A STUDY ON A CALCULUS FOR THE $T_{k,x,y,z}-$ OPERATOR

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Abstract

The present paper deals with the calculus of $T_{k,x,y,z}$ - operator. The operator is a three variable analogue of the operator given earlier by W. A. Al-Salam [1] *and H. B. Mittal* [10]. *The operator is useful for finding operational representations and generating functions of polynomials of three variables and will be dealt in a separate communication.*

Key words: *Differential operators, special functions, generating functions.*

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1 Introduction

In 1964, W. A. Al- Salam [1] defined and studied the properties and applications of the operator

$$\theta = x(1 + xD), D \equiv \frac{d}{dx}. \tag{1.1}$$

He used this operator very elegantly to derive and generalize some known formulae involving some of the classical orthogonal polynomials. He also gave a member of new results and obtained operational representations of the Laguerre, Jacobi, Legendre and other polynomials.

In 1971, H. B. Mittal [10] generalized the operator θ by means of the relation

$$T_k \equiv x(k + xD), D \equiv \frac{d}{dx} \tag{1.2}$$

where k is a constant. He used this operator to obtain results for generalized Laguerre, Jacobi and other polynomials.

In 1992, M. A. Khan [4], the first author of the present paper, introduced q-extension of the operator (1.2) by means of the following relation:

$$T_{k,q,x} \equiv x(1-q)\{[k] + q^k x D_{q,x}\} \tag{1.3}$$

where k is a constant, $\mid q \mid < \mid$. $[k]$ is a q-number and $D_{q,x}$ is a q-derivative with respect to x. Letting $q \to 1$, (1.3) reduces to (1.2).

The paper [4] is a study of a calculus for the $T_{k,q,x}$-operator. Later, in a series of papers M.A. Khan [5, 6, 7] used this operator to obtain operational generating formulae for q-Laguerre, q-Bessel, q-Jacobi and other q-polynomials. In particular, various generating functions and recurrence relations were obtained for q-Laguerre polynomials.

Quite recently in 2004, M. A. Khan and M. P. Singh [8] defined a two variable analogue of the operator (1.2) by means of the following relation:

$$T_{k,x,y} \equiv xy\left\{k + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right\} \tag{1.4}$$

and studied its calculus.

The present papers deals with the calculus of the three variable analogues of the operators (1.1) and (1.2). The aim is to use these operators and the results of this paper to obtain various operational representations, generating functions and recurrence relations of the three variable polynomials in other separate communications.

2 The $T_{k,x,y,z}-$ Operator and its Properties

We define the $T_{k,x,y,z}-$ operator by means of the following relation:

$$T_{k,x,y,z} \equiv xyz\left\{k + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right\} \tag{2.1}$$

which is a three variable analogue of (1.2). For $k = 1$, (2.1) reduces to

$$T_{1,x,y,z} \equiv xyz\left\{1 + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right\} \tag{2.2}$$

which is a three variable analogue of (1.1). It is easy to verify that

$$\begin{aligned} &T^n_{x,y,z}\{x^{\alpha+r}y^{\beta+s}z^{\gamma+t}\} \\ &= 3^n \left(\tfrac{k+\alpha+\beta+\gamma+r+s+t}{3}\right)_n x^{\alpha+r+n}y^{\beta+s+n}z^{\gamma+t+n} \end{aligned} \tag{2.3}$$

where r, s and t are integers, n a positive integer and α, β and γ are arbitrary.

It can easily be seen by induction

$$T^n_{1,x,y,z} \equiv x^n y^n z^n \prod_{j=0}^{n-1} (\delta_1 + \delta_2 + \delta_3 + k + 3j) \tag{2.4}$$

where $\delta_1 \equiv x\dfrac{\partial}{\partial x}$, $\delta_2 \equiv y\dfrac{\partial}{\partial y}$, $\delta_3 \equiv z\dfrac{\partial}{\partial z}$.

Let $F(x)$ be a function which has a Taylor's series expansion, then we have the formal shift rules:

$$\begin{aligned} &F(T_{k,x,y,z})\ \{x^\alpha y^\beta z^\gamma\ f(x,y,z)\} \\ &= x^\alpha y^\beta z^\gamma\, F[T_{k,x,y,z} + (\alpha+\beta+\gamma)\ xyz]\ f(x,y,z) \end{aligned} \tag{2.5}$$

$$\begin{aligned} &F(T_{k,x,y,z})\ \{e^{g(x)+h(y)+p(z)}\ f(x,y,z)\} \\ &= e^{g(x)+h(y)+p(z)} F[T_{k,x,y,z} + x^2yzg'(x) + xy^2zh'(y) + xyz^2p'(z)] \\ &f(x,y,z). \end{aligned} \tag{2.6}$$

It may be noted that (2.4) can also be written as

$$\begin{aligned} &F(T_{k,x,y,z})\{x^\alpha y^\beta z^\gamma\ f(x,y,z)\} \\ &= x^\alpha y^\beta z^\gamma\, F(T_{1+\alpha+\beta+\gamma,x,y,z})\ \ f(x,y,z). \end{aligned} \tag{2.7}$$

The analogues of Leibnitz formula for the operator $T_{k,x,y,z}$ are as follows:

$$\begin{aligned} &T^n_{k,x,y,z}\{x^k u(x,y,z) v(x,y,z)\} \\ &= x^k \sum_{r=0}^{n} \binom{n}{r} \{T^{n-r}_{k,x,y,z}\ v(x,y,z)\}\{T^r_{k,x,y,z}\ u(x,y,z)\} \end{aligned} \tag{2.8}$$

$$T^n_{k,x,y,z}\{y^k\ u(x,y,z)\ v(x,y,z)\}$$
$$= y^k \sum_{r=0}^{n} \binom{n}{r} \{T^{n-r}_{k,x,y,z}\, v(x,y,z)\}\, \{T^r_{k,x,y,z} u(x,y,z)\} \tag{2.9}$$

$$T^n_{k,x,y,z}\ \{z^k\ u(x,y,z)\ v(x,y,z)\}$$
$$= z^k \sum_{r=0}^{n} \binom{n}{r} \{T^{n-r}_{k,x,y,z}\ v(x,y,z)\}\{T^r_{k,x,y,z}\, u(x,y,z)\} \tag{2.10}$$

$$T^n_{o,x,y,z}\{x^k\, y^k\, u(x,y,z)v(x,y,z)\}$$
$$= x^k\, y^k \sum_{r=0}^{n} \binom{n}{r} \{T^{n-r}_{k,x,y,z}\ v(x,y,z)\}\{T^r_{k,x,y,z}\ u(x,y,z)\} \tag{2.11}$$

$$T^n_{o,x,y,z}\{x^k z^k\, u(x,y,z)\ v(x,y,z)\}$$
$$= x^k\, z^k \sum_{r=0}^{n} \binom{n}{r} \{T^{n-r}_{k,x,y,z}\, v(x,y,z)\}\{T^r_{k,x,y,z}\, u(x,y,z)\} \tag{2.12}$$

$$T^n_{o,x,y,z}\{y^k\, z^k\, u(x,y,z)\ v(x,y,z)\}$$
$$= y^k\, z^k \sum_{r=0}^{n} \binom{n}{r} \{T^{n-r}_{k,x,y,z}\ v(x,y,z)\}\{T^r_{k,x,y,z}\ u(x,y,z)\} \tag{2.13}$$

$$T^n_{o,x,y,z}\{x^k\, y^k\, z^k\, u(x,y,z)v(x,y,z)\}$$
$$= x^k\, y^k\, z^k \sum_{r=0}^{n} \binom{n}{r} \{T^{n-r}_{2k,x,y,z}\, v(x,y,z)\}\{T^r_{k,x,y,z}\, u(x,y,z)\} \tag{2.14}$$

$$T^n_{k,x,y,z}\{x^k\, y^k\, z^k\, u(x,y,z)\ v(x,y,z)\}$$
$$= x^k\, y^k\, z^k \sum_{r=0}^{n} \binom{n}{r} \{T^{n-r}_{2k,x,y,z} v(x,y,z)\}\{T^r_{2k,x,y,z} u(x,y,z)\}. \tag{2.15}$$

Here (2.15) can also be written as

$$T^n_{k,x,y,z}\{x^k\, y^k\, z^k\, u(x,y,z) v(x,y,z)\}$$
$$= x^k\, y^k\, z^k \sum_{r=0}^{n} \binom{n}{r} \{T^{n-r}_{3k,x,y,z}\ v(x,y,z)\}\{T^r_{k,x,y,z}\ u(x,y,z)\} \tag{2.16}$$

$$T^n_{k,x,y,z}\{x^k\, y^k\, z^k\, u(x,y,z)\ v(x,y,z)\}$$
$$= x^k\, y^k\, z^k \sum_{r=0}^{n} \binom{n}{r} \{T^{n-r}_{4k,x,y,z} v(x,y,z)\}\{T^r_{o,x,y,z}\ y(x,y,z)\}. \tag{2.17}$$

Further, we have

$$T^n_{k,x,y,z}\{x^\alpha\, y^\beta\, z^\gamma\ u(x,y,z)\ v(x,y,z)\}$$
$$= x^\alpha\, y^\beta\, z^\gamma \sum_{r=0}^{n} \binom{n}{r} \{T^{n-r}_{k,x,y,z}\ v(x,y,z)\}\{T^r_{\alpha,\beta,\gamma,x,y,z}\ u(x,y,z)\}. \tag{2.18}$$

Here (2.18) can also be written as

$$T^n_{k,x,y,z}\{x^\alpha\, y^\beta\, z^\gamma\ u(x,y,z)\ v(x,y,z)\}$$
$$= x^\alpha\, y^\beta\, z^\gamma \sum_{r=0}^{n} \binom{n}{r} \{T^{n-r}_{k+\alpha,x,y,z} v(x,y,z)\}\{T^r_{\beta+\gamma,x,y,z} u(x,y,z)\} \tag{2.19}$$

$$T^n_{k,x,y,z}\{x^\alpha\, y^\beta\, z^\gamma\, u(x,y,z)v(x,y,z)\}$$
$$= x^\alpha\, y^\beta\, z^\gamma \sum_{r=0}^{n} \binom{n}{r} \{T^{n-r}_{k+\alpha+\beta,x,y,z}\, v(x,y,z)\}\{T^r_{\gamma,x,y,z}\, u(x,y,z)\} \tag{2.20}$$

$$T^n_{k,x,y,z}\{x^\alpha\; y^\beta\; z^\gamma\; u(x,y,z)\; v(x,y,z)\}$$
$$= x^\alpha y^\beta z^\gamma \sum_{r=0}^{n} \binom{n}{r} \{T^{n-r}_{k+\alpha+\beta,\gamma,x,y,z} v(x,y,z)\}\{T^r_{O,x,y,z} u(x,y,z)\}. \tag{2.21}$$

From (2.8 - 21), we also have

$$e^{\omega T_{k,x,y,z}}\{x^k\; u(x,y,z)\; v(x,y,z)\}$$
$$= x^k\; e^{\omega T_{k,x,y,z}}\; v(x,y,z)\; e^{\omega T_{k,x,y,z}}\; u(x,y,z) \tag{2.22}$$

$$e^{\omega T_{k,x,y,z}}\{y^k\; u(x,y,z)\; v(x,y,z)\}$$
$$= y^k\; e^{\omega T_{k,x,y,z}}\; v(x,y,z)\; e^{\omega T_{k,x,y,z}}\; u(x,y,z) \tag{2.23}$$

$$e^{\omega T_{k,x,y,z}}\{z^k\; u(x,y,z)\; v(x,y,z)\}$$
$$= z^k\; e^{\omega T_{k,x,y,z}} v(x,y,z)\; e^{\omega T_{k,x,y,z}}\; u(x,y,z) \tag{2.24}$$

$$e^{\omega T_{o,x,y,z}}\{x^k\; y^k\; u(x,y,z)\; v(x,y,z)\}$$
$$= x^k\, y^k\; e^{\omega T_{k,x,y,z}}\; v(x,y,z)\; e^{\omega T_{k,x,y,z}}\; u(x,y,z) \tag{2.25}$$

$$e^{\omega T_{o,x,y,z}}\{x^k\; z^k\; u(x,y,z)\; v(x,y,z)\}$$
$$= x^k\, z^k\; e^{\omega T_{k,x,y,z}}\; v(x,y,z)\; e^{\omega T_{k,x,y,z}}\; u(x,y,z) \tag{2.26}$$

$$e^{\omega T_{o,x,y,z}}\{y^k\; z^k\; u(x,y,z)\; v(x,y,z)\}$$
$$= y^k\, z^k\; e^{\omega T_{k,x,y,z}} v(x,y,z)\; e^{\omega T_{2k,x,y,z}}\; u(x,y,z) \tag{2.27}$$

$$\begin{aligned} & e^{\omega T_{o,x,y,z}}\{x^k\ y^k\ z^k\ u(x,y,z)\ v(x,y,z)\} \\ & = x^k\, y^k\, z^k\ e^{\omega T_{2k,x,y,z}} v(x,y,z)\ e^{\omega T_{k,x,y,z}}\ u(x,y,z) \end{aligned} \tag{2.28}$$

$$\begin{aligned} & e^{\omega T_{k,x,y,z}}\{x^k\ y^k\ z^k\ u(x,y,z)\ v(x,y,z)\} \\ & = x^k\, y^k\ z^k\ e^{\omega T_{2k,x,y,z}} v(x,y,z)\ e^{\omega T_{2k,x,y,z}}\ u(x,y,z). \end{aligned} \tag{2.29}$$

Here (2.29) can also be written as

$$\begin{aligned} & e^{\omega T_{k,x,y,z}}\{x^k\ y^k\ z^k\ u(x,y,z)\ v(x,y,z)\} \\ & = x^k\, y^k\, z^k\ e^{\omega T_{3k,x,y,z}} v(x,y,z)\ e^{\omega T_{k,x,y,z}}\ u(x,y,z) \end{aligned} \tag{2.30}$$

$$\begin{aligned} & e^{\omega T_{k,x,y,z}}\{x^k\ y^k\ z^k\ u(x,y,z)\ v(x,y,z)\} \\ & = x^k\, y^k\ z^k\ e^{\omega T_{4k,x,y,z}} v(x,y,z)\ e^{\omega T_{o,x,y,z}}\ u(x,y,z). \end{aligned} \tag{2.31}$$

Further, we have

$$\begin{aligned} & e^{\omega T_{k,x,y,z}}\{x^\alpha\ y^\beta\ z^\gamma\ u(x,y,z)\ v(x,y,z)\} \\ & = x^\alpha\, y^\beta\ z^\gamma\ e^{\omega T_{k,x,y,z}}\ v(x,y,z)\ e^{\omega T_{\alpha+\beta+\gamma,k,x,y,z}}\ u(x,y,z). \end{aligned} \tag{2.32}$$

Here (2.32) can also be written as

$$\begin{aligned} & e^{\omega T_{k,x,y,z}}\{x^\alpha\ y^\beta\ z^\gamma\ u(x,y,z)\ v(x,y,z)\} \\ & = x^\alpha\ y^\beta\ z^\gamma\ e^{\omega T_{k+\alpha,x,y,z}} v(x,y,z)\ e^{\omega T_{\beta+\gamma,x,y,z}}\ u(x,y,z) \end{aligned} \tag{2.33}$$

$$\begin{aligned} & e^{\omega T_{k,x,y,z}}\{x^\alpha\ y^\beta\ z^\gamma\ u(x,y,z)\ v(x,y,z)\} \\ & = x^\alpha\ y^\beta\ z^\gamma\ e^{\omega T_{k+\alpha+\beta,x,y,z}} v(x,y,z)\ e^{\omega T_{\gamma,x,y,z}}\ u(x,y,z) \end{aligned} \tag{2.34}$$

$$\begin{aligned} & e^{\omega T_{k,x,y,z}}\{x^\alpha\ y^\beta\ z^\gamma\ u(x,y,z)\ v(x,y,z)\} \\ & = x^\alpha\ y^\beta\ z^\gamma\ e^{\omega T_{k+\alpha+\beta+\gamma,x,y,z}} v(x,y,z)\ e^{\omega T_{o,x,y,z}}\ u(x,y,z). \end{aligned} \tag{2.35}$$

From (2.3), we have

$$e^{(\frac{t}{3})T_{3k,x,y}}\{x^{3\alpha}\, y^{3\beta}\, z^{3\gamma}\} = \frac{x^{3\alpha}\, y^{3\beta}\, z^{3\gamma}}{(1-xyzt)^{k+\alpha+\beta+\gamma}} \tag{2.36}$$

and hence, the general formula

$$\begin{aligned} &\exp\left(\tfrac{t}{3}tT_{3k,x,y,z}\right)\{x^{3\alpha}\, y^{3\beta}\, z^{3\gamma}\, f(x^3,y^3,z^3)\} \\ &= \frac{x^{3\alpha}\, y^{3\beta}\, z^{3\gamma}}{(1-xyzt)^{k+\alpha+\beta+\gamma}} f\left(\frac{x^3}{1-xyzt}, \frac{y^3}{1-xyzt}, \frac{z^3}{1-xyzt}\right). \end{aligned} \tag{2.37}$$

It is easy to verify that

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{t^n}{3^n n!}\, T^n_{3k,x,y,z}\{x^{3\alpha-n}\, y^{3\beta-n}\, z^{3\gamma-n}\, f(x^3,y^3,z^3)\} \\ &= x^{3\alpha}\, y^{3\beta}\, z^{3\gamma}(1+t)^{k+\alpha+\beta+\gamma-1} f\{x^3(1+t), y^3(1+t), z^3(1+t)\} \end{aligned} \tag{2.38}$$

$$\begin{aligned} &T^n_{3k,x,y,z}\ {}_rF_s\left[\begin{array}{l} a_1,a_2,---,a_r;\ xyz \\ b_1,b_2,---,b_s; \end{array}\right] \\ &= 3^n(k)_n x^n y^n z^n\ {}_{r+1}F_{s+1}\left[\begin{array}{l} a_1,a_2,---,a_r,k+n;xyz \\ b_1,b_2,---,b_s,k; \end{array}\right] \end{aligned} \tag{2.39}$$

$$\begin{aligned} &\exp\left(\tfrac{1}{3}t\,T_{3k,x,y,z}\right)\ {}_rF_s\left[\begin{array}{l} a_1,a_2,---,a_r;xyz \\ b_1,b_2,---,b_s; \end{array}\right] \\ &= (1-xyzt)^{-k}\ {}_rF_s\left[\begin{array}{ll} a_1,a_2,---,a_r;xyz & \frac{xyz}{(1-xyzt)} \\ b_1,b_2,---,b_s; & \end{array}\right] \end{aligned} \tag{2.40}$$

$$\begin{aligned} &T^n_{3k,x,y,z}\ F^{(3)}\left[\begin{array}{ll} (a)::(b);(b');(b''):(c);(c');(c''); & x^3,y^3z^3 \\ (e)::(g);(g');(g''):(h);(h');(h''); & \end{array}\right] \\ &= 3^n(k)_n x^n y^n z^n\ F^{(3)}\left[\begin{array}{lll} (a),k+n & ::(b);(b');(b''):(c);(c');(c''); & x^3,y^3z^3 \\ (e),k & ::(g);(g');(g''):(h);(h');(h''); & \end{array}\right] \end{aligned} \tag{2.41}$$

where a general triple hypergeometric series $F^{(3)}[x,y,z]$ is defined as [16]:

$$F^{(3)}[x,y,z] \equiv F^{(3)}\left[\begin{array}{ll} (a)::(b);(b');(b''):(c);\ (c');\ (c''); & \\ (e)::(g);(g');(g''):(h);(h');(h''); & x,yz \end{array}\right]$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} A(m,n,p)\ \frac{x^m}{m!}\ \frac{y^n}{n!}\ \frac{z^p}{p!},$$

where for convenience

$$A(m,n,p) = \frac{\prod_{j=1}^{A}(a_j)_{m+n+p}\prod_{j=1}^{B}(b_j)_{m+n}\prod_{j=1}^{B'}(b'_j)_{n+p}\prod_{j=1}^{B''}(b''_j)_{p+m}}{\prod_{j=1}^{E}(e_j)_{m+n+p}\prod_{j=1}^{G}(g_j)_{m+n}\prod_{j=1}^{G'}(g'_j)_{n+p}\prod_{j=1}^{G''}(g''_j)_{p+m}} \tag{2.42}$$
$$\frac{\prod_{j=1}^{C}(c_j)_m\prod_{j=1}^{C'}(c'_j)_n\prod_{j=1}^{C''}(c''_j)_p}{\prod_{j=1}^{H}(h_j)_m\prod_{j=1}^{H'}(h'_j)_n\prod_{j=1}^{H''}(h''_j)_p}$$

where (a) abbreviates the array of A parameters $a_1, a_1, ----, \ a_A$, with similar interpretations for $(b), (b'), (b'')$ et cetera.

$$\exp\left(\tfrac{1}{3}t\ T_{3k,x,y,z}\right)\ F^{(3)}\left[\begin{array}{ll} (a)::(b);(b');(b''):(c);\ (c');\ (c''); & \\ (e)::(g);(g');(g''):(h);(h');(h''); & x^3,y^3z^3 \end{array}\right] \tag{2.43}$$
$$= (1-xyzt)^{-k}F^{(3)}\left[\begin{array}{ll} (a)::(b);(b');(b''):(c);\ (c');\ (c''); & \\ (e)::(g);(g');(g''):(h);(h');(h''); & \frac{x^3}{1-xyzt},\frac{y^3}{1-xyzt},\frac{z^3}{1-xyzt} \end{array}\right].$$

Also, we have

$$ {}_rF_s\left[\begin{array}{l} a_1,a_2,---,a_r;\frac{1}{3}tT_{3k,x,y,z} \\ b_1,b_2,---,b_s; \end{array}\right]\{x^{3\alpha}y^{3\beta}z^{3\gamma}\} $$

$$ = x^{3\alpha}y^{3\beta}z^{3\gamma}\ {}_{r+1}F_s\left[\begin{array}{l} a_1,a_2,---,a_r,k+\alpha+\beta+\gamma; \\ b_1,b_2,---,b_s \qquad ; \end{array} xyzt\right] \tag{2.44} $$

$$ {}_rF_s\left[\begin{array}{l} a_1,a_2,---,a_r;\frac{1}{3}tT_{3k,x,y,z} \\ b_1,b_2,---,b_s; \end{array}\right]\{x^{3\alpha}y^{3\beta}z^{3\gamma}e^{-x^3-y^3-z^3}\} $$

$$ = x^{3\alpha}y^{3\beta}z^{3\gamma}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-x^3)^r(-y^3)^s(-z^3)^j}{r!\ s!\ j!} \tag{2.45} $$

$$ {}_{r+1}F_s\left[\begin{array}{l} a_1,a_2,---,a_r,k+\alpha+\beta+\gamma+r+s+j; \\ b_1,b_2,---,b_s \qquad ; \end{array} x^3\,y^3\,z^3\,t\right]. $$

If $\dfrac{1}{T_{k,x,y,z}}$ is the inverse of the operator $T_{k,x,y,z}$ then

$$ \frac{1}{T_{3k,x,y,z}}\{(k-3)\log x\ \log y\ \log z+\log x\ \log y+\log x\ \log z+\log y\ \log z\} $$

$$ = \left(\frac{\log x}{x}\right)\left(\frac{\log y}{y}\right)\left(\frac{\log z}{z}\right). \tag{2.46} $$

Again, let $S_{l,u,v,w}\equiv uvw\left(l+u\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}+w\frac{\partial}{\partial w}\right)$ and

$T_{k,x,y,z}\equiv xyz\left(k+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)$, then

$$ \left(\frac{S_{3l,u,v,w}}{T_{3k,x,y,z}}\right)^n\ \left\{\frac{u^{3\delta}v^{3\lambda}w^{3\mu}}{x^{3\alpha}y^{3\beta}z^{3\gamma}}\right\} $$

$$ = \frac{(-1)^n(l+\delta+\lambda+\mu)_n}{(1+\alpha+\beta+\gamma-k)_n}\ \frac{u^{3\delta+n}v^{3\lambda+n}w^{3\mu+n}}{x^{3\alpha+n}y^{3\beta+n}z^{3\gamma+n}}\ (\alpha+\beta+\gamma\neq k-1,k-2,---) \tag{2.47} $$

and hence, we have

$$
\begin{aligned}
&{}_rF_s\left[\begin{array}{ll} a_1,a_2,---,a_r; & \frac{S_{3l,u,v,w}}{T_{3k,x,y,z}} \\ b_1,b_2,---,b_s; & \end{array}\right]\left\{\frac{u^{3\delta}v^{3\lambda}w^{3\mu}}{x^{3\alpha}y^{3\beta}z^{3\gamma}}\right\} \\
&= \frac{u^{3\delta}v^{3\lambda}w^{3\mu}}{x^{3\alpha}y^{3\beta}z^{3\gamma}}\ {}_{r+1}F_{s+1}\left[\begin{array}{ll} a_1,a_2,---,a_r,\ l+\delta+\lambda+\mu; & -\frac{t\,u\,v\,w}{x\,y\,z} \\ b_1,b_2,---,b_s,\ 1+\alpha+\beta+\gamma+k; & \end{array}\right].
\end{aligned} \tag{2.48}
$$

In particular, we have

$$
\begin{aligned}
&\left(1-t\frac{S_{3l,u,v,w}}{T_{3k,x,y,z}}\right)^{-\alpha-\beta-\gamma-1+k}\left\{\frac{u^{3\delta}v^{3\lambda}w^{3\mu}}{x^{3\alpha}y^{3\beta}z^{3\gamma}}\right\} \\
&= \frac{u^{3\delta}v^{3\lambda}w^{3\mu}}{x^{3\alpha}y^{3\beta}z^{3\gamma}}\left(1+\frac{t\,u\,v\,w}{x\,y\,z}\right)^{-l-\delta-\lambda-\mu}
\end{aligned} \tag{2.49}
$$

$$
\begin{aligned}
&\left(1-t\frac{S_{3l,u,v,w}}{T_{3k,x,y,z}}\right)^{-C}\left\{\frac{u^{3\delta}v^{3\lambda}w^{3\mu}}{x^{3\alpha}y^{3\beta}z^{3\gamma}}\right\} \\
&= \frac{u^{3\delta}v^{3\lambda}w^{3\mu}}{x^{3\alpha}y^{3\beta}z^{3\gamma}}\ {}_2F_1\left[\begin{array}{ll} C,l+\delta+\lambda+\mu; & -\frac{t\,u\,v\,w}{x\,y\,z} \\ 1+\alpha+\beta+\gamma+k; & \end{array}\right].
\end{aligned} \tag{2.50}
$$

Also, we have

$$
\begin{aligned}
&\exp\left(\frac{t}{3T_{3k,x,y,z}}\right)\{x^{-3\alpha}y^{-3\beta}z^{-3\lambda}\} \\
&= 2^{\alpha+\beta+\gamma-k}\ x^{\frac{1}{2}(\beta+\gamma-k-5\alpha)}\ y^{\frac{1}{2}(\alpha+\gamma-k-5\beta)}\ z^{\frac{1}{2}(\alpha+\beta-k-5\gamma)} \\
&t^{k-\alpha-\beta-\gamma}\ J_{\alpha+\beta+\gamma-k}\left(\frac{t}{\sqrt{x\,y\,z}}\right).
\end{aligned} \tag{2.51}
$$

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Resumen

El presente artículo trata del cálculo del operador $T_{k,x,y,z}$. Este operador de tres variables es análogo al operador dado por W.A. Al-Salam [1] y H. B. Mittal [10]. El operador es útil para hallar representaciones operativas y para funciones generatrices de polinomios de tres variables las que serán tratadas en una comunicación aparte.

Palabras Clave: Operadores diferenciales, funciones especiales, funciones generatrices.

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