A CONJECTURE ABOUT THE NON-TRIVIAL ZEROES OF THE RIEMANN ZETA FUNCTION

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December, 2007

Abstract

Some heuristic arguments are given in support of the following conjecture: If the Riemann Hypothesis(RH) does not hold then the number of zeroes of the Riemann zeta function with real part $\sigma > \frac{1}{2}$ is infinite.

Keywords: *Riemann hypothesis, non-trivial zeroes, conjecture*

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Background and Statement of the Conjecture

If $\pi(n)$ denotes the number of prime numbers in the interval $[1, n]$, $n \in \mathbb{N}$, it seems that from numerical evidence, Gauss conjectured that $\lim[\pi(n)/(n/\ln n)] = 1$.

Using elementary methods Chebyshev proved that

$$\frac{\ln 2}{2} \le \underline{\lim}\frac{\pi(n)}{(n/\ln n)} \le 1 \le \overline{\lim}\frac{\pi(n)}{(n/\ln n)} \le 2\ln 2$$

establishing then that if $\lim[\pi(n)/(n/\ln n)]$ exists, it must be equal to 1.

To prove the existence of this limit Riemann suggested the study of the function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad s = \sigma + it, \quad \sigma > 1, \quad t \in \mathbb{R}$$

afterwards known as the Riemann zeta function. Riemann proved that ζ has analytic continuation to $\mathbb{C}\backslash\{1\}$, still denoted by ζ, that has a pole of first order at $s = 1$ with residue 1. If $Z(\zeta)$ denotes the set of zeroes of ζ (defined on $\mathbb{C}\backslash\{1\}$), then he also proved that

$$Z(\zeta) \cap \{s : \sigma > 1\} = \emptyset,$$

$Z(\zeta) \cap \{s : \sigma < 0\} = \{-2n : n \in \mathbb{N}\}$ and all these zeroes called trivial are simple, the set $Z(\zeta) \cap \{s : 0 \le \sigma \le 1\}$ (all these zeroes are called non-trivial) is infinite, symmetrical with respect to the lines $\left\{s : \sigma = \frac{1}{2}\right\}, \{s : t = 0\}$ and $Z(\zeta) \cap \{s : 0 \le \sigma \le 1\} \cap \mathbb{R} = \emptyset$.

Riemann also formulated the now famous:

Riemann Hypothesis (R.H.)

$$Z(\zeta) \cap \{s : 0 \le \sigma \le 1\} = Z(\zeta) \cap \left\{s : \sigma = \frac{1}{2}\right\}$$

Hardy proved that the set $Z(\zeta) \cap \left\{s : \sigma = \frac{1}{2}\right\}$ is infinite. Levinson proved that more than $\frac{1}{3}$ of the elements in $Z(s) \cap \{s : 0 \leq \sigma \leq 1\}$ are in $Z(s) \cap \left\{s : \sigma = \frac{1}{2}\right\}$ and Conrey showed that in this result $\frac{1}{3}$ could be replaced by $\frac{2}{5}$, [4].

It can be shown that $\lim[\pi(n) / (n/ \ln n)] = 1$ if and only if $Z(\zeta) \cap \{s : \sigma = 1\} = \emptyset$, and Hadamard and Vallee-Poussin, independently verify this last condition, finally proving the conjecture of Gauss that $\lim[\pi(n)/(n/ \ln n)] = 1$, result now known as the Primer Number Theorem. The location of the element of $Z(\zeta) \cap \{s : 0 \leq \sigma \leq 1\}$ closest to the line $\{s : \sigma = 1\}$ determines the order of magnitude of the relative error term $|(\pi(n) - (n/ \ln n))| / (n/ \ln n)$ in The Prime Number Theorem. In some sense this relative error term is minimal if R.H. holds. The proof or disproof of R.H. is one of the most important unsolved problems in Mathematics. R.H. that originally arose in Analytic Number Theory, can also be reformulated as a problem in several other branches of Mathematics. One of the formulations of R.H. as a problem of Functional Analysis is the following, [1]:

Theorem 1. *Let* $[A_\rho f](\theta) = \int_0^1 \rho\left(\frac{\theta}{x}\right) f(x)\,dx$, *where* $\rho(x) = x - [x], x \in \mathbb{R}$, $[x] \in \mathbb{Z}, [x] \leq x < [x] + 1$, *be considered as an integral operator on* $L^2(0, 1)$. *Then R.H. holds if and only if* $\ker A_\rho = \{0\}$ *or if and only if* $h \notin R(A_\rho)$ *where* $h(x) = x$.

The following facts are also established in [1]:

I) A_ρ *is Hilbert-Schmidt, but neither nuclear nor normal.*

II) If $\sigma(A_\rho)$ *is the spectrum of* A_ρ *then* $\lambda \in \sigma(A_\rho)\backslash\{0\}$ *if and only if* $T(\lambda^{-1}) = 0$ *where*

$$T(\mu) = 1 - \mu + \sum_{r=1}^{\infty} (-1)^{r+1} \frac{\prod_{\ell=1}^{r} \zeta(\ell + 1)}{(r + 1)\,!\,(r + 1)} \mu^{r+1}$$

is an entire function of order one and type one. Moreover each non-zero eigenvalue $\lambda = \mu^{-1}$ *has geometric multiplicity one, and associated* *eigenfunction* $\psi_\mu(x) = \mu\, x\, T'(\mu x)$.

III) If $\{\lambda_n\}_{n\geq 1}$, *is the sequence of non-zero eigenvalues of* A_ρ *where the ordering is such that* $|\lambda_n| \geq |\lambda_{n+1}|, \forall n \geq 1$ *and each one of them is repeated according to its algebraic multiplicity, then the first eigenvalue* λ_1 *is positive and has algebraic multiplicity one,* $|\lambda_n| \leq \frac{e}{n}, \forall n \in \mathbb{N}, \sum_{n=1}^{\infty} |\lambda_n| = \infty$ *and* $\lambda_n \notin \mathbb{R}$ *for an infinite number of n's.*

IV) If $D^*(\mu) = \det_2(1 - \mu A_\rho)$ *is the modified Fredholm determinant of* $I - \mu A_\rho$ *then* $D^*(\mu) = e^{\mu} T(\mu)$ *[2]*

V) $\ker A_\rho^* = \{0\}$

In [3] it is given the following characterization of R.H.:

Theorem 2. *If* $[A_\rho(\alpha)](\theta) = \int_0^1 \rho\left(\frac{\alpha\theta}{x}\right) dx$ *where* $\alpha \in]0,1[$, *is considered as an integral operator in* $L^2(0,1)$, *then R.H. holds if and only if* $\overline{R(A_\rho(\alpha)^*)} \supset L^2(0,\alpha)$ *(if it holds for one* $\alpha \in]0,1[$, *it holds for all others* α *in this interval).*

It is also proven there that if $\alpha \in]0,1[$, *then:*

I) $A_\rho(\alpha)$ *is Hilbert-Schmidt, but neither nuclear nor normal.*

II) If $D_\alpha^*(\mu) = \det_2[I - \mu A_\rho(\alpha)]$ *is the modified Fredholm determinant of* $I - \mu A_\rho(\alpha)$ *then*

$$D_\alpha^*(\mu) = e^{\alpha\mu} T_\alpha(\mu)([2]),$$

where

$$T_\alpha(\mu) = 1 - \alpha\mu + \sum_{r=1}^{\infty} \frac{(-1)^{r+1} \alpha^{(r+1)(r+2)/2}}{(r+1)!\,(r+1)} \prod_{\ell=1}^{r} \zeta(\ell+1)\mu^{r+1}$$

is an entire function of order zero and D_α^* *has order one and type* α

III) The set of eigenvectors and generalized eigenvectors associated to the non-zero eigenvalues of $A_\rho(\alpha)$ *(that are determined explicitly)is total in* $L^2(0,1)$, *but it is not part of a Markushevich basis in* $L^2(0,1)$ *and therefore it is not a Schauder basis in* $L^2(0,1)$.

IV) $\dim \ker A_\rho(\alpha) = \infty$ *and* $\ker A_\rho^*(\alpha) = \{0\}$.

It is not difficult to show that if $0 < \beta < \alpha < 1$ then $\ker A_\rho(\alpha) \subset \ker A_\rho(\beta)$ and from the dominated convergence theorem of Lebesgue it follows that

$$\bigcap_{0<\alpha<1} \ker A_\rho(\alpha) = \ker A_\rho$$

These considerations lead us to formulate the following conjectures:

Conjecture 1. *Does the last relation and the condition* $\ker A_\rho^* = \{0\}$ *imply that either* $\ker A_\rho = \{0\}$ *or* $\dim \ker A_\rho = \infty$ *holds?*

Conjecture 2. $\dim \ker A_\rho = \infty$ *if and only if* ζ *has infinite zeroes in the half plane* $\left\{s : \sigma > \frac{1}{2}\right\}$

Since $A_\rho h^r = \frac{h}{r} - \frac{\zeta(r+1)}{r+1} h^{r+1}$ if $Re\, r > -1$ then $Im(r\, h^r) \in \ker A_\rho \backslash \{0\}$ if $r + 1 \in Z(\zeta)$ (recall that $r + 1 \notin \mathbb{R}$), one of the implications in conjecture 2 is trivial. From these two conjectures we derive our

MAIN CONJECTURE: If R.H. does not hold then the set $Z(\zeta) \cap \left\{s : \sigma > \frac{1}{2}\right\}$ is infinite.

References

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Resumen

Se dan algunos argumentos heurísticos para validar la siguiente conjetura: Si la Hipótesis de Riemann (RH) no se cumple entonces la función zeta de Riemann tiene infinitos ceros con parte real $\sigma > \frac{1}{2}$.

Palabras Clave: Hipótesis de Riemann, ceros no triviales, conjetura.

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