A SET OF ALMOST PERIODIC DISCONTINUOUS FUNCTIONS

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Abstract

In this paper the non density of **AP**, *the set of almost periodic functions in the sense of Bohr, in the space* **S** *of almost periodic functions in the sense of Stepanov is proven.*

Key words: *Bohr and Stepanov almost periodic functions; almost periodic sequences.*

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1 Introduction

In order to become aware with the spirit of almost periodic functions let us consider the following problems.

Periodic functions are almost periodic. It is a well known fact that the function $f(t) = \sin t$ is periodic with period 2π: $|\sin(t+2\pi)-\sin x| = 0$, for every $t \in \mathbb{R}$. From this identity follows that every real number $2k\pi$, where $k \in \mathbb{Z}$, is also a period. Thus this function has a set of periods uniformly located on the real line. But, what happens if instead of the identity $|\sin(t+2\pi) - \sin t| = 0$ we look for numbers τ such that $|\sin(t+\tau) - \sin t| < \varepsilon$, where ε is positive number, preferable small? The set of such a numbers is called as the set of almost periods of sin and is denoted by $T(\sin, \varepsilon)$. It is clear that $2\pi\mathbb{Z}$, the set of periods of sin is included in $T(\sin, \varepsilon)$. From the uniform continuity of sin we see that there exists a positive number $\delta = \delta(\varepsilon)$ such that

$$\cup_{k\in\mathbb{Z}}(2k\pi - \delta, 2k\pi + \delta) \subset T(\sin, \varepsilon).$$

But, the surprising fact is that there exists a set E $\subset \mathbb{Z}$, in some sense uniformly located on $\mathbb{R}$ such that $E \subset T(\sin, \varepsilon)$. This is a deep result of the theory of almost periodic functions with a nontrivial proof [3].

Almost periodic functions are present in the physical world. It is a well-known fact that all oscillatory phenomena observed in physics, engeenering and bio-sciences cannot be described by periodic functions. For example, we find this situation in the pendulum equation

$$x'' + m^2 x = \sin \alpha t, \quad m^2 - \alpha^2 \neq 0,$$

whose general solution, $x(t) = A \sin mt + B \cos mt + \frac{1}{m^2-\alpha^2} \sin \alpha t$ is not periodic if α/m is an irrational number and $A^2 + B^2 \neq 0$. In reply to this situation, the theory of almost periodic functions allows to describe the solution with a behavior close to periodic functions.

These questions started to be studied to the end of 19^{th} century by the works of H. Bohr [3] who generalized the notion of a periodic function to an almost periodic function: A number $\tau \in \mathbb{R}$ is called an ε-almost period for a function $f : \mathbb{R} \to \mathbb{R}$ if $|f(t+\tau) - f(t)| < \varepsilon$ holds for every value $t \in \mathbb{R}$. The set of all ε-almost periods for a function f is denoted by $T(f, \varepsilon)$.

A second fundamental notion in the Bohr theory was the notion of a relatively dense set of real numbers: A set $A \subset \mathbb{R}$ will be called relatively dense if there exists a positive number L such that $[a, a+L] \cap A \neq \emptyset$ for all $a \in \mathbb{R}$.

Bohr gave the following definition: A function $f : \mathbb{R} \to \mathbb{R}$ is said to be almost periodic if the following properties hold

(C) *f is continuous,*

(B) *For any $\varepsilon > 0$ the set $T(f, \varepsilon)$ is relatively dense in $\mathbb{R}$.*

An almost periodic functions in the sense of Borh is uniformly continuous and bounded. The set of almost periodic functions is usually denoted by **AP**. This set can be considered as a linear space with the norm $||f||_{\mathrm{B}} = \sup\{|f(t)| : t \in \mathbb{R}\}$.

In Bohr definitions the requirement of continuity of an almost periodic function is fundamental. If this assumption is omitted then the consequences for this theory are catastrophic. After Bohr others definitions or generalizations of the notion of almost periodicity were given using functions which are subject to conditions of integrability (see Appendix I in [3]). For example, Stepanov gave the following definition: A locally integrable function $f : \mathbb{R} \to \mathbb{R}$ is said to be almost periodic in the sense of Stepanov iff

$$||f||_{\mathrm{S}} = \sup_{t\in\mathbb{R}} \int_t^{t+1} |f(s)|ds < \infty$$

and for every $\varepsilon > 0$ there exists a relatively dense set of real numbers $S(f, \varepsilon)$ such that for every $\tau \in S(f, \varepsilon)$ one has $||f_{(\tau)} - f||_{\mathrm{S}} < \varepsilon$, where $f_{(\tau)}$

denotes the function $f_{(\tau)}(t) = f(t+\tau)$. According to this definition, **S** will denote the set of almost periodic functions in the sense of Stepanov. This set can be viewed as a linear normed space with the norm $||f||_{\mathrm{S}}$.

The main concern of this paper is the following question: Is the set **AP** dense in the space of almost periodic functions in the Stepanov sense? In other words, given $\varepsilon > 0$ and $f \in \mathbf{S}$, is there exists a function $g \in \mathbf{AP}$ such that $||f - g||_{\mathrm{S}} < \varepsilon$? We will show that the answer is negative. Besides we will give a description of the closure of **AP** in the Stepanov space. Although we have not found any reference to this problem in the some classics texts on almost periodic functions [3, 4, 6], we feel that this is an interesting question. Our answer rely on the notion of almost periodic sequences in the sense of Borh.

We will use the following theorem (see theorem 48.III in [3]).

Theorem 1. *If f and g are contained in* **AP**, *then for every $\varepsilon > 0$ the set $T(f,\varepsilon) \cap T(g,\varepsilon)$ is relatively dense in $\mathbb{R}$.*

By closing this introduction we must prevent the reader about the elementary character of this work. This paper has been written for a senior student level. Theorem 2 is known, although we include a proof with some novelties. Theorem 4 is new.

2 Some notations and previous results

In what follows, $\mathbb{Z}\{m,n\}$ will denote the set of integral numbers $\{m, m+1, \ldots, n\}$. We recall that every real t can be written in the form $t = [t] + \theta$, where $[t]$ is an integer and $0 \le \theta < 1$. The almost periodic sequences is the main device of this paper [1].

Definition 1. *A function $\varphi : \mathbb{Z} \to \mathbb{R}$ is called an almost periodic sequence if for any $\varepsilon > 0$ there corresponds an integer $N = N(\varepsilon)$ such that* *in every set* $\mathbb{Z}\{m, m+N\}$ *there exists an integer* p *with the property*

$$|\varphi(n+p) - \varphi(n)| < \varepsilon, \qquad n = 0, \pm 1, \pm 2, \ldots$$

The integer p is called as an ε-almost period of φ. The set of ε-almost periods of the sequence φ will be denoted by $\mathcal{T}(\varphi, \varepsilon)$. $\mathcal{AP}(\mathbb{Z}, \mathbb{R})$ will denote the space of almost periodic sequences with the norm $|\varphi|_{\mathrm{B}} = \sup\{|\varphi(n)| : n \in \mathbb{Z}\}$. The following theorem is known from the theory of almost periodic functions. We give a proof adapted to the aim of this paper [4, 6].

Theorem 2. *A necessary and sufficient condition for a sequence* $\varphi = \{\varphi(n)\}$ *to be almost periodic is the existence of an almost periodic function* f *such that* $\varphi(n) = f(n)$, $n = 0, \pm 1, \pm 2, \ldots$

Proof. For $c \in \mathbb{R}_+ = (0, \infty)$, $0 < \delta < c$ and $\varphi \in \mathcal{AP}(\mathbb{Z}, \mathbb{R})$, let us define the function

$$\varphi_{c,\delta}(t) = \begin{cases} \varphi(p), \; t \in [pc, (p+1)c - \delta), \; p \in \mathbb{Z}, \\ \\ \frac{\varphi(p+1) - \varphi(p)}{\delta}(t - (p+1)c + \delta) + \varphi(p), \; t \in [(p+1)c - \delta, (p+1)c) \end{cases}$$

Beside the bound

$$||\varphi_{c,\delta}||_{\mathrm{B}} \leq 2|\varphi|_{\mathrm{B}},$$

we emphasize that $\varphi_{c,\delta}$ is almost periodic in the sense of Borh. In fact, the function $\varphi_{c,\delta}$ is continuous and we will see that $\varphi_{c,\delta}$ satisfies the condition **(B)**. For any $\varepsilon > 0$, $a \in \mathbb{R}$ there corresponds an integer $N = N(\varepsilon)$, such that there exists an integer $k \in \mathbb{Z}\{[ac^{-1}] + 1, [ac^{-1}] + 1 + N\}$ such that

$$|\varphi(p+k) - \varphi(p)| < \varepsilon, \qquad p \in \mathbb{Z}.$$

Let $\tau = ck \in [a, a + c(1+N)]$ and $x \in \mathbb{R}$. If $x \in [pc, (p+1)c - \delta)$ then $x + \tau \in [(k+p)c, (k+p+1)c - \delta)$. Hence

$$|\varphi_{c,\delta}(t+\tau) - \varphi_{c,\delta}(t)| = |\varphi(p+k) - \varphi(p)| < \varepsilon.$$

If $t \in [(p+1)c - \delta, (p+1)c)$, then $t + \tau \in [(k+p+1)c - \delta, (k+p+1)c]$. Therefore we can write

$$\begin{aligned} \varphi_{c,\delta}(t) &= \tfrac{\varphi(p+1)-\varphi(p)}{\delta}(t - (p+1)c + \delta) + \varphi(p), \\ \varphi_{c,\delta}(t+\tau) &= \tfrac{\varphi(k+p+1)-\varphi(k+p)}{\delta}(t + kc - (p+k+1)c + \delta) - \varphi(p+k)|, \end{aligned}$$

from whence

$$|\varphi_{c,\delta}(t) - \varphi_{c,\delta}(t+\tau)| \leq |\varphi(p) - \varphi(p+k)| +$$

$$| \left(\tfrac{\varphi(p+1)-\varphi(p)}{\delta} - \tfrac{\varphi(k+p+1)-\varphi(k+p)}{\delta} \right) (t - (p+1)c + \delta)| < \varepsilon + \tfrac{2\varepsilon\delta}{\delta} = 3\varepsilon.$$

Thus if $k \in \mathcal{T}(\varphi, \varepsilon)$, then $\tau = ck \in T(\varphi_{c,\delta}, 3\varepsilon) \cap [a, a + c(1+N)]$.

Conversely, let f be an almost periodic function. We will prove that the sequence $\varphi = \{f(n)\}$ is almost periodic. Following the ideas in [6] we consider the 2-periodic function g defined by $g(t) = 1 - |t-1|$ on $[0,2]$. This is an almost periodic function. From $|g(t+\tau) - g(t)| < \delta$ we deduce that $|2k + \tau| < \delta$ for some integral number k. This implies that

$$T(g, \delta) = \bigcup_{p \in \mathbb{Z}} (2p - \delta, 2p + \delta).$$

From theorem 1 for every $\delta > 0$ the set $T(f, \delta) \cap T(g, \delta)$ is relatively dense in $\mathbb{R}$. But every $\tau \in T(f, \delta) \cap T(g, \delta)$ can be written in the form $\tau = 2p + w, p \in \mathbb{Z}, |w| < \delta$. Hence

$$|f(t + 2p + w) - f(t)| < \delta, t \in \mathbb{R}.$$

For a given ε, we can fix the number δ of the above arguments with the following properties:

$$0 < \delta < \varepsilon, |f(u) - f(v)| < \varepsilon \text{ if } |u - v| < \delta.$$

Now, from

$$|f(t+2p) - f(t)| < |f(t+2p+w) - f(t)| + |f(t+2p) - f(t+2p+w)| < \delta + \varepsilon$$

it follows $2p \in \mathcal{T}(f, 2\varepsilon)$. It is easy to see that the set of integers

$$\{2p : \tau = 2p + w \in \mathcal{T}(f, \delta) \cap \mathcal{T}(g, \delta), |w| < \delta\} \subset \mathcal{T}(\varphi, 2\varepsilon)$$

is relatively dense in $\mathbb{Z}$. $\square$

Now, we introduce a set of discontinuous functions. Let $\varphi \in \mathcal{AP}(\mathbb{Z}, \mathbb{R})$. For $c \in \mathbb{R}_+$, we define $\varphi_c : \mathbb{R} \to \mathbb{R}$ by $\varphi_c(t) = \varphi(n)$, if $t \in [nc, (n+1)c)$, $n \in \mathbb{Z}$.

Theorem 3. *The function φ_c satisfies the property* **(B)**.

Proof. Let $\varepsilon > 0$. Since $\varphi \in \mathcal{AP}(\mathbb{Z}, \mathbb{R})$, there exists a positive integer N such that there is an ε-almost period of φ in $\mathbb{Z}\{m, m+N\}$ for every $m \in \mathbb{Z}$. Notice that $[c^{-1}a] \in \mathbb{Z}$ for all $a \in \mathbb{R}$. Consequently, let $p \in \mathcal{T}(\varphi, \varepsilon)$ such that $p \in \mathbb{Z}\{[c^{-1}a]+1, [c^{-1}a]+1+N\}$, then $cp \in [a, a+\ell)$, where $\ell = cN$. Hence

$$t \in [mc, (m+1)c),\ m = m(t) \in \mathbb{Z} \implies t + cp \in [(m+p)c, (m+p+1)c)$$

implies

$$|\varphi_c(t+cp) - \varphi_c(t)| = |\varphi(m+p) - \varphi(m)| < \varepsilon.$$

$\square$

If ϕ is not a constant sequence, the function φ_c is not almost periodic in the sense of Bohr because is not continuous. In [5] it is proven that the sum $\varphi_c + \psi_d$ satisfies the condition **(B)**, when $c, d \in \mathbb{Q}_+$. In general this property is not valid for irrational numbers c and d.

It is easy to prove that the function φ_c is almost periodic in the sense of Stepanov. From the general theory of Stepanov [4] follows that that the sum $\varphi_c + \psi_d$ is contained in **S**. But looking after the simplity of this exposition, we will deduce this result from the following

Lemma 1. *The function φ_c is almost periodic in the sense of Stepanov. Besides for a given $\varepsilon > 0$ there exists a $\delta \in (0, c)$, such that*

$$\int_t^{t+1} |(\varphi_c - \varphi_{c,\delta})(s)| ds < \varepsilon$$

holds uniformly with respect to $t \in \mathbb{R}$.

Proof. The function φ is locally integrable. From theorem 3 φ is almost periodic in the sense of Stepanov. We notice that on each interval $[rc, (r+1)c]$, $r \in \mathbb{Z}$, we have the estimate

$$\sup_{[rc,(r+1)c]|} |(\varphi_c - \varphi_{c,\delta})(t)| \leq 2|\varphi|_{\mathrm{B}}.$$

This estimate does not depend on the values of the integer r, neither on the real numbers δ and c. If $1 = pc + \eta$, $0 \leq \eta < c$, $t = kc + \theta$, $0 \leq \theta < c$, then

$$\Delta := \int_t^{t+1} |(\varphi_c - \varphi_{c,\delta})(s)| ds \leq \int_{kc}^{(k+p+2)c} |(\varphi_c - \varphi_{c,\delta})(s)| ds.$$

Hence

$$\Delta \leq \sum_{j=0}^{p+1} \int_{kc+jc+c-\delta}^{kc+jc+c} |(\varphi_c - \varphi_{c,\delta})(s)| ds \leq 2\delta(p+2)|\varphi|_{\mathrm{B}},$$

where the integer p does not depend on t. This last bound proves the lemma. $\square$

We will prove now that the product $\varphi_c \psi_d$ is almost periodic in the sense of Stepanov. According to lemma 1 for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $||\varphi_{c,\delta} - \varphi_c||_{\mathrm{S}} < \varepsilon$ and $||\psi_{d,\delta} - \psi_d||_{\mathrm{S}} < \varepsilon$. Let $\tau \in$

$T(\varphi_{c,\delta},\varepsilon)\cap T(\psi_{d,\delta},\varepsilon)$, then

$$||\varphi_c\psi_d-(\varphi_c\psi_c)_{(\tau)}||_{\mathrm{S}}\le|\psi|_{\mathrm{B}}||\varphi_c-(\varphi_c)_{(\tau)}||_{\mathrm{S}}+|\varphi|_{\mathrm{B}}||\psi_d-(\psi_d)_{(\tau)}||_{\mathrm{S}}\le$$

$$|\psi|_{\mathrm{B}}\left(||\varphi_{c,\delta}-(\varphi_{c,\delta})_{(\tau)}||_{\mathrm{S}}+||\varphi_c-\varphi_{c,\delta}||_{\mathrm{S}}+||(\varphi_c)_{(\tau)}-(\varphi_{c,\delta})_{(\tau)}||_{\mathrm{S}}\right)+$$

$$|\varphi|_{\mathrm{B}}\left(||\psi_{d,\delta}-(\psi_{d,\delta})_{(\tau)}||_{\mathrm{S}}+||\psi_c-\psi_{d,\delta}||_{\mathrm{S}}+||(\psi_d)_{(\tau)}-(\psi_{d,\delta})_{(\tau)}||_{\mathrm{S}}\right)<$$

$$3(|\varphi|_{\mathrm{B}}+|\psi|_{\mathrm{B}})\varepsilon.$$

This implies that if $\tau\in T(\varphi_{c,\delta},\varepsilon)\cap T(\psi_{d,\delta},\varepsilon)$ then

$$\tau\in T(\varphi_c\psi_d,3(|\varphi|_{\mathrm{B}}+|\psi|_{\mathrm{B}})\varepsilon\}.$$

that is the product $\varphi_c\psi_d$ is almost periodic in the Stepanov sense.

We have promoted this proof because in analogous form we can prove that the scalar multiple of φ_c, the sum $\varphi_c+\phi_d$, and the quotient φ_c/ϕ_d, provided that ϕ_d does not vanish on $\mathbb{R}$, are almost periodic in the sense of Stepanov.

Lemma 2. *If $f\in$ **AP**, then, for every $\varepsilon>0$, there exists a sequence $\varphi\in\mathcal{AP}(\mathbb{Z},\mathbb{R})$ and $c>0$ such that*

$$|\varphi|_B\le||f||_B,\ \ ||f-\varphi_c||_B\le\varepsilon\ \textit{and}\ \ ||f-\varphi_c||_{\ S}\le\varepsilon.$$

Proof. From the uniform continuity of $f\in$ **AP** there exists a positive number c such that $|f(t+v)-f(t)|<\varepsilon$ if $v\in[0,c]$, for every $t\in\mathbb{R}$. From theorem 2 the sequence $\varphi(n)=f(n)$ is almost periodic. Hence $|\varphi|_{\mathrm{B}}\le||\varphi_c||_{\mathrm{B}}\le||f||_{\mathrm{B}}$, and $||f-\varphi_c||_{\mathrm{B}}\le\varepsilon$. The definition of the norm $||\cdot||_{\mathrm{S}}$ implies $||f-\varphi_c||_{\mathrm{S}}\le\varepsilon$. □

3 The APD-space

Let us denote $\widetilde{\mathbf{APD}}$ the algebra over $\mathbb{R}$ generated by all the functions φ_c. Further on, we define the space **APD** of all functions $f:\mathbb{R}\to\mathbb{R}$

obtained in the following way: There exists a sequence $\{g_n\} \in \widetilde{\mathbf{APD}}$ such that

(i) $\sup_n ||g_n||_{\mathrm{B}} < \infty$.

(ii) $||f - g_n||_{\mathrm{S}} \to 0$ as $n \to \infty$.

According to the definition of the space **APD**, a function $f \in \mathbf{APD}$ is almost periodic in the Stepanov's sense. Moreover **APD** is a closed subset in **S**. The space **APD** is large enough, from lemma 2 it contains the space **AP.** **DAP** has some remarkable properties, for example

Lemma 3.

$$f \in \mathbf{APD} \Longrightarrow \textit{ess-sup}_{\mathbb{R}}|f| < \infty.$$

Proof. Let $\{g_n\}$ be a uniformly bounded sequence in $\widetilde{\mathbf{APD}}$ such that $||g_n - f||_{\mathrm{S}} \to 0$ if $n \to \infty$. From theorems 7.1.2 and 7.1.5 in [2] for every compact interval $[-T, T]$, $T > 0$, there exists a subsequence $\{g_{n_i}\}$ of $\{g_n\}$ such that $\{g_{n_i}\}$ converges everywhere to f. Therefore,

$$\sup x \in [-T, T]|f(x)| \leq \sup_n \sup_{t \in \mathbb{R}} |g_n(t)| < \infty.$$

The proof is complete, since this estimate is independent of T. □

Theorem 4.

$$\overline{\mathbf{AP}} = \mathbf{APD} \neq \mathbf{S}.$$

The closure is understood in the sense of Stepanov.

Proof of $\overline{\mathbf{AP}} = \mathbf{APD}$. Lemma 2 implies that each function $f \in \mathbf{AP}$ is a limit of a sequence $\{g_n\}$ contained in $\widetilde{\mathbf{APD}}$ and $||g_n||_{\mathrm{B}} \leq ||f||_{\mathrm{B}}$ for every index n. This means that $\overline{\mathbf{AP}} \subset \mathbf{APD}$. On the other direction, from lemma 1, each function $f \in \widetilde{\mathbf{APD}}$ can be approximated by functions from **AP** and therefore $\widetilde{\mathbf{APD}} \subset \overline{\mathbf{AP}}$, from whence $\mathbf{APD} \subset \overline{\mathbf{AP}}$.

Proof of APD $\neq$ S. Let us define the intervals I_n and the set A by

$$I_n = \left[\frac{1}{n} - \frac{1}{2^{n+1}}, \frac{1}{n}\right],\ n = 1, 2, 3, \ldots, A = \bigcup_{n \geq 1} I_n.$$

We define the function $f : [0, 1] \to \mathbb{R}$ as follows $f(t) = n$ if $t \in I_n$ and $f(t) = 0$ if $t \in [01] \setminus A$. Further on, we extend the definition of f to all of $\mathbb{R}$ by the property $f(t+1) = f(t)$ for every $x \in \mathbb{R}$. It is easy to see that f is locally integrable. Since f is a periodic function then f is almost periodic in the sense of Stepanov. From lemma 3 $f \notin$ **APD** because $\text{ess-sup}_{\mathbb{R}}|f| = \infty$.

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Resumen

En este artículo se demuestra que **AP**, el conjunto de las funciones casi periódicas según Bohr, no es denso en **S**, el espacio de las funciones casi periódicas según Stepanov.

Palabras Clave: Casi periodicidad en el sentido de Bohr y en el sentido de Stepanov; funciones y sucesiones casi periódicas.

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