}(t)|^3\,|u'_{\varepsilon m}(t)|.
\end{aligned}
\tag{2.14}
$$

Second Estimate. Taking $w = u_{\varepsilon m}(t)$ in (2.5) we have:

$$
\begin{aligned}
&(a\ u''_{\varepsilon m}(t), u_{\varepsilon m}(t)) + (b\,\Delta u_{\varepsilon m}(t), \Delta u_{\varepsilon m}(t)) \\
&+(M(t,||u_{\varepsilon m}(t)||^2)\ \nabla u_{\varepsilon m}(t), \nabla u_{\varepsilon m}(t)) \\
&+(\nabla_x M(t,||u_{\varepsilon m}(t)||^2)\cdot \nabla u_{\varepsilon m}(t), u_{\varepsilon m}(t)) \\
&+\delta(\ u'_{\varepsilon m}(t), u_{\varepsilon m}(t)) + \frac{1}{\varepsilon}(\chi(t)u'_{\varepsilon m}(t), u_{\varepsilon m}(t)) \\
&+\frac{1}{\varepsilon}(\chi(t)\nabla u_{\varepsilon m}(t), \nabla u_{\varepsilon m}(t)) = 0
\end{aligned}
\tag{2.15}
$$

From (2.15), we get

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \{a(x)\; u'_{\varepsilon m}(x,t)\, u_{\varepsilon m}(t) + \frac{1}{2}\delta\, |u_{\varepsilon m}(x,t)|^2\}dx \\
&\quad - \int_\Omega a(x)|\; u'_{\varepsilon m}(x,t)|^2\, dx + \int_\Omega b(x)\, |\Delta u_{\varepsilon m}(x,t)|^2 dx \\
&\quad + \int_\Omega M(x,t,||u_{\varepsilon m}(t)||^2)\; |\nabla u_{\varepsilon m}(t)|^2 dx \qquad (2.16)\\
&\quad + \int_\Omega [\nabla_x M(x,t,||u_{\varepsilon m}(x,t)||^2)\cdot \nabla u_{\varepsilon m}(x,t)] u_{\varepsilon m}(x,t)\, dx \\
&+\frac{1}{\varepsilon}(\chi(t)\, u'_{\varepsilon m}(t), u_{\varepsilon m}(t)) + \frac{1}{\varepsilon}|\chi(t)\nabla u_{\varepsilon m}(t)|^2 = 0
\end{aligned}
$$

From (1.2), (2.13) and the Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
&\left|\int_\Omega [\nabla_x M(x,t,||u_{\varepsilon m}(x,t)||^2)\cdot \nabla u_{\varepsilon m}(x,t)] u_{\varepsilon m}(x,t)\, dx\right| \\
&\quad \le C_1 ||u_{\varepsilon m}(t)||\; f(||u_{\varepsilon m}(t)||^2) \int_\Omega |\nabla u_{\varepsilon m}(x,t)|\, |u_{\varepsilon m}(x,t)|\, dx \qquad (2.17)\\
&\quad \le C_1\; f(||u_{\varepsilon m}(t)||^2)\; ||u_{\varepsilon m}(t)||^2 |u_{\varepsilon m}(t)| \\
&\quad \le C_0^3 \widetilde{C}_0 C_1\; f(||u_{\varepsilon m}(t)||^2)\; |\Delta u_{\varepsilon m}(t)|^3
\end{aligned}
$$

where we have used the inequality

$$
|u_{\varepsilon m}(t)| \le \widetilde{C}_0\; ||u_{\varepsilon m}(t)||. \qquad (2.18)
$$

From (2.16) and (2.17) we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \{a(x)\; u'_{\varepsilon m}(x,t)\, u_{\varepsilon m}(t) + \frac{1}{2}\delta\, |u_{\varepsilon m}(x,t)|^2\}dx \\
&-\int_\Omega a(x)|\; u'_{\varepsilon m}(x,t)|^2\, dx + \int_\Omega b(x)\, |\Delta u_{\varepsilon m}(x,t)|^2 dx \\
&+\int_\Omega M(x,t,||u_{\varepsilon m}(t)||^2)\; |\nabla u_{\varepsilon m}(t)|^2 dx \\
&\quad +\frac{1}{\varepsilon}(\chi(t)\, u'_{\varepsilon m}(t), u_{\varepsilon m}(t)) + \frac{1}{\varepsilon}|\chi(t)\nabla u_{\varepsilon m}(t)|^2 \\
&\quad \leq C_0^3 \widetilde{C}_0 C_1\; f(||u_{\varepsilon m}(t)||^2)\, |\Delta u_{\varepsilon m}(t)|^3.
\end{aligned}
\tag{2.19}
$$

Multiplying (2.19) by $\dfrac{\delta}{4}$ and summing to (2.14), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \{a(x)|\; u'_{\varepsilon m}(x,t)|^2 + b(x)\, |\Delta u_{\varepsilon m}(x,t)|^2 \\
&+M(x,t,||u_{\varepsilon m}(t)||^2)\; |\nabla u_{\varepsilon m}(t)|^2 + \frac{\delta}{2}a(x)\; u'_{\varepsilon m}(x,t)\, u_{\varepsilon m}(t) \\
&+\frac{\delta^2}{4}|u_{\varepsilon m}(x,t)|^2\}\, dx - \frac{\delta}{4}\int_\Omega a(x)|\; u'_{\varepsilon m}(x,t)|^2\, dx \\
&+\frac{\delta}{4}\int_\Omega b(x)\, |\Delta u_{\varepsilon m}(x,t)|^2 dx \\
&+\frac{\delta}{4}\int_\Omega M(x,t,||u_{\varepsilon m}(t)||^2)\; |\nabla u_{\varepsilon m}(t)|^2 dx \\
&+\frac{1}{\varepsilon}|\chi(t)\, u'_{\varepsilon m}(t)|^2 + \frac{1}{\varepsilon}(\chi(t)\, \nabla u_{\varepsilon m}(t), \nabla u'_{\varepsilon m}(t)) \\
&+\frac{\delta}{4\varepsilon}(\chi(t)\, u'_{\varepsilon m}(t), u_{\varepsilon m}(t)) + \frac{\delta}{4\varepsilon}|\chi(t)\nabla u_{\varepsilon m}(t)|^2
\end{aligned}
\tag{2.20}
$$

$$\leq C_0^2 C_1 \ f(||u_{\varepsilon m}(t)||^2)\,|\Delta u_{\varepsilon m}(t)|^2 |u'_{\varepsilon m}(t)|$$

$$+\tfrac{1}{2} C_0^2 C_2 \ g(||u_{\varepsilon m}(t)||^2)\,|\Delta u_{\varepsilon m}(t)|^2$$

$$+C_0^2 C_3 \ h(||u_{\varepsilon m}(t)||^2)\,|\Delta u_{\varepsilon m}(t)|^3 |u'_{\varepsilon m}(t)|$$

$$+\tfrac{\delta}{4} C_0^3 \widetilde{C}_0 C_1 \ f(||u_{\varepsilon m}(t)||^2)\,|\Delta u_{\varepsilon m}(t)|^3.$$

Denoting by

$$\begin{aligned} H_m(t) = H_m(u_{\varepsilon m}(t)) &= \frac{1}{2}\int_\Omega \{a(x)|\ u'_{\varepsilon m}(x,t)|^2 \\ &+ b(x)\,|\Delta u_{\varepsilon m}(x,t)|^2 + M(x,t,||u_{\varepsilon m}(t)||^2)\ |\nabla u_{\varepsilon m}(t)|^2 \\ &+ \frac{\delta}{2} a(x)\ u'_{\varepsilon m}(x,t)\, u_{\varepsilon m}(x,t) + \frac{\delta^2}{4}|u_{\varepsilon m}(x,t)|^2\}\, dx \end{aligned} \tag{2.21}$$

and using the hypothesis $a(x) \leq 1$, we get

$$\begin{aligned} &|\frac{\delta}{2}\int_\Omega a(x)\ u'_{\varepsilon m}(x,t)\, u_{\varepsilon m}(x,t)\, dx| \\ &\leq \frac{\delta}{2}\int_\Omega a(x)\ |u'_{\varepsilon m}(x,t)|\,|\,u_{\varepsilon m}(x,t)|\, dx \\ &\leq \frac{1}{2}\int_\Omega a(x)\ |u'_{\varepsilon m}(x,t)|^2\, dx + \frac{\delta^2}{8}\int_\Omega |u_{\varepsilon m}(x,t)|^2\, dx. \end{aligned}$$

From this we have

$$\begin{aligned} &\frac{\delta}{2}\int_\Omega a(x)\ u'_{\varepsilon m}(x,t)\, u_{\varepsilon m}(x,t)\, dx \\ &\geqslant -\frac{1}{2}\int_\Omega a(x)\ |u'_{\varepsilon m}(x,t)|^2\, dx - \frac{\delta^2}{8}\int_\Omega |u_{\varepsilon m}(x,t)|^2\, dx. \end{aligned} \tag{2.22}$$

From (2.21) and (2.22) we get

$$\begin{aligned} H_m(t) \geqslant \int_\Omega \{ \frac{1}{4} a(x)\ |u'_{\varepsilon m}(x,t)|^2 + \frac{1}{2} b(x)\ |\Delta u_{\varepsilon m}(x,t)|^2 \\ + \frac{1}{2} M(x,t,||u_{\varepsilon m}(t)||^2)\ |\nabla u_{\varepsilon m}(t)|^2 \}\, dx. \end{aligned} \tag{2.23}$$

From (2.2) and (2.23) we can write

$$\begin{aligned} H_m(t) &\geqslant \int_\Omega \{ \frac{a_0}{4}\ |u'_{\varepsilon m}(x,t)|^2 + \frac{b_0}{2}\ |\Delta u_{\varepsilon m}(x,t)|^2 \\ &+ \frac{1}{2} M(x,t,||u_{\varepsilon m}(t)||^2)\ |\nabla u_{\varepsilon m}(t)|^2 \}\, dx \\ &\geqslant C_4 \{ |u'_{\varepsilon m}(x,t)|^2 + |\Delta u_{\varepsilon m}(x,t)|^2 \\ &+ \int_\Omega M(x,t,||u_{\varepsilon m}(t)||^2)\ |\nabla u_{\varepsilon m}(t)|^2\, dx \} \end{aligned}$$

where

$$C_4 = \min \left\{ \frac{a_0}{4}, \frac{b_0}{2},\ \frac{1}{2} \right\}. \tag{2.24}$$

From (2.20) and (2.21), we have

$$\begin{aligned} &\frac{d}{dt} H_m(t) + \delta\, |u'_{\varepsilon m}(x,t)|^2 - \frac{\delta}{4} \int_\Omega a(x)\ |u'_{\varepsilon m}(x,t)|^2\, dx \\ &+ \frac{\delta}{4} \int_\Omega b(x)\ |\Delta u_{\varepsilon m}(x,t)|^2\, dx \\ &+ \frac{\delta}{4} \int_\Omega M(x,t,||u_{\varepsilon m}(t)||^2)\ |\nabla u_{\varepsilon m}(t)|^2\, dx \\ &+ \frac{1}{\varepsilon} |\chi(t)\, u'_{\varepsilon m}(t)|^2 + \frac{1}{\varepsilon} (\chi(t)\, \nabla u_{\varepsilon m}(t), \nabla u'_{\varepsilon m}(t)) \\ &+ \frac{\delta}{4\varepsilon} (\chi(t)\, u'_{\varepsilon m}(t), u_{\varepsilon m}(t)) + \frac{\delta}{4\varepsilon} |\chi(t)\, \nabla u_{\varepsilon m}(t)|^2 \end{aligned}$$

$$
\begin{aligned}
&\leq C_0^2 C_1\, f(||u_{\varepsilon m}(t)||^2)\ |\nabla u_{\varepsilon m}(t)|^2\, |u'_{\varepsilon m}(t)|^2 \\
&+\frac{1}{2} C_0^2 C_2\ g(||u_{\varepsilon m}(t)||^2)\, |\Delta u_{\varepsilon m}(t)|^2 \\
&+C_0^2 C_3\ h(||u_{\varepsilon m}(t)||^2)\, |\Delta u_{\varepsilon m}(t)|^3 |u'_{\varepsilon m}(t)| \\
&+\frac{\delta}{4} C_0^3 \widetilde{C}_0 C_1\ f(||u_{\varepsilon m}(t)||^2)\, |\Delta u_{\varepsilon m}(t)|^3.
\end{aligned}
\tag{2.25}
$$

Using again the hypothesis $a(x) \leq 1$, we get

$$
\begin{aligned}
&\delta\, |u'_{\varepsilon m}(x,t)|^2 - \frac{\delta}{4} \int_\Omega a(x)\ |u'_{\varepsilon m}(x,t)|^2\, dx \\
&= \delta \int_\Omega \{\, |u'_{\varepsilon m}(x,t)|^2 - \frac{1}{4} a(x)\ |u'_{\varepsilon m}(x,t)|^2 \}\, dx \\
&= \frac{\delta}{4} \int_\Omega [\, 4 - a(x)]\ |u'_{\varepsilon m}(x,t)|^2\, dx \geqslant \frac{3\delta}{4}\ |u'_{\varepsilon m}(x,t)|^2 \geqslant 0
\end{aligned}
\tag{2.26}
$$

Thus, from (2.2), (2.25) and (2.26), we obtain

$$
\begin{aligned}
&\frac{d}{dt} H_m(t) + |\Delta u_{\varepsilon m}(t)|^2\, [\frac{\delta\, b_0}{2} - C_0^2 C_1\, f(||u_{\varepsilon m}(t)||^2)\ |u'_{\varepsilon m}(x,t)|^2 \\
&-\frac{C_0^2 C_2}{2}\ g(||u_{\varepsilon m}(t)||^2) - C_0^2 C_3\ h(||u_{\varepsilon m}(t)||^2)\, |\Delta u_{\varepsilon m}(t)|\, |u'_{\varepsilon m}(t)| \\
&-\frac{\delta}{4} C_0^3 \widetilde{C}_0 C_1\ f(||u_{\varepsilon m}(t)||^2)\, |\Delta u_{\varepsilon m}(t)|\,] + \frac{1}{\varepsilon} |\chi(t)\, u'_{\varepsilon m}(t)|^2 \\
&+\frac{1}{\varepsilon} (\chi(t)\, \nabla u_{\varepsilon m}(t), \nabla u'_{\varepsilon m}(t)) + \frac{\delta}{4\varepsilon} (\chi(t)\, u'_{\varepsilon m}(t), u_{\varepsilon m}(t)) \\
&+\frac{\delta}{4\varepsilon} |\chi(t)\ \nabla u_{\varepsilon m}(t)|^2 \leq 0.
\end{aligned}
\tag{2.27}
$$

We define

$$\gamma_m(t) = \gamma_m(u_{\varepsilon m}(t)) = C_0^2 C_1\, f(||u_{\varepsilon m}(t)||^2)|u'_{\varepsilon m}(t)|$$

$$+\frac{C_0^2 C_2}{2}\, g(||u_{\varepsilon m}(t)||^2) + C_0^2 C_3\, h(||u_{\varepsilon m}(t)||^2)\,|\Delta u_{\varepsilon m}(t)|\,|u'_{\varepsilon m}(t)| \quad (2.28)$$

$$+\frac{\delta}{4} C_0^3 \widetilde{C}_0 C_1\, f(||u_{\varepsilon m}(t)||^2)\,|\Delta u_{\varepsilon m}(t)|, \quad \text{for every } t \geqslant 0.$$

From (2.27) and (2.28), we get

$$\frac{d}{dt} H_m(t) + |\Delta u_{\varepsilon m}(t)|^2\,[\frac{\delta\, b_0}{4} - \gamma_m(t)] + \frac{1}{\varepsilon}|\chi(t)\, u'_{\varepsilon m}(t)|^2$$

$$+\frac{1}{\varepsilon}(\chi(t)\,\nabla u_{\varepsilon m}(t), \nabla u'_{\varepsilon m}(t)) \quad (2.29)$$

$$+\frac{\delta}{4\varepsilon}(\chi(t)\, u'_{\varepsilon m}(t), u_{\varepsilon m}(t)) + \frac{\delta}{4\varepsilon}|\chi(t)\,\nabla u_{\varepsilon m}(t)|^2 \leq 0,$$

for every $t \geqslant 0$.

From (2.24), we have

$$|u'_{\varepsilon m}(t)| \leq \frac{H_m(t)^{\frac{1}{2}}}{\sqrt{C_4}}, \quad |\Delta u_{\varepsilon m}(t)| \leq \frac{H_m(t)^{\frac{1}{2}}}{\sqrt{C_4}},$$

$$||u_{\varepsilon m}(t)|| \leq C_0|\Delta u_{\varepsilon m}(t)| \leq \frac{C_0}{\sqrt{C_4}} H_m(t)^{\frac{1}{2}} \quad (2.30)$$

From (2.28), (2.30), and since f, g, and h are increasing C^1 functions we have

$$\gamma_m(t) \leq \frac{C_0^2 C_1}{\sqrt{C_4}} f(\frac{C_0^2}{C_4} H_m(t))\, H_m(t)^{\frac{1}{2}} + \frac{C_0^2 C_2}{2} g(\frac{C_0^2}{C_4} H_m(t))$$

$$+\frac{C_0^2 C_3}{C_4} h(\frac{C_0^2}{C_4} H_m(t))\, H_m(t) + \frac{\delta C_0^3 \widetilde{C}_0 C_1}{4\sqrt{C_4}} f(\frac{C_0^2}{C_4} H_m(t))\, H_m(t)^{\frac{1}{2}}. \quad (2.31)$$

We define

$$F(s) = \frac{C_0^2 C_1}{\sqrt{C_4}} f(\frac{C_0^2}{C_4} s) + \frac{C_0^2 C_2}{2} g(\frac{C_0^2}{C_4} s) + \frac{C_0^2 C_3}{C_4} h(\frac{C_0^2}{C_4} s)\, s$$
$$+ \frac{\delta C_0^3 \widetilde{C}_0 C_1}{4\sqrt{C_4}} f(\frac{C_0^2}{C_4} s)\, s^{\frac{1}{2}}. \tag{2.32}$$

From (2.31) and (2.32), we obtain

$$\gamma_m(t) \leq F(H_m(t)) \quad \text{for every } t \geqslant 0 \tag{2.33}$$

On the other hand, from (2.2),(2.13), (2.18) and (2.21), we get

$$H_m(t) \leq \frac{1}{2}\ |u'_{\varepsilon m}(t)|^2 + \frac{b_1}{2}|\Delta u_{\varepsilon m}(t)|^2 + \frac{1}{2}\phi(||u_{\varepsilon m}(t)||^2)||u_{\varepsilon m}(t)||^2$$
$$+\frac{\delta}{4}\ |u'_{\varepsilon m}(x,t)|\ |u_{\varepsilon m}(t)| + \frac{\delta^2}{8}\ |u_{\varepsilon m}(t)|^2 \tag{2.34}$$
$$\leq C_5[\ |u'_{\varepsilon m}(t)|^2 + |\Delta u_{\varepsilon m}(t)|^2 + \phi(||u_{\varepsilon m}(t)||^2)||u_{\varepsilon m}(t)||^2]$$

where $C_5 = \max\left\{\frac{5}{8},\ \frac{b_1}{2} + \frac{\delta^2 C_0^2 \widetilde{C}_0^2}{4},\ \frac{1}{2}\right\}$.

We define

$$H_0 = C_5[|u_1|^2 + |\Delta u_0|^2 + \phi(||u_0||^2)||u_0||^2]. \tag{2.35}$$

From (2.34), we have

$$H_m(0) \leq C_5[|u_{1m}|^2 + |\Delta u_{0m}|^2 + \phi(||u_{0m}||^2)||u_{0m}||^2]. \tag{2.36}$$

Since F is increasing we get from (2.36)

$$K(H_m(0)) \leq K(C_5[|u_{1m}|^2 + |\Delta u_{0m}|^2 + \phi(||u_{0m}||^2)||u_{0m}||^2]). \tag{2.37}$$

Taking the limit in (2.37), as $m \to \infty$ and from (1.2) and (2.3) we obtain

$$\lim_{m\to\infty} F(H_m(0)) \leq F(H_0) < \frac{\delta\, b_0}{5}.$$

Then there exists $m_0 \in \mathbb{Z}^+$, such that

$$F(H_m(0)) < \frac{\delta\, b_0}{5} \quad \text{for every } m \geqslant m_0 \tag{2.38}$$

For the rest of this article we shall assume that the Galerkin approximation satisfy $m \geqslant m_0$.

Now we are going to prove that

$$\gamma(t) \leq \frac{\delta\, b_0}{4} \quad \text{for every } t \geqslant 0 \tag{2.39}$$

Otherwise we have $\gamma(t) > \dfrac{\delta\, b_0}{4}$ for some t_0, and thus we will prove that this hypothesis implies a contradiction. In fact from (2.33) and (2.38) we have

$$\gamma_m(0) < F(H_m(0) < \frac{\delta\, b_0}{5} \tag{2.40}$$

As the function $\gamma(t)$ is continuous, for $t > 0$ there exists $t^* > 0$, such that $t^* = \min\{t > 0;\ \gamma(t) = \dfrac{\delta\, b_0}{4}\}$, such that

$$\gamma_m(t) < \frac{\delta\, b_0}{4} \qquad \text{for every } 0 \leq t < t^* \text{ and } \gamma_m(t^*) = \frac{\delta\, b_0}{5} \tag{2.41}$$

Integrating (2.29) from 0 to t^*, we get

$$\begin{aligned} H_m(t^*) &+ \frac{1}{\varepsilon}\int_0^{t^*} (\chi(t)\, \nabla u_{\varepsilon m}(t), \nabla u'_{\varepsilon m}(t))\, dt \\ &+ \frac{\delta}{4}\int_0^{t^*} (\chi(t)\, u'_{\varepsilon m}(t), u_{\varepsilon m}(t))\, dx \leq H_m(0). \end{aligned} \tag{2.42}$$

From the Lemma of Nakao and Narazaki [11], we have

$$\begin{aligned}\int_0^{t^*}(\chi(t)\,\nabla u_{\varepsilon m}(t),\nabla u'_{\varepsilon m}(t))\,dt &\geqslant |\chi(t^*)\,u_{em}(t)|^2-|\chi\,\widetilde{u}_0|^2\\ \int_0^{t^*}(\chi(t)\,u'_{\varepsilon m}(t),u_{\varepsilon m}(t))\,dt &\geqslant |\chi(t^*)\nabla\,u_{em}(t)|^2-|\chi\,\nabla\widetilde{u}_0|^2.\end{aligned} \tag{2.43}$$

But from the definition of $\widetilde{u}_0$ and $\chi(x,t)$ we have

$$\begin{aligned}|\chi\,\widetilde{u}_0|^2 &= \int_\Omega \chi(x,0)\,\widetilde{u}_0(x)\,dx=0\\ |\chi\,\nabla\widetilde{u}_0|^2 &= \int_\Omega \chi(x,0)\nabla\,\widetilde{u}_0(x)\,dx=0.\end{aligned} \tag{2.44}$$

From (2.42), (2.43) and (2.44) we obtain

$$H_m(t^*)\le H_m(0). \tag{2.45}$$

From (2.33), (2.40) and (2.45) we get

$$\gamma_m(t^*)\le F(H_m(0))<\frac{\delta\,b_0}{5}<\frac{\delta\,b_0}{4}$$

which is a contradiction with (2.41).

Therefore,

$$0<\gamma_m(t)<\frac{\delta\,b_0}{4}\ \text{for every}\ t\geqslant 0 \tag{2.46}$$

From (2.29), (2.35) and (2.39), it follows that

$$H_m(t)\le H_0\ \text{for every}\ t\geqslant 0$$

Then

$$|u'_{\varepsilon m}(t)|^2+|\Delta u_{\varepsilon m}(t)|^2\le C,\ \text{for every}\ t\geqslant 0,\ \text{for every}\ \varepsilon>0 \tag{2.47}$$

From (2.47), we can obtain a subsequence, still represented by $(u_{\varepsilon m})_{m\in\mathbb{N}}$, such that

$$u_{\varepsilon m} \rightharpoonup u_\varepsilon \text{ weak star in } L^\infty(0,T;H_0^2(\Omega)), \text{ as } m \to \infty, \tag{2.48}$$

$$u'_{\varepsilon m} \rightharpoonup u'_\varepsilon \text{ weak star in } L^\infty(0,T;L^2(\Omega)), \text{ as } m \to \infty. \tag{2.49}$$

From (2.47) and the Aubin-Lions Theorem, [8], we obtain a subsequence, still represented by $(u_{\varepsilon m})_{m\in\mathbb{N}}$, such that

$$u_{\varepsilon m} \to u_\varepsilon \text{ strongly in } L^2(0,T;H_0^1(\Omega)), \text{ as } m \to \infty, \tag{2.50}$$

We also have

$$\chi\, u'_{\varepsilon m} \rightharpoonup \chi\, u'_\varepsilon \text{ weak star in } L^\infty(0,T;L^2(\Omega)), \text{ as } m \to \infty, \tag{2.51}$$

$$\chi\, \nabla u_{\varepsilon m} \rightharpoonup \chi\, \nabla u_\varepsilon \text{ weak star in } L^\infty(0,T;L^2(\Omega)), \text{ as } m \to \infty, \tag{2.52}$$

From (2.6)and using (2.48)-(2.52), we can take the limit as $m \to \infty$ obtaining a weak solution of the penalized problem (2.4).

From (2.48), (2.49), (2.50) and the Banach-Steinhaus Theorem, we obtain a net $(u_\varepsilon)_{0<\varepsilon<1}$ and a function $w : Q \to \mathbb{R}$, such that

$$u_\varepsilon \rightharpoonup w \text{ weak star in } L^\infty(0,T;H_0^2(\Omega)), \text{ as } \varepsilon \to 0 \tag{2.53}$$

$$u'_\varepsilon \rightharpoonup w' \text{ weak star in } L^\infty(0,T;L^2(\Omega)), \text{ as } \varepsilon \to 0 \tag{2.54}$$

$$u_\varepsilon \to w \text{ strongly in } L^2(0,T;H_0^1(\Omega)), \text{ as } \varepsilon \to 0 \text{ and a.e. in } Q. \tag{2.55}$$

Integrating (2.29) from 0 to T, we get

$$\int_Q \chi(x,t)\, |u'_{\varepsilon m}(x,t)|^2 dx\, dt \leq \varepsilon\, C \tag{2.56}$$

$$\int_Q \chi(x,t)\,|\nabla u_{\varepsilon m}(x,t)|^2 dx\,dt \le \varepsilon\, C \tag{2.57}$$

where $C = (1 + \frac{4}{\delta})H_0$ is a real positive constant independent of ε and m.

From (2.51) and (2.56) we have

$$|\chi\, u'_\varepsilon|_{L^2(0,T;L^2(\Omega))} \le \underline{\lim}\, |\chi\, u'_{\varepsilon m}|_{L^2(0,T;L^2(\Omega))} \le \varepsilon\, C$$

Then,

$$\chi u'_\varepsilon \to 0 \text{ strongly in } L^2(0,T;L^2(\Omega)),\ \text{ as } \varepsilon \to 0 \tag{2.58}$$

From (2.54) and (2.58) we obtain

$$\int_Q \chi(x,t)\,|w'(x,t)|^2 dx\,dt = 0 \tag{2.59}$$

By similar argument, we obtain from (2.52) and (2.56), that

$$\int_Q \chi(x,t)\,|\nabla w(x,t)|^2 dx\,dt = 0. \tag{2.60}$$

From (2.59), we have

$$\chi(x,t)\, w'(x,t) = 0 \text{ a.e. in } Q \tag{2.61}$$

and then

$$w'(x,t) = 0 \text{ in } Q - \widehat{Q} \cup \Omega_0 \times \{0\} \tag{2.62}$$

But $\widehat{Q}$ is increasing, then we get

$$\int_0^t w'(x,s)\,ds = 0 \text{ for every } 0 < t < T, \quad x \in \Omega - \Omega_0. \tag{2.63}$$

Since $w(x,0) = \widetilde{u}_0 = 0$ in $\Omega - \Omega_0$, we have

$$w(x,t) = 0 \text{ a.e. in } Q - \widehat{Q} \cup \Omega_0 \times \{0\}. \tag{2.64}$$

From (2.60) we obtain

$$\chi(x,t)\,|\nabla w(x,t)| = 0 \text{ a.e. in } Q - \widehat{Q} \cup \Omega_0 \times \{0\} \tag{2.65}$$

and then implies that

$$\frac{\partial w}{\partial x_i}(x,t) = 0 \text{ a.e. in } Q - \widehat{Q} \cup \Omega_0 \times \{0\} \tag{2.66}$$

Let u be the restriction of w to $\widehat{Q}$. From (2.64) and (2.66), we have that

$$u \in L^\infty(0,T; H_0^2(\Omega_t)) \tag{2.67}$$

From (2.54) and (2.62) we get

$$u' \in L^\infty(0,T; L^2(\Omega_t)) \tag{2.68}$$

Taking $\phi \in L^2(0,T; H_0^2(\Omega_t))$ with $\phi' \in L^2(0,T; L^2(\Omega_t))$ in (2.4), the penalized terms are null. From the convergence (2.53), (2.54), (2.55) and passing to the limit as $\varepsilon \to 0$, we obtain the weak solution of Theorem 2.1.

References

[1] J. Ball, *Initial boundary value problem for an extensible beam.* Journal of Mathematical Analysis and Aplications; **42** (1973), 61-90.

[2] P. Biler, *Remark on the decay for damped string and beam equations.* Nonlinear Analysis, TMA; **10** (1986), 839-842.

[3] E. H. Brito, *Decay estimates for generalized damped extensible string and beam equations.* Nonlinear Analysis, TMA; **8** (1984), 1489-1496.

[4] R. W. Dickey, *Free vibrations and dynamic buckling of the extensible beam.* Journal of Mathematical Analysis and Aplications; **29** (1970), 443-454.

[5] J. G. Eiesley, *Nonlinear vibrations of beams and rectangular plates.* Zeitschrift fur Angewandle Mathematik und Physik; **15** (1964), 167-175.

[6] J. Límaco, H.R. Clark and A.J. Feitosa, *Beam evolution equation with variable coefficient* Math. Meth. Appl. Sci. n^o **28** (2005), 457-478.

[7] J. Límaco, H.R. Clark and L.A. Medeiros, *Remarks on nonlinear biharmonic evolution equation of Kirchhoff type in noncylindrical domain.* International Journal of Mathematics and Mathematical Sciences; n^o **32** (2003), 2035-2052.

[8] J.L. Lions, *Quelques Méthodes d'Resolution des Problèmes aux Limites Non Linaires.* Dunod: Paris, France, 1969.

[9] J.L. Lions, *Une remarque sur les problèmes d' èvolution linéaires dans des domaines non cylindriques.* Revue Roumaine de Math. Pures et Appliquèes; V. **9**, (1964), 11-18.

[10] L. A. Medeiros, *On a new class of nonlinear wave equations.* Journal of Mathematical Analysis and Applications; **69** (1979), 252-262.

[11] M. Nakao and Narazaki, *Existence and Decay of solutions of some nonlinear wave equations in noncylindrical domains.* Math. Rep. Kyushu Univ. **11** no. 2, (1978), 117-125.

[12] D. C. Pereira, *Existence, uniqueness and asymtotic behavior for solutions of the nonlinear beam equation.* Nonlinear Analysis; **8** (1990), 613-623.

Resumen

En este artículo presentamos resultados relacionados con la existencia de soluciones para una ecuación de evolución de una viga, con coeficientes variables en dominios no cilíndricos.

Key words: Ecuación de evolución de una viga, existencia, método de penalidad, dominios no cilíndricos.

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