ON λ-CLOSURE SPACES

Miguel Caldas [1] *Erdal Ekici* [2]

Saeid Jafari [3]

Abstract

In this paper, we show that a pointwise λ-symmetric λ-isotonic λ-closure function is uniquely determined by the pairs of sets it separates. We then show that when the λ-closure function of the domain is λ-isotonic and the λ-closure function of the codomain is λ-isotonic and pointwise-λ-symmetric, functions which separate only those pairs of sets which are already separated are λ-continuous.

Key words: *pointwise λ-closed sets, λ-closure function , λ-continuous functions.*

[1] *Departamento de Matemática Aplicada, Universidade Federal Fluminense. Brasil*

[2] *Department of Mathematics, Canakkale Onsekiz Mart University. Turkey*

[3] *College of Vestsjaelland South. Denmark*

1 Introduction

Throughout the paper (X, τ) (or simply X) will always denote a topological space. For a subset A of X, the closure, interior and complement of A in X are denoted by $Cl(A), Int(A)$ and $X \backslash A$, respectively. By $\lambda O(X, \tau)$ and $\lambda C(X, \tau)$ we denote the collection of all λ-open sets and the collection of all λ-closed sets of (X, τ), respectively. Let B be a subset of a space (X, τ). B is a λ-set [2] if $B = B^{\Lambda}$, where: $B^{\Lambda} = \bigcap\{U \mid U \supset B, U \in \tau\}$. A subset A of a topological space (X, τ) is called λ-closed [1] if $A = B \cap C$, where B is a λ-set and C is a closed set. A is λ-open if $X \backslash A$ is Λ-closed. The intersection of all λ-closed sets containing A is called the λ-preclosure of A and is denoted by $Cl_{\lambda}(A)$.

Definition 1. *(1) A generalized λ-closure space is a pair (X, Cl_{λ}) consisting of a set X and a λ-closure function Cl_{λ}, a function from the power set of X to itself.*

(2) The λ-closure of a subset A of X, denoted Cl_{λ}, is the image of A under Cl_{λ}.

(3) The λ-exterior of A is $Ext_{\lambda}(A) = X \backslash Cl_{\lambda}(A)$, and the λ-Interior of A is $Int_{\lambda}(A) = X \backslash Cl_{\lambda}(X \backslash A)$.

(4) We say that A is λ-closed if $A = Cl_{\lambda}(A)$, A is λ-open if $A = Int_{\lambda}(A)$ and N is a λ-neighborhood of x if $x \in Int_{\lambda}(N)$.

Definition 2. *We say that a λ-closure function Cl_{λ} defined on X is:*

(1) λ-grounded if $Cl_{\lambda}(\emptyset) = \emptyset$.

(2) λ-isotonic if $Cl_{\lambda}(A) \subseteq Cl_{\lambda}(B)$ whenever $A \subseteq B$.

(3) λ-enlarging if $A \subseteq Cl_{\lambda}(A)$ for each subset A of X.

(4) λ-idempotent if $Cl_{\lambda}(A) = Cl_{\lambda}(Cl_{\lambda}(A))$ for each subset A of X.

(5) λ-sub-linear if $Cl_{\lambda}(A \cup B) \subseteq Cl_{\lambda}(A) \cup Cl_{\lambda}(B)$ for all $A, B \subseteq X$.

Definition 3. *(1) Subsets A and B of X are said to be λ-closure-separated in a generalized λ-closure space (X, Cl_λ) (or simply, Cl_λ-separated) if $A \cap Cl_\lambda(B) = \emptyset$ and $Cl_\lambda(A) \cap B = \emptyset$, or equivalently, if $A \subseteq Ext_\lambda(B)$ and $B \subseteq Ext_\lambda(A)$.*

(2) λ-Exterior points are said to be λ-closure-separated in a generalized λ-closure space (X, Cl_λ) if for each $A \subseteq X$ and for each $x \in Ext_\lambda(A)$, $\{x\}$ and A are Cl_λ-separated.

Theorem 1.1. *Let (X, Cl_λ) be a generalized λ-closure space in which λ-Exterior points are Cl_λ-separated and let S be the pairs of Cl_λ-separated sets in X. Then, for each subset A of X, the λ-closure of A is $Cl_\lambda(A) = \{x \in X : \{\{x\}, A\} \notin S\}$.*

Proof. In any generalized λ-closure space $Cl_\lambda(A)) \subseteq \{x \in X : \{\{x\}, A\} \notin S\}$. Really suppose that $y \notin \{x \in X : \{\{x\}, A\} \notin S\}$, that is, $\{\{y\}, A\} \in S$. Then $\{y\} \cap Cl_\lambda(A) = \emptyset$, and so $y \notin Cl_\lambda(A)$.
Suppose now that $y \notin Cl_\lambda(A)$. By hypothesis, $\{\{y\}, A\} \in S$, and hence, $y \notin \{x \in X : \{\{x\}, A\} \notin S\}$.

2 Some Fundamental Properties

Definition 4. *A λ-closure function Cl_λ defined on a set X is said to be pointwise λ-symmetric when, for all $x, y \in X$, if $x \in Cl_\lambda(\{y\})$, then $y \in Cl_\lambda(\{x\})$.*

A generalized λ-closure space (X, Cl_λ) is said to be λ-R_0 when, for all $x, y \in X$, if x is in each λ-neighborhood of y, then y is in each λ-neighborhood of x.

Corollary 2.1. *Let (X, Cl_λ) a generalized λ-closure space in which λ-Exterior points are Cl_λ -separated. Then Cl_λ is pointwise λ-symmetric and (X, Cl_λ) is λ-R_0.*

Proof. Suppose that λ-Exterior points are Cl_λ-separated in (X, Cl_λ). If $x \in Cl_\lambda(\{y\})$, then $\{x\}$ and $\{y\}$ are not Cl_λ-separated and hence, $y \in Cl_\lambda(\{x\})$. Hence, Cl_λ is pointwise λ-symmetric.
Suppose that x belongs to every λ-neighborhood of y, that is, $x \in M$ whenever $y \in Int_\lambda(M)$. Letting $A = X \backslash M$ and rewriting contrapositively, $y \in Cl_\lambda(A)$ whenever $x \in A$.
Suppose $x \in Int_\lambda(N)$. $x \notin Cl_\lambda(X \backslash N)$, so x is Cl_λ-separated from $X \backslash N$. Hence $Cl_\lambda(\{x\}) \subseteq N$. $x \in \{x\}$, so $y \in Cl_\lambda(\{x\}) \subseteq N$. Hence (X, Cl_λ) is λ-R_0.

While these three axioms are not equivalent in general, they are equivalent when the λ-closure function is λ-isotonic:

Theorem 2.2. *Let (X, Cl_λ) be a generalized λ-closure space with Cl_λ λ-isotonic. Then the following are equivalent:*

(1) λ Exterior points are Cl_λ-separated.

(2) Cl_λ is pointwise λ-symmetric.

(3) (X, Cl_λ) is λ-R_0.

Proof. Suppose that (2) is true. Let $A \subseteq X$, and suppose $x \in Ext_\lambda(A)$. Then, as Cl_λ is λ-isotonic, for each $y \in A$, $x \notin Cl_\lambda(\{y\})$, and hence, $y \notin Cl_\lambda(\{x\})$. Hence $A \cap Cl_\lambda(\{x\}) = \emptyset$. Hence (2) implies (1), and by the previous corollary, (1) implies (2).
Suppose now that (2) is true and let $x, y \in X$ such that x is in every λ-neighborhood of y, that is, $x \in N$ whenever $y \in Int_\lambda(N)$. Then $y \in Cl_\lambda(A)$ whenever $x \in A$, and in particular, since $x \in \{x\}$, $y \in Cl_\lambda(\{x\})$. Hence $x \in Cl_\lambda(\{y\})$. Thus if $y \in B$, then $x \in Cl_\lambda(\{y\}) \subseteq Cl_\lambda(B)$, as Cl_λ is λ-isotonic. Hence, if $x \in Int_\lambda(C)$, then $y \in C$, that is, y is in every λ-neighborhood of x. Hence, (2) implies (3).
Finally, suppose that (X, Cl_λ) is λ-R_0 and suppose that $x \in Cl_\lambda(\{y\})$. Since Cl_λ is λ-isotonic, $x \in Cl_\lambda(B)$ whenever $y \in B$, or, equivalently, y is in every λ-neighborhood of x. Since (X, Cl_λ) is λ-R_0, $x \in N$ whenever

$y \in Int_\lambda(N)$. Hence, $y \in Cl_\lambda(A)$ whenever $x \in A$, and in particular, since $x \in \{x\}$, $y \in Cl_\lambda(\{x\})$. Hence (3) implies (2).

Theorem 2.3. *Let S be a set of unordered pairs of subsets of a set X such that, for all $A, B, C \subseteq X$,*

(1) if $A \subseteq B$ and $\{B, C\} \in S$, then $\{A, C\} \in S$ and

(2) if $\{\{x\}, B\} \in S$ for each $x \in A$ and $\{\{y\}, A\} \in S$ for each $y \in B$, then $\{A, B\} \in S$.

Then there exists a unique pointwise λ-symmetric λ-isotonic λ-closure function Cl_λ on X which λ-closure-separates the elements of S.

Proof. Define Cl_λ by $Cl_\lambda(A) = \{x \in X : \{\{x\}, A\} \notin S\}$ for every $A \subseteq X$. If $A \subseteq B \subseteq X$ and $x \in Cl_\lambda(A)$, then $\{\{x\}, A\} \notin S$. Hence, $\{\{x\}, B\} \notin S$, that is, $x \in Cl_\lambda(B)$. Hence Cl_λ is λ-isotonic. Also, $x \in Cl_\lambda(\{y\})$ if and only if $\{\{x\}, \{y\}\} \notin S$ if and only if $y \in Cl_\lambda(\{x\})$, and thus Cl_λ is pointwise λ-symmetric.
Suppose that $\{A, B\} \in S$. Then $A \cap Cl_\lambda(B) = A \cap \{x \in X : \{\{x\}, B\} \notin S\} = \{x \in A : \{\{x\}, A\} \notin S\} = \emptyset$. Similarly, $Cl_\lambda(A) \cap B = \emptyset$. Hence, if $\{A, B\} \in S$, then A and B are Cl_λ-separated.
Now suppose that A and B are Cl_λ-separated. Then $\{x \in A : \{\{x\}, B\} \notin S\} = A \cap Cl_\lambda(B) = \emptyset$ and $\{x \in B : \{\{x\}, A\} \notin S\} = Cl_\lambda(A) \cap B = \emptyset$. Hence, $\{\{x\}, B\} \in S$ for each $x \in A$ and $\{\{y\}, A\} \in S$ for each $y \in B$, and thus, $\{A, B\} \in S$.

Furthermore, many properties of λ-closure functions can be expressed in terms of the sets they separate:

Theorem 2.4. *Let S be the pairs of Cl_λ-separated sets of a generalized λ-closure space (X, Cl_λ) in which λExterior points are λ-closure-separates. Then Cl_λ is*

(1) λ-grounded if and only if for all $x \in X$ $\{\{x\}, \emptyset\} \in S$.

(2) λ-enlarging if and only if for all $\{A, B\} \in S$, A and B are disjoint.

(3) λ*-sub-linear if and only if* $\{A, B \cup C\} \in S$ *whenever* $\{A, B\} \in S$ *and* $\{A, C\} \in S$.

Moreover, if Cl_λ is λ-enlarging and for all $A, B \subseteq X$, $\{\{x\}, A\} \notin S$ whenever $\{\{x\}, B\} \notin S$ and $\{\{y\}, A\} \notin S$ for each $y \in B$, then Cl_λ is λ-idempotent. Also, if Cl_λ is λ-isotonic and λ-idempotent, then $\{\{x\}, A\} \notin S$ whenever $\{\{x\}, B\} \notin S$ and $\{\{y\}, A\} \notin S$ for each $y \in B$.

Proof. Recall that by Theorem 1.1, $Cl_\lambda(A) = \{x \in X : \{\{x\}, A\} \notin S\}$ for every $A \subseteq X$. Suppose that for all $x \in X$, $\{\{x\}, \emptyset\} \in S$. Then $Cl_\lambda(\emptyset) = \{x \in X : \{\{x\}, \emptyset\} \notin S\} = \emptyset$. Hence Cl_λ is λ-grounded. Conversely, if $\emptyset = Cl_\lambda(\emptyset) = \{x \in X : \{\{x\}, \emptyset\} \notin S\}$, then $\{\{x\}, \emptyset\} \in S$, for all $x \in X$.

Suppose that for all $\{A, B\} \in S$, A and B are disjoint. Since $\{\{a\}, A\} \notin S$ if $a \in A$, $A \subseteq Cl_\lambda(A)$ for each $A \subseteq X$. Hence, Cl_λ is λ-enlarging. Conversely, suppose that Cl_λ is λ-enlarging and $\{A, B\} \in S$. Then $A \cap B \subseteq Cl_\lambda(A) \cap B = \emptyset$. Suppose that $\{A, B \cup C\} \in S$ whenever $\{A, B\} \in S$ and $\{A, C\} \in S$. Let $x \in X$ and $B, C \subseteq X$ such that $\{\{x\}, B \cup C\} \notin S$. Then $\{\{x\}, B\} \notin S$ or $\{\{x\}, C\} \notin S$. Hence $Cl_\lambda(B \cup C) \subseteq Cl_\lambda(B) \cup Cl_\lambda(C)$, and therefore, Cl_λ is λ-sub-linear. Conversely, suppose that Cl_λ is λ-sub-linear and let $\{A, B\}, \{A, C\} \in S$. Then $Cl_\lambda(B \cup C) \cap A \subseteq (Cl_\lambda(B) \cup Cl_\lambda(C)) \cap A = (Cl_\lambda(B) \cap A) \cup (Cl_\lambda(C)) \cap A) = \emptyset$ and $(B \cup C) \cap Cl_\lambda(A) = (B \cap Cl_\lambda(A)) \cup (C \cap Cl_\lambda(A)) = \emptyset$. Suppose that Cl_λ is λ-enlarging and suppose that $\{\{x\}, A\} \notin S$ whenever $\{\{x\}, B\} \notin S$ and $\{\{y\}, A\} \notin S$ for each $y \in B$. Then $Cl_\lambda(Cl_\lambda(A)) \subseteq Cl_\lambda(A)$: If $x \in Cl_\lambda(Cl_\lambda(A))$, then $\{\{x\}, Cl_\lambda(A)\} \notin S$. $\{\{y\}, A\} \notin S$, for each $y \in Cl_\lambda(A)$; hence $\{\{x\}, A\} \notin S$. And since Cl_λ is λ-enlarging, $Cl_\lambda(A) \subseteq Cl_\lambda(Cl_\lambda(A))$. Thus $Cl_\lambda(Cl_\lambda(A)) = Cl_\lambda(A)$, for each $A \subseteq X$. Finally, suppose that Cl_λ is λ-isotonic and λ-idempotent. Let $x \in X$ and $A, B \subseteq X$ such that $\{\{x\}, B\} \notin S$ and, for each $y \in B$, $\{\{y\}, A\} \notin S$. Then $x \in Cl_\lambda(B)$ and for each $y \in B$, $y \in Cl_\lambda(A)$, that is, $B \subseteq Cl_\lambda(A)$. Hence, $x \in Cl_\lambda(B) \subseteq Cl_\lambda(Cl_\lambda(A)) = Cl_\lambda(A)$.

Definition 5. *If $(X,(Cl_\lambda)_X)$ and $(Y,(Cl_\lambda)_Y)$ are generalized λ-closure spaces, then a function $f : X \to Y$ is said to be*

(1) λ-closure-preserving if $f((Cl_\lambda)_X(A)) \subseteq (Cl_\lambda)_Y(f(A))$ for each $A \subseteq X$.

(2) λ-continuous if $(Cl_\lambda)_X(f^{-1}(B)) \subseteq f^{-1}((Cl_\lambda)_Y(B))$ for each $B \subseteq Y$.

In general, neither condition implies the other. However, we easily obtain the following result:

Theorem 2.5. *Let $(X,(Cl_\lambda)_X)$ and $(Y,(Cl_\lambda)_Y)$ be generalized λ-closure spaces and let $f : X \to Y$.*

(1) If f is λ-closure-preserving and $(Cl_\lambda)_Y$ is λ-isotonic, then f is λ-continuous.

(2) If f is λ-continuous and $(Cl_\lambda)_X$ is λ-isotonic, then f is λ-closure-preserving.

Proof. Suppose that f is λ-closure-preserving and $(Cl_\lambda)_Y$ is λ-isotonic. Let $B \subseteq Y$. $f(Cl_\lambda)_X(f^{-1}(B))) \subseteq (Cl_\lambda)_Y(f(f^{-1}(B))) \subseteq (Cl_\lambda)_Y(B)$ and hence,
$(Cl_\lambda)_X(f^{-1}(B)) \subseteq f^{-1}(f((Cl_\lambda)_X(f^{-1}(B)))) \subseteq f^{-1}((Cl_\lambda)_Y(B))$.
Suppose that f is λ-continuous and $(Cl_\lambda)_X$ is λ-isotonic. Let $A \subseteq X$. $(Cl_\lambda)_X(A) \subseteq (Cl_\lambda)_X(f^{-1}(f(A))) \subseteq f^{-1}((Cl_\lambda)_Y(f(A)))$ and hence $f((Cl_\lambda)_X(A)) \subseteq f(f^{-1}((Cl_\lambda)_Y(f(A)))) \subseteq (Cl_\lambda)_Y(f(A))$.

Definition 6. *Let $(X,(Cl_\lambda)_X)$ and $(Y,(Cl_\lambda)_Y)$ be generalized λ-closure spaces and let $f : X \to Y$ be a function. If for all $A, B \subseteq X$, $f(A)$ and $f(B)$ are not $(Cl_\lambda)_Y$-separated whenever A and B are not $(Cl_\lambda)_X$-separated, then we say that f is non-λ-separating.*

Note that f is non-λ-separating if and only if A and B are $(Cl_\lambda)_X$-separated whenever $f(A)$ and $f(B)$ are $(Cl_\lambda)_Y$-separated.

Theorem 2.6. *Let $(X, (Cl_\lambda)_X)$ and $(Y, (Cl_\lambda)_Y)$ be generalized λ-closure spaces and let $f : X \to Y$.*

(1) If $(Cl_\lambda)_Y$ is λ-isotonic and f is non-λ-separating, then $f^{-1}(C)$ and $f^{-1}(D)$ are $(Cl_\lambda)_X$-separated whenever C and D are $(Cl_\lambda)_Y$-separated.

(2) If $(Cl_\lambda)_X$ is λ-isotonic and $f^{-1}(C)$ and $f^{-1}(D)$ are $(Cl_\lambda)_X$-separated whenever C and D are $(Cl_\lambda)_Y$-separated, then f is non-λ-separating.

Proof. Let C and D be $(Cl_\lambda)_Y$-separated subsets, where $(Cl_\lambda)_Y$ is λ-isotonic. Let $A = f^{-1}(C)$ and let $B = f^{-1}(D)$. $f(A) \subseteq C$ and $f(B) \subseteq D$ and since $(Cl_\lambda)_Y$ is λ-isotonic, $f(A)$ and $f(B)$ are also $(Cl_\lambda)_Y$-separated. Hence, A and B are $(Cl_\lambda)_X$-separated in X.
Suppose that $(Cl_\lambda)_X$ is λ-isotonic and let $A, B \subseteq X$ such that $C = f(A)$ and $D = f(B)$ are $(Cl_\lambda)_X$-separated. Then $f^{-1}(C)$ and $f^{-1}(D)$ are $(Cl_\lambda)_X$-separated and since $(Cl_\lambda)_X$ is λ-isotonic, $A \subseteq f^{-1}(f(A)) = f^{-1}(C)$ and $B \subseteq f^{-1}(f(B)) = f^{-1}(D)$ are $(Cl_\lambda)_X$-separated as well.

Theorem 2.7. *Let $(X, (Cl_\lambda)_X)$ and $(Y, (Cl_\lambda)_Y)$ be generalized λ-closure spaces and let $f : X \to Y$ be a function. If f is λ-closure-preserving, then f is non-λ-separating.*

Proof. Suppose that f is λ-closure-preserving and $A, B \subseteq X$ are not $(Cl_\lambda)_X$-separated. Suppose that $(Cl_\lambda)_X(A) \cap B \neq \emptyset$. Then $\emptyset \neq f((Cl_\lambda)_X(A) \cap B) \subseteq f((Cl_\lambda)_X(A)) \cap f(B) \subseteq (Cl_\lambda)_Y(f(A)) \cap f(B)$. Similarly, if $A \cap (Cl_\lambda)_X(B) \neq \emptyset$, then $f(A) \cap (Cl_\lambda)_Y(f(B)) \neq \emptyset$. Hence $f(A)$ and $f(B)$ are not $(Cl_\lambda)_Y$-separated.

Corollary 2.8. *Let $(X, (Cl_\lambda)_X)$ and $(Y, (Cl_\lambda)_Y)$ be generalized λ-closure spaces with $(Cl_\lambda)_X$ λ-isotonic and let $f : X \to Y$. If f is λ-continuous, then f is non-λ-separating.*

Proof. If f is λ-continuous and $(Cl_\lambda)_X$ λ-isotonic, then by Theorem 2.5 (2) f is λ-closure-preserving. Hence by Theorem 2.7, f is non-λ-separating.

Theorem 2.9. *Let $(X, (Cl_\lambda)_X)$ and $(Y, (Cl_\lambda)_Y)$ be generalized λ-closure spaces which λ-Exterior points $(Cl_\lambda)_Y$-separated in Y and let $f : X \to Y$ be a function. Then f is λ-closure-preserving if and only if f non-λ-separating.*

Proof. By Theorem 2.7, if f is λ-closure-preserving, then f is non-λ-separating. Suppose that f is non-λ-separating and let $A \subseteq X$. If $(Cl_\lambda)_X = \emptyset$, then $f((Cl_\lambda)_X(A)) = \emptyset \subseteq (Cl_\lambda)_Y(f(A))$.
Suppose $(Cl_\lambda)_X(A) \neq \emptyset$. Let S_X and S_Y denote the pairs of $(Cl_\lambda)_X$-separated subsets of X and the pairs of $(Cl_\lambda)_Y$-separated subsets of Y, respectively. Let $y \in f((Cl_\lambda)_X(A))$ and let $x \in (Cl_\lambda)_X(A) \cap f^{-1}(\{y\})$. Since $x \in (Cl_\lambda)_X(A)$, $\{\{x\}, A\} \notin S_X$ and since f non-λ-separating, $\{\{y\}, f(A)\} \notin S_Y$. Since λ-Exterior points are $(Cl_\lambda)_Y$-separated, $y \in (Cl_\lambda)_Y(f(A))$. Thus $f((Cl_\lambda)_X(A)) \subseteq (Cl_\lambda)_Y(f(A))$ for each $A \subseteq X$.

Corollary 2.10. *Let $(X, (Cl_\lambda)_X)$ and $(Y, (Cl_\lambda)_Y)$ be generalized λ-closure spaces with λ-isotonic closure functions and with $(Cl_\lambda)_Y$ -pointwise- λ-symmetric and let $f : X \to Y$. Then f is λ-continuous if and only if f non-λ-separating.*

Proof. Since $(Cl_\lambda)_Y$ is λ-isotonic and pointwise-λ-symmetric, λ-Exterior points are λ-closure separated in $(Y, (Cl_\lambda)_Y)$ (Theorem 2.2 (1)). Since both λ-closure functions are λ-isotonic, f is λ-closure-preserving (Theorem 2.5) if and only if f is λ-continuous. Hence, we can apply the Theorem 2.9.

3 λ-Connected Generalized λ-Closure Spaces

Definition 7. *Let (X, Cl_λ) be a generalized λ-closure space. X is said to be λ-connected if X is not a union of disjoint nontrivial λ-closure-separated pair of sets.*

Theorem 3.1. *Let (X, Cl_λ) be a generalized λ-closure space with λ-grounded λ-isotonic λ-enlarging Cl_λ. Then, the following are equivalent:*

(1) (X, Cl_λ) is λ-connected,

(2) X can not be a union of nonempty disjoint λ-open sets.

Proof. (1)⇒(2): Let X be a union of nonempty disjoint λ-open sets A and B. Then, $X = A \cup B$ and this implies that $B = X \backslash A$ and A is a λ-open set. Thus, B is λ-closed and hence $A \cap Cl_\lambda(B) = A \cap B = \varnothing$. By using similar way, we obtain $Cl_\lambda(A) \cap B = \varnothing$. Hence, A and B are λ-closure-separated and hence X is not λ-connected. This is a contradiction.

(2)⇒(1): Suppose that X is not λ-connected. Then $X = A \cup B$, where A, B are disjoint λ-closure-separated sets, i.e $A \cap Cl_\lambda(B) = Cl_\lambda(A) \cap B = \varnothing$. We have $Cl_\lambda(B) \subset X \backslash A \subset B$. Since Cl_λ is λ-enlarging, we obtain $Cl_\lambda(B) = B$ and hence, B is λ-closed. By using $Cl_\lambda(A) \cap B = \varnothing$ and similar way, it is obvious that A is λ-closed. This is a contradiction.

Definition 8. *Let (X, Cl_λ) be a generalized λ-closure space with λ-grounded λ-isotonic Cl_λ. Then, (X, Cl_λ) is called a T_1-λ-grounded λ-isotonic space if $Cl_\lambda(\{x\}) \subset \{x\}$ for all $x \in X$.*

Theorem 3.2. *Let (X, Cl_λ) be a generalized λ-closure space with λ-grounded λ-isotonic Cl_λ. Then, the following are equivalent:*

(1) (X, Cl_λ) is λ-connected,

(2) Any λ-continuous function $f : X \to Y$ is constant for all T_1-λ-grounded λ-isotonic spaces $Y = \{0, 1\}$.

Proof. (1)⇒(2): Let X be λ-connected. Suppose that $f : X \to Y$ is λ-continuous and it is not constant. Then there exists a set $U \subset X$ such that $U = f^{-1}(\{0\})$ and $X \backslash U = f^{-1}(\{1\})$. Since f is λ-continuous and Y

is T_1-λ-grounded λ-isotonic space, then we have $Cl_\lambda(U) = Cl_\lambda(f^{-1}(\{0\})) \subset f^{-1}(Cl_\lambda\{0\}) \subset f^{-1}(\{0\}) = U$ and hence $Cl_\lambda(U) \cap (X\backslash U) = \varnothing$. By using similar way we have $U \cap Cl_\lambda(X\backslash U) = \varnothing$. This is a contradiction. Thus, f is constant.

(2)$\Rightarrow$(1): Suppose that X is not λ-connected. Then there exist λ-closure-separated sets U and V such that $U \cup V = X$. We have $Cl_\lambda(U) \subset U$ and $Cl_\lambda(V) \subset V$ and $X\backslash U \subset V$. Since Cl_λ is λ-isotonic and U and V are λ-closure-separated, then $Cl_\lambda(X\backslash U) \subset Cl_\lambda(V) \subset X\backslash U$. If we consider the space (Y, Cl_λ) by $Y = \{0,1\}$, $Cl_\lambda(\varnothing) = \varnothing$, $Cl_\lambda(\{0\}) = \{0\}$, $Cl_\lambda(\{1\}) = \{1\}$ and $Cl_\lambda(Y) = Y$, then the space (Y, Cl_λ) is a T_1-λ-grounded λ-isotonic space. We define the function $f : X \to Y$ as $f(U) = \{0\}$ and $f(X\backslash U) = \{1\}$. Let $A \neq \varnothing$ and $A \subset Y$. If $A = Y$, then $f^{-1}(A) = X$ and hence $Cl_\lambda(X) = Cl_\lambda(f^{-1}(A)) \subset X = f^{-1}(A) = f^{-1}(Cl_\lambda(A))$. If $A = \{0\}$, then $f^{-1}(A) = U$ and hence $Cl_\lambda(U) = Cl_\lambda(f^{-1}(A)) \subset U = f^{-1}(A) = f^{-1}(Cl_\lambda(A))$. If $A = \{1\}$, then $f^{-1}(A) = X\backslash U$ and hence $Cl_\lambda(X\backslash U) = Cl_\lambda(f^{-1}(A)) \subset X\backslash U = f^{-1}(A) = f^{-1}(Cl_\lambda(A))$. Hence, f is λ-continuous. Since f is not constant, this is a contradiction.

Theorem 3.3. *Let $f : (X, Cl_\lambda) \to (Y, Cl_\lambda)$ and $g : (Y, Cl_\lambda) \to (Z, Cl_\lambda)$ be λ-continuous functions. Then, $gof : X \to Z$ is λ-continuous.*

Proof. Suppose that f and g are λ-continuous. For all $A \subset Z$ we have $Cl_\lambda(gof)^{-1}(A) = Cl_\lambda(f^{-1}(g^{-1}(A))) \subset f^{-1}(Cl_\lambda(g^{-1}(A))) \subset f^{-1}(g^{-1}(Cl_\lambda(A))) = (gof)^{-1}(Cl_\lambda(A))$. Hence, $gof : X \to Z$ is λ-continuous.

Theorem 3.4. *Let (X, Cl_λ) and (Y, Cl_λ) be generalized λ-closure spaces with λ-grounded λ-isotonic Cl_λ and $f : (X, Cl_\lambda) \to (Y, Cl_\lambda)$ be a λ-continuous function onto Y. If X is λ-connected, then Y is λ-connected.*

Proof. Suppose that $\{0,1\}$ is a generalized λ-closure spaces with λ-grounded λ-isotonic Cl_λ and $g : Y \to \{0,1\}$ is a λ-continuous function. Since f is λ-continuous, by Theorem 3.3, $gof : X \to \{0,1\}$ is

λ-continuous. Since X is λ-connected, gof is constant and hence g is constant. By Theorem 3.2, Y is λ-connected.

Definition 9. *Let (Y, Cl_λ) be a generalized λ-closure space with λ-grounded λ-isotonic Cl_λ and more than one element. A generalized λ-closure space (X, Cl_λ) with λ-grounded λ-isotonic Cl_λ is called Y-λ-connected if any λ-continuous function $f : X \to Y$ is constant.*

Theorem 3.5. *Let (Y, Cl_λ) be a generalized λ-closure space with λ-grounded λ-isotonic λ-enlarging Cl_λ and more than one element. Then every Y-λ-connected generalized λ-closure space with λ-grounded λ-isotonic is λ-connected.*

Proof. Let (X, Cl_λ) be a Y-λ-connected generalized λ-closure space with λ-grounded λ-isotonic Cl_λ. Suppose that $f : X \to \{0,1\}$ is a λ-continuous function, where $\{0,1\}$ is a T_1-λ-grounded λ-isotonic space. Since Y is a generalized λ-closure space with λ-grounded λ-isotonic λ-enlarging Cl_λ and more than one element, then there exists a λ-continuous injection $g : \{0,1\} \to Y$. By Theorem 3.3, $gof : X \to Y$ is λ-continuous. Since X is Y-λ-connected, then gof is constant. Thus, f is constant and hence, by Theorem 3.2, X is λ-connected.

Theorem 3.6. *Let (X, Cl_λ) and (Y, Cl_λ) be generalized λ-closure spaces with λ-grounded λ-isotonic Cl_λ and $f : (X, Cl_\lambda) \to (Y, Cl_\lambda)$ be a λ-continuous function onto Y. If X is Z-λ-connected, then Y is Z-λ-connected.*

Proof. Suppose that $g : Y \to Z$ is a λ-continuous function. Then $gof : X \to Z$ is λ-continuous. Since X is Z-λ-connected, then gof is constant. This implies that g is constant. Thus, Y is Z-λ-connected.

Definition 10. *A generalized λ-closure space (X, Cl_λ) is strongly λ-connected if there is no countable collection of pairwise λ-closure-separated sets $\{A_n\}$ such that $X = \cup A_n$.*

Theorem 3.7. *Every strongly λ-connected generalized λ-closure space with λ-grounded λ-isotonic Cl_λ is λ-connected.*

Theorem 3.8. *Let (X, Cl_λ) and (Y, Cl_λ) be generalized λ-closure spaces with λ-grounded λ-isotonic Cl_λ and $f : (X, Cl_\lambda) \to (Y, Cl_\lambda)$ be a λ-continuous function onto Y. If X is strongly λ-connected, then Y is strongly λ-connected.*

Proof. Suppose that Y is not strongly λ-connected. Then, there exists a countable collection of pairwise λ-closure-separated sets $\{A_n\}$ such that $Y = \cup A_n$. Since $f^{-1}(A_n) \cap Cl_\lambda(f^{-1}(A_m)) \subset f^{-1}(A_n) \cap f^{-1}(Cl_\lambda(A_m)) = \varnothing$ for all $n \neq m$, then the collection $\{f^{-1}(A_n)\}$ is pairwise λ-closure-separated. This is a contradiction. Hence, Y is strongly λ-connected.

Theorem 3.9. *Let $(X, (Cl_\lambda)_X)$ and $(Y, (Cl_\lambda)_Y)$ are generalized λ-closure spaces. Then the following are equivalent for a function $f : X \to Y$:*

(1) f is λ-continuous,

(2) $f^{-1}(Int_\lambda(B)) \subseteq Int_\lambda(f^{-1}(B))$ for each $B \subseteq Y$.

Theorem 3.10. *Let (X, Cl_λ) be a generalized λ-closure space with λ-grounded λ-isotonic Cl_λ. Then (X, Cl_λ) is strongly λ-connected if and only if (X, Cl_λ) is Y-λ-connected for any countable T_1-λ-grounded λ-isotonic space (Y, Cl_λ).*

Proof. ($\Rightarrow$): Let (X, Cl_λ) be strongly λ-connected. Suppose that (X, Cl_λ) is not Y-λ-connected for some countable T_1-λ-grounded λ-isotonic space (Y, Cl_λ). There exists a λ-continuous function $f : X \to Y$ which is not constant and hence $K = f(X)$ is a countable set with more than one element. For each $y_n \in K$, there exists $U_n \subset X$ such that $U_n = f^{-1}(\{y_n\})$ and hence $Y = \cup U_n$. Since f is λ-continuous and Y is λ-grounded, then for each $n \neq m$, $U_n \cap Cl_\lambda(U_m) = f^{-1}(\{y_n\}) \cap$

$Cl_\lambda(f^{-1}(\{y_m\})) \subset f^{-1}(\{y_n\}) \cap f^{-1}(Cl_\lambda(\{y_m\})) \subset f^{-1}(\{y_n\}) \cap f^{-1}(\{y_m\}) = \varnothing$. This contradict with the strong λ-connectedness of X. Thus, X is Y-λ-connected.

($\Leftarrow$): Let X be Y-λ-connected for any countable T_1-λ-grounded λ-isotonic space (Y, Cl_λ). Suppose that X is not strongly λ-connected. There exists a countable collection of pairwise λ-closure-separated sets $\{U_n\}$ such that $X = \cup U_n$. We take the space (Z, Cl_λ), where Z is the set of integers and $Cl_\lambda : P(Z) \to P(Z)$ is defined by $Cl_\lambda(K) = K$ for each $K \subset Z$. Clearly (Z, Cl_λ) is a countable T_1-λ-grounded λ-isotonic space. Put $U_k \in \{U_n\}$. We define a function $f : X \to Z$ by $f(U_k) = \{x\}$ and $f(X \backslash U_k) = \{y\}$ where x, $y \in Z$ and $x \neq y$. Since $Cl_\lambda(U_k) \cap U_n = \varnothing$ for all $n \neq k$, then $Cl_\lambda(U_k) \cap \cup_{n \neq k} U_n = \varnothing$ and hence $Cl_\lambda(U_k) \subset U_k$. Let $\varnothing \neq K \subset Z$. If x, $y \in K$ then $f^{-1}(K) = X$ and $Cl_\lambda(f^{-1}(K)) = Cl_\lambda(X) \subset X = f^{-1}(K) = f^{-1}(Cl_\lambda(K))$. If $x \in K$ and $y \notin K$, then $f^{-1}(K) = U_k$ and $Cl_\lambda(f^{-1}(K)) = Cl_\lambda(U_k) \subset U_k = f^{-1}(K) = f^{-1}(Cl_\lambda(K))$. If $y \in K$ and $x \notin K$ then $f^{-1}(K) = X \backslash U_k$. Since $Cl_\lambda(K) = K$ for each $K \subset Z$, then $Int_\lambda(K) = K$ for each $K \subset Z$. Also, $X \backslash U_k \subset \cup_{n \neq k} U_n \subset X \backslash Cl_\lambda(U_k) = Int_\lambda(X \backslash U_k)$. Therefore, $f^{-1}(Int_\lambda(K)) = X \backslash U_k = f^{-1}(K) \subset Int_\lambda(X \backslash U_k) = Int_\lambda(f^{-1}(K))$. Hence we obtain that f is λ-continuous. Since f is not constant, this is a contradiction with the Z-λ-connectedness of X. Hence, X is strongly λ-connected.

References

[1] F. G. Arenas, J. Dontchev and M. Ganster, *On λ-sets and dual of generalized continuity.* Questions Answers Gen. Topology, 15 (1997) 3-13.

[2] H. Maki, *Generalized Λ-sets and the associated closure operator.* The Special Issue in Commemoration of Prof. Kazusada IKEDA' Retirement, 1. Oct. 1986, 139-146.

Resumen
En este artículo, demostramos que una función puntualmente λ simétrica, λ-isotónica, λ-clausurada es determinada únicamente por los pares de conjuntos que ella separa. Luego probamos que, cuando la función λ-clausurada del dominio es λ-isotónica y la función λ-clausurada del codominio es λ-isotónica y puntualmente λ-simétrica, las funciones que separan solamente aquellos pares de conjuntos que están ya separados son λ-continuas.

Palabras Clave: conjuntos puntualmente λ-clausurados, función λ-clausurada, funciones λ-continuas.

Miguel Caldas
Departamento de Matemática Aplicada,
Universidade Federal Fluminense,
Rua Mario Santos Braga, s/n
24020-140, Niteroi, RJ Brasil.
gmamccs@vm.uff.br

Erdal Ekici
Department of Mathematics,
Canakkale Onsekiz Mart University,
Terzioglu Campus, 17020 Canakkale-Turkey.
eekici@comu.edu.tr

Saeid Jafari
College of Vestsjaelland South,
Herrestraede 11,
4200. Slagelse, Denmark.
jafari@stofanet.dk