ESTIMATING MULTIDIMENSIONAL DENSITY FUNCTIONS USING THE MALLIAVIN-THALMAIER FORMULA

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Abstract

The Malliavin-Thalmaier formula was introduced for simulation of high dimensional probability density functions. But when this integration by parts formula is applied directly in computer simulations, we show that it is unstable. We propose an approximation to the Malliavin-Thalmaier formula. In this paper, we find the order of the bias and the variance of the approximation error. And we obtain an explicit Malliavin-Thalmaier formula for the calculation of Greeks in finance. The weights obtained are free from the curse of dimensionality.

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1 Introduction

Let $(\Omega = C([0,T];\mathbf{R}^d), \mathcal{F}, P)$ denote the canonical Wiener probability space whose canonical process denoted by W is a Wiener process. The σ-field on Ω, $\mathcal{F}$, is given by the smallest σ-field generated by the sets of the type $\{x \in \Omega;\ x(t_1) \in A_1, ..., x(t_n) \in A_n\}$ for $n \in \mathbb{N}$, $t_1, ..., t_n \in [0,T]$ and $A_1, ..., A_n$ are Borel sets in $\mathbf{R}^d$. The measure P is such that the canonical process $W : \Omega \times [0,T] \to \mathbf{R}^d$ is a measurable map from the product σ-field $\mathcal{F} \otimes \mathcal{B}([0,T])$ where $\mathcal{B}([0,T])$ denotes the Borelian σ-field on $[0,T]$ such that $W(x,t) = x(t)$ satisfies

1. $W(x,\cdot) : [0,T] \to \mathbf{R}^d$ is continuous in $[0,T]$ for P-almost all $x \in \Omega$. By an abuse of notation, we sometimes write $W(t)$, W_t to emphasize that these are random variables when the time variable is fixed.

2. Given $0 = t_0 < t_1 < ... < t_n = T$ the random vector $(W(t_n) - W(t_{n-1}), ..., W(t_1) - W(t_0))$ is Gaussian distributed with mean zero and diagonal covariance matrix with i-th element $t_{n-i+1} - t_{n-i}$.

In this set-up one defined the filtration associated with W as the family of σ-algebras $(\mathcal{F}_t)_{t\in[0,T]}$. $\mathcal{F}_t$ is the smallest σ-field that makes $W(s)$, $s \leq t$ random variables with respect to this filtration.

The goal of Malliavin Calculus is to set up conditions under which a random variable $F : \Omega \to \mathbf{R}^d$, $F = (F_1, ..., F_d)$ $(d \geq 2)$ has a smooth density. In the particular case that F is the final value of a diffusion process, the Malliavin result leads to the celebrated Hörmander theorem that states the existence of a fundamental solution of a parabolic partial differential equation.

Instead of following the path of a rigorous mathematical sequence of definitions that can be found in [7], we will just give the heuristics behind the definitions. The idea is to define a differential calculus on the space Ω. The problem is that $\Omega = C([0,T])$ is not a Hilbert space so defining a derivative is not an easy matter. Furthermore, most of the interesting functionals will not be differentiable is the derivative is defined in the Frechet sense, as is the case of e.g. solutions of stochastic differential equations. In fact, the derivative is defined in an a.s. sense and only with respect to directions given in

$H^1 = \{x \in C([0,T]); x$ is differentiable for almost all $t \in [0,T]$ and $\dot{x} \in L^2[0,T]\}$.

In probability this is known as the Cameron-Martin space. Then once the derivative denoted by D is defined one proves a chain rule property stating that $Df(F) =< \nabla f(F), DF >$, that this operator is closable and therefore its adjoint D^* is well defined but it is an unbounded operator. From here one obtains the important integration by parts formula which is given in Proposition 1.1 below.

Although the goal during the first 20 years of Malliavin Calculus was the theoretical study of the densities of random variables. It was in an article authored by a group led by P.L. Lions [4] that they discovered the applications of this formula to estimation of sensitivity quantities related to risk control in financial institutions. This is further related to the approximation of densities in Wiener space.

The goal of the present article is to estimate through simulations the probability density function of F using Malliavin Calculus and discuss some of its applications, particularly in Finance.

This problem has attracted some interest due to its financial applications although we frame it here as a general density estimation problem.

Usually, the result applied to estimate a density is the classical integration by parts formula of Malliavin Calculus that can be stated as follows. For definitions and results we refer the reader to Section 2 of this article or Nualart [7], Theorem 2.1.4 and Proposition 2.1.5 p.102-103 or Sanz [8], proposition 5.4 p.67.

Proposition 1.1 *Let $G \in \mathbb{D}^\infty$, $F = (F_1, ..., F_d) \in (\mathbb{D}^\infty)^d$ be a nondegenerate random vector. Then*

$$p_F(\mathbf{x}) = E\left[\prod_{i=1}^{d} \mathbf{1}_{[0,\infty)}(F_i - x_i) H_{(1,2,\ldots,d)}(F; G)\right], \tag{1.1}$$

where

$$H_{(1)}(F;G) = \sum_{j=1}^{d} D^*\Big((\gamma_F^{-1})^{1j} DF^j G\Big),$$

$$H_{(1,\dots,i)}(F;G) = \sum_{j=1}^{d} D^*\Big(H_{(1,\dots,i-1)}(F;G)(\gamma_F^{-1})^{ij} DF^j\Big). \quad (i = 2,\dots,d)$$

Here D^* denotes the adjoint operator of the Malliavin derivative operator D and $\gamma_F^{ij} =< DF^i, DF^j >$ is the so called Malliavin covariance matrix.

The Malliavin covariance matrix is a matrix with random elements that replaces the concept of covariance matrix in Gaussian settings. We do not give the exact definition of a nondegenerate random vector but it comprises the fact that the random vector F has to be differentiable and that the Malliavin covariance matrix is invertible and its inverse has all moments and is differentiable too.

Expression (1.1) has lead to various results concerning theoretical estimates of the density, its support etc. Nevertheless this expression is not a tractable expression for computer simulation as it may seem at first.

One can prove that D^* is some sort of integral which extends the Itô stochastic integral. Therefore, $H_{(1,\dots,d)}(F;G)$ is expressed using a d-iterated Skorohod integral.

The Skorohod integral being a non-adapted integral is not easy to simulate in iterative form and therefore the above expression takes a relatively large amount of time to be simulated when d is big unless an explicit expression for $H_{(1,\dots,d)}(F;G)$ is obtained. Besides this problem, one often finds also problems of high variance and therefore variance reduction methods have to be incorporated making the problem even less tractable from an applied point of view.

We try to explain this with some experiment. First, we consider as the density to simulate the two dimensional lognormal distribution whose graph is given by

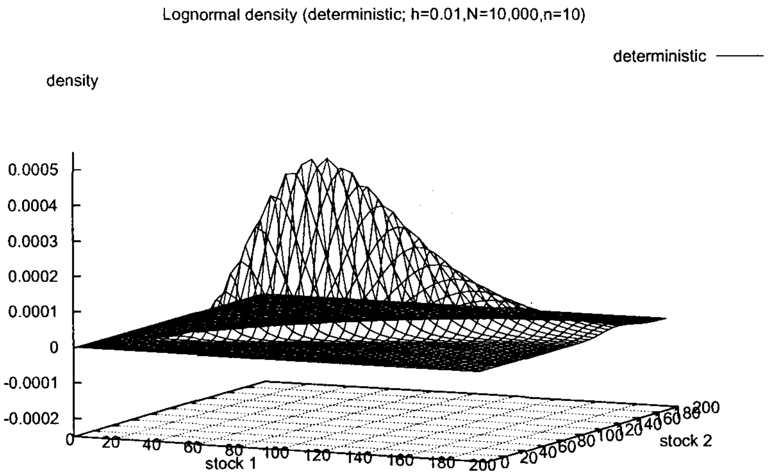


Figure 1: 2-dim. Lognormal Density

That is, the above is the density of (X_1^1, X_1^2) solution of

$$X_t^1 = 100 + 0.01 \int_0^t X_s^1 ds + 0.2 \int_0^t X_s^1 dW_s^1 + 0.3 \int_0^t X_s^1 dW_s^2,$$
$$X_t^2 = 100 + 0.02 \int_0^t X_s^2 ds + 0.2 \int_0^t X_s^2 dW_s^1 + 0.1 \int_0^t X_s^2 dW_s^2. \quad (1.2)$$

Here (W^1, W^2) is a 2-dimensional Wiener process. The above equation being linear has an explicit solution given by

$$X_t^1 = 100 \exp\left((0.01 - \frac{0.2^2 + 0.3^2}{2})t + 0.2 W_t^1 + 0.3 W_t^2\right)$$
$$X_t^2 = 100 \exp\left((0.02 + \frac{0.2^2 + 0.1^2}{2})t + 0.2 W_t^1 + 0.1 W_t^2\right).$$

Therefore the exact density can be computed which is given in the figure above. The parameters h, N and n are the simulation parameters that will be used in all the examples so that they are easy to compare but in this case they do not play any role.

Now we use Proposition 1.1 in order to approximate the same density. Guided by the law of large numbers we know that

$$p_F(\mathbf{x}) \approx \frac{1}{N}\sum_{j=1}^{N}\left(\prod_{i=1}^{d}\mathbf{1}_{[0,\infty)}(F_i^{(j)}-x_i)\right)H_{(1,\ldots,d)}(F;1)^{(j)}$$

The index j in the above formula indicates independent copies of the respective random variables.

Once this simulation is done for (1.3) one obtains[1]

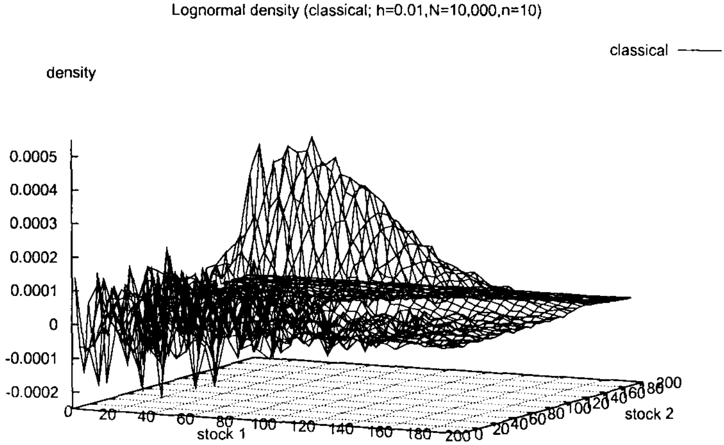


Figure 2: Classical integration by parts formula of Malliavin Calculus

[1]In general, one can not expect to solve a general non-linear stochastic differential equation as we did in (1.3). Therefore, one usually uses the Euler-Maruyama approximation of (1.3). That is, for $n \in \mathbb{N}$, define $t_i = \frac{i}{n}$. Then $\bar{X}_0^1 = 100, \bar{X}_0^2 = 100$ and

$$\begin{aligned}\bar{X}^1_{t_{i+1}} &= \bar{X}^1_{t_i}\left(1+0.01(t_{i+1}-t_i)+0.2(W^1_{t_{i+1}}-W^1_{t_i})+0.3(W^2_{t_{i+1}}-W^2_{t_i})\right),\\ \bar{X}^2_{t_{i+1}} &= \bar{X}^2_{t_i}\left(1+0.02(t_{i+1}-t_i)+0.2(W^1_{t_{i+1}}-W^1_{t_i})+0.1(W^2_{t_{i+1}}-W^2_{t_i})\right). \qquad (1.3)\end{aligned}$$

So to carry out this simulation, we need to simulate the increments of the Wiener process (which have Gaussian laws) and carry out an iterative procedure as the above formula shows.

In this case the parameters needed to carry out this simulation are N and n. h does not play any role in this approximation. These two values have been chosen after general theoretical results that assure that the Monte Carlo effects are minimized in the above formula. So what we see in the above picture is due to the effect on n the number of intervals taken in the discretization of the multiple Skorohod integrals defined in H. Then the expression within H is computed explicitely in all detail which after simulation gives the above graph. The conclusion is clear: The approximation is not very good for values close to zero. Nevertheless the approximation toward higher values is not so bad. This effect is due to the use of the function $\mathbf{1}_{[0,\infty)]}$. Using $-\mathbf{1}_{(-\infty,0]}$ leads to the reverse result. Using $0.5\,(\mathbf{1}_{[0,\infty)]} - \mathbf{1}_{(-\infty,0]})$ leads to uniform results on the whole space but the errors are still big.

In fact, this result has been known for some time and this has purported the use of variance reduction methods. Still the amount of calculations seem to be too high to think that a variance reduction method is the final solution to the problem.

Recently, Malliavin and Thalmaier [6] (Section 4.5.) gave a new integration by parts formula that seems to alleviate the computational burden for simulation of densities in high dimension. In fact, Malliavin and Thalmaier express the multi-dimensional delta function as

$$\delta_0(\mathbf{x}) = \Delta Q_d(\mathbf{x}), \tag{1.4}$$

where $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator and Q_d is the fundamental solution of Poisson equation. That is,

$$Q_2(\mathbf{x}) = a_2^{-1} \ln|\mathbf{x}| \quad \textit{and} \quad Q_d(\mathbf{x}) = -a_d^{-1} \frac{1}{|\mathbf{x}|^{d-2}} \quad (d \geq 3).$$

Here a_d is the area of the unit sphere in $\mathbf{R}^d$.

Then they obtain the following representation for the density of F

$$p_F(\mathbf{x}) = E\left[\sum_{i=1}^{d} \frac{\partial}{\partial x_i} Q_d(F - \mathbf{x}) H_{(i)}(F; G)\right]. \tag{1.5}$$

Therefore one needs to simulate $H_{(i)}(F;G)$ which involves only one Skorohod integral instead of the previous d Skorohod integrals.

In fact, if we partition the time interval in N intervals in order to carry out simulations of the increments of the Wiener process, then the iterated Skorohod integrals appearing in (1.1) will require the calculation over N^d cross-intervals. Instead formula (1.5) only requires Nd. Graphically the situation is as follows:

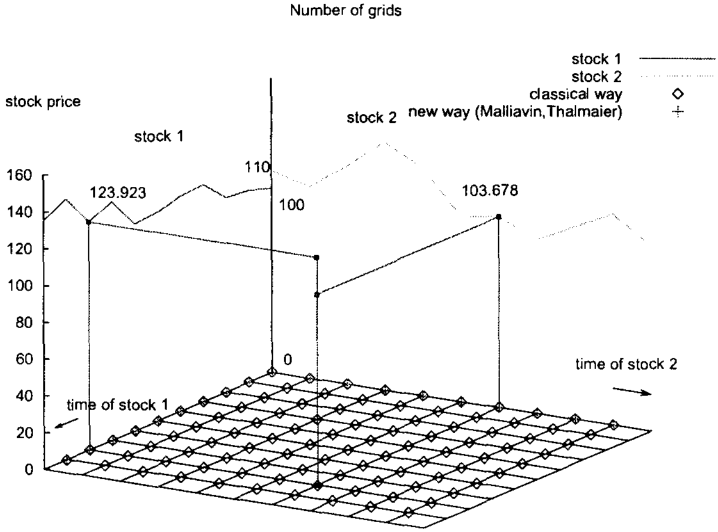


Figure 3: Number of simulation grids

That is, in the procedure indicated by formula (1.1) we need to simulate increments in every blue diamond point indicated in the picture. In comparison the Malliavin-Thalmaier formula suggests that only with the simulation in the points marked with red diamonds (in both time axis) is enough to carry out the simulation. This introduces a reduction from n^2 to $2n$ in the two dimensional case.

In principle, one expects then that the calculation time will be highly reduced. Nevertheless, the high variance problem in formula (1.1) is taken to an extreme as the variance of the estimator in (1.5) is infinite. This problem appears because the limit of $\frac{\partial}{\partial x_i} Q_d(\mathbf{x})$ at $\mathbf{x} = 0$ is ∞.

In fact, the derivatives of the Poisson kernel are

$$\frac{\partial}{\partial x_i} Q_d(\mathbf{x}) = A_d \frac{x_i}{|\mathbf{x}|^d}, \tag{1.6}$$

where $i = 1, ..., d$, $A_2 := a_2^{-1}$ and for $d \geq 3$, $A_d := a_d^{-1}(d-2)$.

If one carries the simulation of the Malliavin-Thalmaier formula through the calculation of

$$p_F(\mathbf{x}) \approx \frac{1}{N} \sum_{j=1}^{N} \sum_{i=1}^{d} \frac{\partial}{\partial x_i} Q_d(F^{(j)} - \mathbf{x}) H_{(i)}(F, 1)^{(j)},$$

one obtains the following result

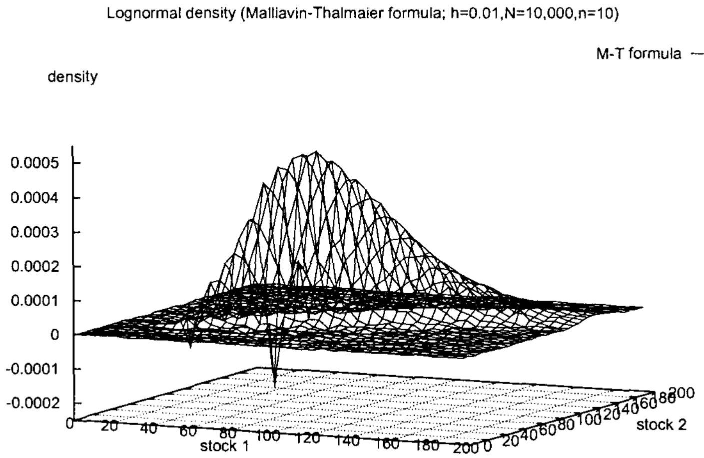
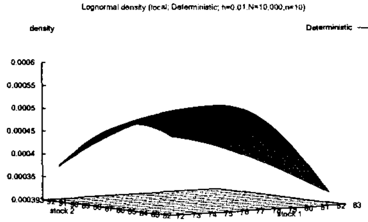
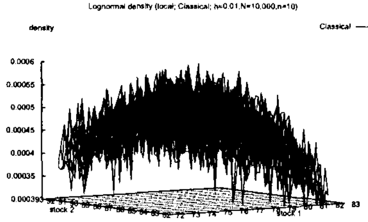


Figure 4: Simulation of the Malliavin-Thalmaier formula

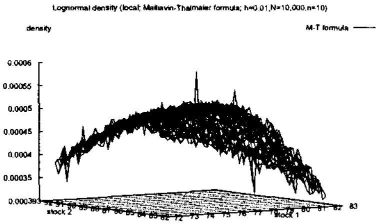
As one can see from the above result, overall this approximation behaves better than the classical formula (1.1) but still one has some undesirable spikes that come from the unstability of the term $\dfrac{\partial}{\partial x_i} Q_d(F^{(j)} - \mathbf{x})$ in the above formula. These issues do not only appear in localized points as it seems in the above picture. This effect also appears in the local level as the following pictures can show.



(a) Exact answer



(b) Classical ibp



(c) Malliavin-Thalmaier formula

Figure 5: Local Comparison

As we can see the same conclusions that were drawn from the global picture can also be drawn at a local level.

Therefore to solve this problem, we propose a slightly modified estimator that depends on a modification parameter h which will tend to the function $\frac{\partial}{\partial x_i} Q_d(\mathbf{x})$ as $h \to 0$.

In fact, we propose:

For $i = 1, ..., d$, define the following approximation to $\frac{\partial}{\partial x_i} Q_d$,

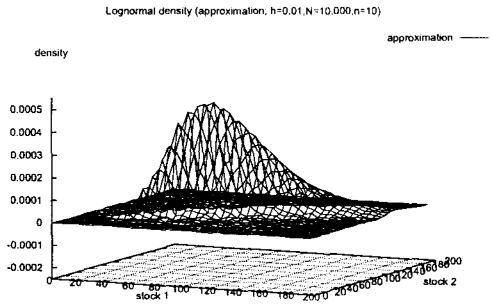
$$\frac{\partial}{\partial x_i} Q_d^h(\mathbf{x}) := A_d \frac{x_i}{|\mathbf{x}|_h^d}.$$

Here $|\mathbf{x}|_h := \sqrt{\sum_{i=1}^d x_i^2 + h}$ $(h > 0,\ \mathbf{x} \in \mathbf{R}^d)$. This approach will generate a small bias and a large variance which is not infinite. Then we try to control the explosive behavior of the variance using the number of simulations. This type of calculation is common in kernel density estimation methods although here the problem differs in the fact that the modification is not of the same type as in kernel density estimation.

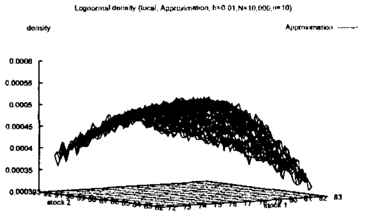
The simulation results for our proposed approximation are as follows.

After obtaining these error estimations and the corresponding optimal parameter h, we apply the Malliavin-Thalmaier formula to finance, especially to the calculation of Greeks. In the one dimensional case, a method to obtain Greeks by the integration by parts formula was introduced by Fournié *et al* [4]. Here we focus our attention to the high dimensional case. We give an expression of Greeks, which is derived using the Malliavin-Thalmaier formula. In particular, the weights are free from the curse of dimensionality, that is, the expression does not have a d-iterated Skorohod integral.

In section 2, we set up our problem. In section 3, we estimate the difference of (1.5) and modified density of (1.5). In section 4, we obtain the rate of divergence of the variance of the approximative estimator. In section 5, we obtain the central limiting theorem for the error of approximation. In section 6, we consider as an example the two dimensional Geometric Brownian motion in order to show the high variance of the Malliavin-Thalmaier estimator and the performance of the corrected estimator. In section 7, we apply the Malliavin-Thalmaier formula to the calculation of Greeks in Finance and we compute the Delta of a bivariate digital European type option on the final stock value and the volatility value under the Heston model.



(a) Global view



(b) Local view

Figure 6: The approximation of the Malliavin-Thalmaier formula

As this article is pedagogical in nature, we have done this long introduction to give a first glimpse of our results from the practical point of view. Next we give the sequence of theorems that back the simulation results. In this article we only quote the results. For the proofs of technical lemmas, we refer the reader to the full paper that will appear elsewhere.

Also note that the expression in (1.1) corresponds to a density only in the case that $G = 1$. In general, it represents a conditional expectation multiplied by the density (see Lemma 3.3). To avoid introducing further terminology, we will keep referring to $p_F(x)$ as the "density".

2 Preliminaries

Let us introduce some notations. For a multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \{1, ..., d\}^n$, we denote by $|\alpha| = n$ the length of the multi-index.

2.1 Malliavin Calculus

Let $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ be a filtered probability space. Here $\{\mathcal{F}_t\}$ satisfies the usual conditions. That is, it is right-continuous and $\mathcal{F}_0$ contains all the P-negligible events in $\mathcal{F}$. Suppose that H is a real separable Hilbert space whose norm and inner product are denoted by $\|\cdot\|_H$ and $< \cdot, \cdot >_H$ respectively. Let $W(h)$ denote a Wiener process on H.

We denote by $C_p^\infty(\mathbb{R}^n)$ the set of all infinitely differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that f and all of its partial derivatives have at most polynomial growth.

Let $\mathcal{S}$ denote the class of *smooth* random variables of the form

$$F = f(W(h_1), ..., W(h_n)), \tag{2.1}$$

where $f \in C_p^\infty(\mathbf{R}^n)$, $h_1, ..., h_n \in H$, and $n \geq 1$.

If F has the form (2.1) we define its derivative DF as the H-valued random variable given by

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(h_1), ..., W(h_n))h_i.$$

We will denote the domain of D in $L^p(\Omega)$ by $\mathbb{D}^{1,p}$. This space is the closure of the class of smooth random variables $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p} = \left\{E\left[|F|^p\right] + E\left[\|DF\|_H^p\right]\right\}^{\frac{1}{p}}.$$

We can define the iteration of the operator D in such a way that for a smooth random variable F, the derivative $D^k F$ is a random variable with values on $H^{\otimes k}$. Then for every $p \geq 1$ and $k \in \mathbb{N}$ we introduce a seminorm on $\mathcal{S}$ defined by

$$\|F\|_{k,p}^p = E\left[|F|^p\right] + \sum_{j=1}^{k} E\left[\|D^j F\|_{H^{\otimes j}}^p\right].$$

For any real $p \geq 1$ and any natural number $k \geq 0$, we will denote by $\mathbb{D}^{k,p}$ the completion of the family of smooth random variables $\mathcal{S}$ with respect to the norm $\|\cdot\|_{k,p}$. Note that $\mathbb{D}^{j,p} \subset \mathbb{D}^{k,q}$ if $j \geq k$ and $p \geq q$.

Consider the intersection

$$\mathbb{D}^{\infty} = \bigcap_{p \geq 1} \bigcap_{k \geq 1} \mathbb{D}^{k,p}.$$

Then $\mathbb{D}^{\infty}$ is a complete, countably normed, metric space.

We will denote by D^* *the adjoint of the operator* D as an unbounded operator from $L^2(\Omega)$ into $L^2(\Omega; H)$. That is, the domain of D^*, denoted by Dom(D^*), is the set of H-valued square integrable random variables u such that

$$|E[< DF, u >_H]| \leq c\|F\|_2,$$

for all $F \in \mathbb{D}^{1,2}$, where c is some constant depending on u. (here $\|\cdot\|_2$ denotes the $L^2(\Omega)$-norm.)

Suppose that $F = (F_1, ..., F_d)$ is a random vector whose components belong to the space $\mathbb{D}^{1,1}$. We associate with F the following random symmetric nonnegative definite matrix:

$$\gamma_F = \Big(< DF_i, DF_j >_H \Big)_{1 \leq i,j \leq d}.$$

This matrix is called the *Malliavin covariance matrix* of the random vector F.

Definition 2.1 *We will say that the random vector $F = (F_1, ..., F_d) \in (\mathbb{D}^{\infty})^d$ is nondegenerate if the matrix γ_F is invertible a.s. and*

$$(\det \gamma_F)^{-1} \in \bigcap_{p \geq 1} L^p(\Omega). \tag{2.2}$$

2.2 Malliavin-Thalmaier Representation of Multi-Dimensional Density Functions

We represent the delta function by

$$\delta_0(\mathbf{x}) = \Delta Q_d(\mathbf{x}) \quad (\mathbf{x} \in \mathbf{R}^d,\ d \geq 2),$$

in the following sense. If f is a smooth function then the solution of the Poisson equation $\Delta u = f$ is given by the convolution $Q_d * f$.

Definition 2.2 *Given the $\mathbf{R}^d$-valued random vector F and the $\mathbf{R}$-valued random variable G, a multi-index α and a power $p \geq 1$ we say that there is an integration by parts formula (IBP formula) in Malliavin sense if there exists a random variable $H_\alpha(F; G) \in L^p(\Omega)$ such that*

$$IP_{\alpha,p}(F, G):\ E\left[\frac{\partial^\alpha}{\partial \mathbf{x}^\alpha} f(F) G\right] = E\big[f(F) H_\alpha(F; G)\big] \text{ for all } f \in C_0^{|\alpha|}(\mathbf{R}^d). \tag{2.3}$$

Related to the Malliavin-Thalmaier formula, Bally and Caramellino [2], have obtained the following result

Proposition 2.3 *(Bally, Caramellino [2]) Suppose that for some $p > 1$*

$$\sup_{|\mathbf{a}| \leq R} E\left[\left|\frac{\partial}{\partial x_i} Q_d(F - \mathbf{a})\right|^{\frac{p}{p-1}} + \left|Q_d(F - \mathbf{a})\right|^{\frac{p}{p-1}}\right] < \infty \quad \text{for all } R > 0,\ \mathbf{a} \in \mathbf{R}^d \tag{2.4}$$

(i). If $IP_{i,p}(F; G)$ $(i = 1, ..., d)$ holds then the law of F is absolutely continuous with respect to the Lebesgue measure on $\mathbf{R}^d$ and the density p_F is represented as

$$p_F(\mathbf{x}) = E\left[\sum_{i=1}^{d} \frac{\partial}{\partial x_i} Q_d(F - \mathbf{x}) H_{(i)}(F; G)\right]. \tag{2.5}$$

(ii). If $IP_{\alpha,p}(F;G)$ holds for every multi-index α with $|\alpha| \leq m+1$ then $p_F \in C^m(\mathbf{R}^d)$ and for every multi-index ρ with $|\rho| \leq m$ one has

$$\frac{\partial^\rho}{\partial \mathbf{x}^\rho} p_F(\mathbf{x}) = E\left[\sum_{i=1}^{d} \frac{\partial}{\partial x_i} Q_d(F-\mathbf{x}) H_{(i,\rho)}(F;G)\right].$$

The heuristic idea of the above proof is to use the integration by parts formula in Malliavin sense as follows

$$\begin{aligned} p_F(\mathbf{x}) &= E\left[\Delta Q_d(F-\mathbf{x})G\right] = \sum_{i=1}^{d} E\left[\frac{\partial^2}{\partial x_i^2} Q_d(F-\mathbf{x})G\right] \\ &= E\left[\sum_{i=1}^{d} \frac{\partial}{\partial x_i} Q_d(F-\mathbf{x}) H_{(i)}(F;G)\right]. \end{aligned}$$

Next we impose conditions to assure that the assumptions of proposition 2.3 are satisfied. The proof is given in the Appendix.

Corollary 2.4 *If $G \in \mathbb{D}^\infty$, $F = (F_1, ..., F_d) \in (\mathbb{D}^\infty)^d$ is a nondegenerate random vector, then the probability density function of the random vector F is*

$$p_F(\mathbf{x}) = E\left[\sum_{i=1}^{d} \frac{\partial}{\partial x_i} Q_d(F-\mathbf{x}) H_{(i)}(F;G)\right].$$

3 Error Estimation

In this section, we find the rate of convergence of the modified estimator of the density at $\mathbf{x} \in \mathbf{R}^d$. Through this section, we always assume $G \in \mathbb{D}^\infty$, $F = (F_1, ..., F_d) \in (\mathbb{D}^\infty)^d$ is d-dimensional nondegenerate random variable. Therefore $IP_{\alpha,p}(F;G)$ will always hold (see Nualart [7], Proposition 2.1.4, p.100 or Sanz [8], Proposition 5.4 p.67).

We start with some definitions and notations to be used in what follows.

Definitions and Notation

1. Define $|\cdot|_h$ by

$$|\mathbf{x}|_h := \sqrt{\sum_{i=1}^{d} x_i^2 + h} \quad (h > 0,\ \mathbf{x} \in \mathbf{R}^d).$$

2. For $i = 1, ..., d$, define the following approximation to $\dfrac{\partial}{\partial x_i} Q_d$,

$$\frac{\partial}{\partial x_i} Q_d^h(\mathbf{x}) := A_d \frac{x_i}{|\mathbf{x}|_h^d}.$$

3. Then we define the approximation to the density function of F as;

$$p_F^h(\mathbf{x}) := E\left[\sum_{i=1}^{d} \frac{\partial}{\partial x_i} Q_d^h(F - \mathbf{x}) H_{(i)}(F; G)\right]. \tag{3.1}$$

4. Consider a function η which satisfies;

$$\begin{cases} (i). & \eta \in C_0^\infty(\mathbf{R}^d), \quad \eta(\mathbf{x}) \geq 0 \quad (\mathbf{x} \in \mathbf{R}^d), \\ (ii). & \mathrm{supp}(\eta) \subset \{\mathbf{x} \in \mathbf{R}^d |\ |\mathbf{x}| \leq 1\}, \\ (iii). & \int_{\mathbf{R}^d} \eta(\mathbf{x}) d\mathbf{x} = 1, \\ (iv). & \eta(\mathbf{x}) \text{ is constant on } \mathbf{x} \in \partial B(0, r). \end{cases}$$

5. For each $\varepsilon > 0$, we define $\eta_\varepsilon(\mathbf{x})$ as

$$\eta_\varepsilon(\mathbf{x}) := \frac{1}{\varepsilon^d} \eta\Big(\frac{\mathbf{x}}{\varepsilon}\Big). \tag{3.2}$$

6. We define $\tilde{\eta}_\varepsilon(\mathbf{x})$;

$$\tilde{\eta}_\varepsilon(\mathbf{x}) := \int_{-\infty}^{x_d} \cdots \int_{-\infty}^{x_1} \eta_\varepsilon(\mathbf{y}) dy_1 ... dy_d. \ (\leq 1 \quad \text{from } \mathbf{4}.) \tag{3.3}$$

Remark 3.1 *$\tilde{\eta}_\varepsilon(\mathbf{x})$ has the following property;*

$$\tilde{\eta}_\varepsilon(\mathbf{x}) \longrightarrow \prod_{i=1}^{d} \mathbf{1}(F_i \leq x_i) + \frac{1}{2^d} \mathbf{1}_{\{0\}}(\mathbf{x}) \quad (\text{as } \varepsilon \to 0).$$

Lemma 3.2 *Let $\alpha \in \{1,...,d\}^n$, $n \in \mathbf{N} \cup \{0\}$, be any multi-index. Suppose $G \in \mathbb{D}^\infty$.*

$$\lim_{\varepsilon\to 0} \frac{\partial^\alpha}{\partial \mathbf{y}^\alpha} E\Big[\eta_\varepsilon(F-\mathbf{x})G\Big] = E\left[\left(\prod_{i=1}^d \mathbf{1}(F_i \le x_i)\right) H_{(1,...,d,\alpha)}(F;G)\right] < +\infty.$$

Proof. By IBP formula, Proposition 2.3 (i), (3.3) and dominated convergence theorem,

$$\begin{aligned}\lim_{\varepsilon\to 0} \frac{\partial^\alpha}{\partial \mathbf{y}^\alpha} E\Big[\eta_\varepsilon(F-\mathbf{x})G\Big] &= \lim_{\varepsilon\to 0} E\left[\tilde{\eta}_\varepsilon(F-\mathbf{x})H_{(1,...,d,\alpha)}(F;G)\right] \\ &= E\left[\left(\prod_{i=1}^d \mathbf{1}(F_i \le x_i) + \tfrac{1}{2^d}\mathbf{1}_{\{0\}}(F-\mathbf{x})\right) H_{(1,...,d,\alpha)}(F;G)\right] \\ &= E\left[\left(\prod_{i=1}^d \mathbf{1}(F_i \le x_i)\right) H_{(1,...,d,\alpha)}(F;G)\right] < +\infty.\end{aligned}$$

■

Lemma 3.3 *Let $G \in \mathbb{D}^\infty$. Then*

$$\lim_{\varepsilon\to 0} E\Big[\eta_\varepsilon(F-\mathbf{z})G\Big] = E[G|F=\mathbf{z}]p_F(\mathbf{z}).$$

Proof. By IBP formula, for $\varphi \in C_0^\infty(\mathbf{R}^d)$,

$$\int_{\mathbf{R}^d} E\left[\left(\prod_{i=1}^d \mathbf{1}(F_i \le z_i)\right) H_{(1,...,d)}(F;G)\right]\varphi(\mathbf{z})d\mathbf{z}$$

$$= \int_{\mathbf{R}^d} E\left[\left(\frac{\partial}{\partial x_1}\cdots\frac{\partial}{\partial x_d}\prod_{i=1}^d \mathbf{1}(F_i \le z_i)\right)G\right]\varphi(\mathbf{z})d\mathbf{z} = E[\varphi(F)G]$$

$$= \int_{\mathbf{R}^d} E[G|F=\mathbf{z}]p_F(\mathbf{z})\varphi(\mathbf{z})d\mathbf{z}.$$

Therefore from Lemma 3.2,

$$\lim_{\varepsilon\to 0} E\Big[\eta_\varepsilon(F-\mathbf{z})G\Big] = E\left[\left(\prod_{i=1}^d \mathbf{1}(F_i \le z_i)\right) H_{(1,...,d)}(F;G)\right] = E[G|F=\mathbf{z}]p_F(\mathbf{z}).$$

■

3.1 Error Estimation

The next result gives the order of the error of the approximation to the density.

Theorem 3.4 *Let F be a nondegenerated random vector, then*

$$p_F(x) - p_F^h(x) = C_1^x h \ln \frac{1}{h} + C_2^x h + o(h), \tag{3.4}$$

where

$$C_1^x := \sum_{i=1}^{d} C_{1,i}^x \quad \text{and} \quad C_2^x := \sum_{i=1}^{d} \left\{ C_{2,i}^x + \sum_{j,k=1}^{d} C_{3,i,j,k}^x + C_{4,i}^x \right\}$$

and the constants appearing above are defined in Lemmas 8.3, 8.4 and 8.5 in the Appendix.

Proof. Denote by $s_i = \sin\theta_i$ and $c_i = \cos\theta_i$ $(i = 1, ..., d)$. As we will have to change from rectangular to spherical coordinates, to avoid long expressions we define $\Theta := (\Theta_1, ..., \Theta_d)^*$ as the coordinate change

$$\begin{aligned} z_1 - x_1 &= r\cos(\theta_1)\cos(\theta_2)\cdots\cos(\theta_{d-2})\cos(\theta_{d-1}) =: r\Theta_1 \\ z_i - x_i &= r\cos(\theta_1)\cdots\cos(\theta_{d-i})\sin(\theta_{d-i+1}) =: r\Theta_i \quad (i = 2, ..., d) \end{aligned}$$

where $0 \le r < \infty$, $-\frac{\pi}{2} \le \theta_j \le \frac{\pi}{2}$, $i = 1, ..., d-2$, $0 \le \theta_{d-1} \le 2\pi$.
First note that Q_d^h and its derivatives are bounded for fixed h. By using Lemma 3.3, Taylor expansion and spherical coordinates,

$$p_F(\mathbf{x}) - p_F^h(\mathbf{x}) = E\left[\sum_{i=1}^d \left(\frac{\partial}{\partial x_i} Q_d(F - \mathbf{x}) - \frac{\partial}{\partial x_i} Q_d^h(F - \mathbf{x})\right) H_{(i)}(F; G)\right]$$

$$= A_d \sum_{i=1}^d \int_{\mathbf{R}^d} \left(\frac{z_i - x_i}{|\mathbf{z}-\mathbf{x}|^d} - \frac{z_i - x_i}{|\mathbf{z}-\mathbf{x}|_h^d}\right)\left(\lim_{\varepsilon\to\infty} E\Big[\eta_\varepsilon(F - \mathbf{z}) H_{(i)}(F; G)\Big]\right) dz_1 \cdots dz_d \tag{3.5}$$

$$= A_d \sum_{i=1}^d \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdots \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\int_0^1 + \int_1^\infty\right) \frac{(r^2+h)^{\frac{d}{2}} - r^d}{(r^2+h)^{\frac{d}{2}}} \Theta_i c_1^{d-2} \cdots c_{d-2}$$

$$\times \left(\lim_{\varepsilon\to 0} \Phi_{i,\varepsilon}^F (r\Theta + \mathbf{x})\right) dr d\theta_1 ... d\theta_{d-1}.$$

where $\Phi^F_{i,\varepsilon}(\mathbf{z}) = E[\eta_\varepsilon(F-\mathbf{z})H_{(i)}(F;G)]$ $(i=1,...,d)$. Here note that the limits appearing in the above formula exist due to Lemma 3.2.

Next, we consider the integral for $r \in [0,1]$ where the following Taylor formula is used

$$\Phi^F_{i,\varepsilon}(\mathbf{z}) = \Phi^F_{i,\varepsilon}(\mathbf{x}) + \sum_{j=1}^{d} r\Theta_j \frac{\partial}{\partial y_j}\Phi^F_{i,\varepsilon}(\mathbf{x}) + \frac{1}{2}\sum_{j,k=1}^{d} r^2\Theta_j\Theta_k \int_0^1 \frac{\partial^2}{\partial y_k \partial y_j}\Phi^F_{i,\varepsilon}(\mathbf{x}+\gamma r\Theta)d\gamma.$$

This leads to three terms, whose order of convergence are analyzed respectively in Lemmas 8.2, 8.3 and 8.4 and in the Appendix. Finally, the integral term for $r \in [1,+\infty)$ is analyzed in Lemma 8.5 in the Appendix. Therefore one obtains that

$$p_F(x) - p^h_F(x) = \sum_{i=1}^{d}\left\{C^x_{1,i}h\ln\frac{1}{h} + C^x_{2,i}h + o(h) + \sum_{j,k=1}^{d} C^x_{3,i,j,k}h + o(h) + C^x_{4,i}h + o(h)\right\}.$$

The constants are explicitly given in the Appendix. ∎

4 Estimation of the Variance of the Approximation

In this section, we try to estimate the rate at which the variance of the estimator using Q^h_d diverges. That is,

$$\begin{aligned} &E\left[\left(\sum_{i=1}^{d}\frac{\partial}{\partial x_i}Q^h_d(F-\mathbf{x})H_{(i)}(F;G) - p_F(\mathbf{x})\right)^2\right] \\ &= E\left[\left(\sum_{i=1}^{d}\frac{\partial}{\partial x_i}Q^h_d(F-\mathbf{x})H_{(i)}(F;G)\right)^2\right] + 2p_F(\mathbf{x})\left\{p_F(\mathbf{x}) - p^h_F(\mathbf{x})\right\} - p_F(\mathbf{x})^2. \end{aligned} \tag{4.1}$$

Note that therefore is enough to estimate the rate of divergence of the first term in (4.1) as the second term converges to 0 (proven in Section 3.1) and the third is a constant. The term we will estimate is then

$$E\left[\left(\sum_{i=1}^{d}\frac{\partial}{\partial x_i}Q_d^h(F-\mathbf{x})H_{(i)}(F;G)\right)^2\right]$$
$$=\sum_{i,j=1}^{d}E\left[\frac{\partial}{\partial x_i}Q_d^h(F-\mathbf{x})\frac{\partial}{\partial x_j}Q_d^h(F-\mathbf{x})H_{(i)}(F;G)H_{(j)}(F;G)\right].$$

Let $\hat{\Phi}^F_{i,j,\varepsilon}(\mathbf{z}) := E[\eta_\varepsilon(F-\mathbf{z})H_{(i)}(F;G)H_{(j)}(F;G)]$.

4.1 Case $d = 2$

Theorem 4.1 *Let F be a non-degenerate random vector. Then*

$$E\left[\left(\sum_{i=1}^{2}\frac{\partial}{\partial x_i}Q_2^h(F-\mathbf{x})H_{(i)}(F;G)-p_F(\mathbf{x})\right)^2\right]=C_3^x\ln\frac{1}{h}+O(1),$$

where $C_3^x := \sum_{i=1}^{2}C_{5,i}^x$ and the constants $C_{5,i}^x$ are defined in Lemma 8.6 in the Appendix.

Proof. For $i,j=1,2$, by using Lemma 3.3, Taylor expansion and spherical coordinates,

$$E\left[\frac{\partial}{\partial x_i}Q_2^h(F-\mathbf{x})\frac{\partial}{\partial x_j}Q_2^h(F-\mathbf{x})H_{(i)}(F;G)H_{(j)}(F;G)\right]$$

$$=A_2^2\int_{\mathbf{R}^2}\frac{(z_i-x_i)(z_j-x_j)}{|\mathbf{z}-\mathbf{x}|_h^4}\left(\lim_{\varepsilon\to 0}\hat{\Phi}^F_{i,j,\varepsilon}(\mathbf{z})\right)dz_1dz_2$$

$$=A_2^2\int_0^{2\pi}\int_0^{2|\mathbf{x}|+1}r\frac{r^2\Theta_i\Theta_j}{(r^2+h)^2} \tag{4.2}$$

$$\times\left\{\lim_{\varepsilon\to 0}\left(\hat{\Phi}^F_{i,j,\varepsilon}(\mathbf{x})+\sum_{k=1}^{2}r\Theta_k\int_0^1\frac{\partial}{\partial y_k}\hat{\Phi}^F_{i,j,\varepsilon}(\mathbf{x}+\gamma r\Theta)d\gamma\right)\right\}drd\theta$$

$$+A_2^2\int_0^{2\pi}\int_{2|\mathbf{x}|+1}^{\infty}r\frac{r^2\Theta_i\Theta_j}{(r^2+h)^2}\left(\lim_{\varepsilon\to 0}\hat{\Phi}^F_{i,j,\varepsilon}(r\Theta+\mathbf{x})\right)drd\theta.$$

Then by using Lemma 8.6, Lemma 8.7 and Lemma 8.9, we obtain

$$(4.2) = \sum_{i=1}^{2} C^x_{5,i} \ln \frac{1}{h} + O(1).$$

■

4.2 Case $d \geq 3$

Theorem 4.2 *Let F be a nondenegerated random vector, then for $d \geq 3$,*

$$E\left[\left(\sum_{i=1}^{d} \frac{\partial}{\partial x_i} Q^h_d(F - \mathbf{x}) H_{(i)}(F; G) - p_F(\mathbf{x})\right)^2\right] = C^x_4 \frac{1}{h^{\frac{d}{2}-1}} + o\left(\frac{1}{h^{\frac{d}{2}-1}}\right),$$

where $C^x_4 = \sum_{i=1}^{d} C^x_{8,i}$ and the constants $C^x_{8,i}$ are defined in Lemma 8.11.

Proof. For $i, j = 1, ..., d$, by using Lemma 3.3, Taylor expansion and spherical coordinates,

$$\begin{aligned}
&E\left[\frac{\partial}{\partial x_i} Q^h_d(F - \mathbf{x}) \frac{\partial}{\partial x_j} Q^h_d(F - \mathbf{x}) H_{(i)}(F; G) H_{(j)}(F; G)\right] \\
&= A_d^2 \int_{\mathbf{R}^d} \frac{(z_i - x_i)(z_j - x_j)}{|\mathbf{z} - \mathbf{x}|_h^{2d}} \left(\lim_{\varepsilon \to 0} \hat{\Phi}^F_{i,j,\varepsilon}(\mathbf{z})\right) dz_1 ... dz_d \\
&= A_d^2 \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdots \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 \frac{r^2 \Theta_i \Theta_j}{(r^2 + h)^d} r^{d-1} c_1^{d-2} \cdots c_{d-2} \\
&\times \left\{\lim_{\varepsilon \to 0} \left(\hat{\Phi}^F_{i,j,\varepsilon}(\mathbf{x}) + \sum_{k=1}^{d} r\Theta_k \int_0^1 \frac{\partial}{\partial y_k} \hat{\Phi}^F_{i,j,\varepsilon}(\mathbf{x} + \gamma r \Theta) d\gamma\right)\right\} dr d\theta_1 ... d\theta_{d-1} \qquad (4.3) \\
&+ A_d^2 \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdots \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_1^{\infty} \frac{r^2 \Theta_i \Theta_j}{(r^2 + h)^d} r^{d-1} c_1^{d-2} \cdots c_{d-2} \\
&\times \left(\lim_{\varepsilon \to 0} \hat{\Phi}^F_{i,j,\varepsilon}(r\Theta + \mathbf{x})\right) dr d\theta_1 ... d\theta_{d-1}.
\end{aligned}$$

Then from Lemma 8.11, Lemma 8.12 and Lemma 8.13, we can obtain

$$(4.3) = \sum_{i=1}^{d} C_{8,i}^{x} \frac{1}{h^{\frac{d}{2}-1}} + o(\frac{1}{h^{\frac{d}{2}-1}}) + O\left(\frac{1}{h^{\frac{d-2}{2}}}\right) + O(1).$$

■

Remark. In particular, for $h = 0$ one obtains that the variance of the Malliavin-Thalmaier estimator is infinite.

5 The Central Limit Theorem

Obviously when performing simulations, one is also interested in obtaining confidence intervals and therefore the Central Limit Theorem is useful in such a situation. In what follows $\Rightarrow$ denotes weak convergence and the index $j = 1, ..., N$ denote independent copies of the respective random variables. For different j they are independent.

Theorem 5.1 *Let G be a random variable with standard normal distribution.* **(i).** *When $d = 2$, set $n = \frac{C}{h \ln \frac{1}{h}}$ and $N = \frac{C^2}{h^2 \ln \frac{1}{h}}$ for some constant C fixed throughout. Then*

$$n\left(\frac{1}{N}\sum_{j=1}^{N}\sum_{i=1}^{d}\frac{\partial}{\partial x_i}Q_d^h(F^{(j)} - \mathbf{x})H_{(i)}^{(j)}(F^{(j)}; G) - p_F(x)\right) \quad \Longrightarrow \quad \sqrt{C_3^x}G + C_1^x C.$$

(ii). *When $d \geq 3$, set $n = \frac{C}{h \ln \frac{1}{h}}$ and $N = \frac{C^2}{h^{\frac{d}{2}+1}(\ln \frac{1}{h})^2}$ for some constant C fixed throughout. Then*

$$n\left(\frac{1}{N}\sum_{j=1}^{N}\sum_{i=1}^{d}\frac{\partial}{\partial x_i}Q_d^h(F^{(j)} - \mathbf{x})H_{(i)}^{(j)}(F^{(j)}; G) - p_F(x)\right) \quad \Longrightarrow \quad \sqrt{C_4^x}G + C_1^x C.$$

Proof. Consider

$$n\left(\frac{1}{N}\sum_{j=1}^{N}\sum_{i=1}^{d}\frac{\partial}{\partial x_i}Q_d^h(F^{(j)} - \mathbf{x})H_{(i)}^{(j)}(F^{(j)}; G) - p_F(\mathbf{x})\right)$$

$$= \frac{n}{N}\sum_{j=1}^{N}\left\{\sum_{i=1}^{d}\frac{\partial}{\partial x_i}Q_d^h(F^{(j)} - \mathbf{x})H_{(i)}^{(j)}(F^{(j)}; G) - p_F^h(\mathbf{x})\right\} + n\left(p_F^h(\mathbf{x}) - p_F(\mathbf{x})\right).$$

Due to the definition of n and Theorem 3.4 we have that the second term above converges to $C_1^x C$. Therefore it only remains to prove a central limit theorem for $\frac{n}{N}\sum_{j=1}^N \zeta_j^{n,N,h}$ where

$$\zeta_j^{n,N,h} := \sum_{i=1}^d \frac{\partial}{\partial x_i} Q_d^h(F^{(j)} - \mathbf{x}) H_{(i)}^{(j)}(F^{(j)}; G) - p_F^h(\mathbf{x}).$$

To prove this, we compute the characteristic function of $\frac{n}{N}\sum_{j=1}^N \zeta_j^{n,N,h}$. Set $i := \sqrt{-1}$. By Taylor expansion, Lemma 8.14 and Lemma 8.15,

$$\begin{aligned}
&E\left[\exp\left(\frac{iun}{N}\sum_{j=1}^N \zeta_j^{n,N,h}\right)\right]\\
&= \left\{1 - \frac{1}{N}\left(\frac{1}{2}\frac{u^2 n^2}{N} E\left[\left(\zeta_1^{n,N,h}\right)^2\right] - N\mathcal{R}\right)\right\}^N \longrightarrow \exp\left(-\frac{u^2}{2}C_x'\right),
\end{aligned}$$

where when $d = 2$, $C_x' = C_3^x$ and when $d \geq 3$, $C_x' = C_4^x$ and we define the remainder term $\mathcal{R}$ as

$$\mathcal{R} := E\left[\exp\left(\frac{iun}{N}\zeta_1^{n,N,h}\right)\right] - \left\{1 - \frac{1}{2}\frac{u^2 n^2}{N^2} E\left[\left(\zeta_1^{n,N,h}\right)^2\right]\right\}.$$

■

6 Example

In this section, we apply our approximation result to the multi-dimensional log-normal density, that is, the solution of the following stochastic differential equation,

$$\frac{dX_t^i}{X_t^i} = \mu_i dt + \sum_{j=1}^d \sigma_{ij} dW_t^j, \quad X_0^i = x_i. \tag{6.1}$$

where $W = (W^1, ..., W^d)$ is a standard d-dimensional Brownian motion, μ_i and σ_{ij} are constants.

The goal in this section is to rewrite the integration by parts formula in various ways so as to compare the different formulations.

6.1 The approximation to the Malliavin-Thalmaier formula (3.1)

The only element needed to write the formula (3.1) explicitly is to find an expression for the weight $H_{(i)}(F;G)$. In our settings, we have to consider multi-dimensional settings.

(1). Set a complete probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$, then we consider d-dimensional Brownian motion $B_t = (B_t^1, ..., B_t^d)^*$.
(2). Next we define $h := (h_1, ..., h_d) \in H := L^2([0,T]; \mathbf{R}^d) : \ [0,T] \to \mathbf{R}^d$, where $h_i := \mathbf{1}(\cdot \leq T) \in \hat{H} := L^2([0,T]; \mathbf{R}) : [0,T] \to \mathbf{R}$.
(3). Then we put, for $i = 1, ..., d$, $W_i(\cdot) := \int_0^\infty \cdot \, dB_s^i$: $\hat{H} \to \mathbf{R}$.
(4). Let

$$\begin{aligned} f_i(x_1, ..., x_d) &:= y_i^0 \exp\left(\left(\mu_i - \frac{\sum_{j=1}^d \sigma_{ij}^2}{2}\right) T + \sum_{j=1}^d \sigma_{ij} x_i \right), \\ F^i &:= f_i(W_1(h_1), ..., W_d(h_d)). \end{aligned}$$

(5). We define the Malliavin derivatives $D^1, ..., D^d$ (D^i is a map from smooth random variables to $\hat{H}$-valued random variables);

$$D^j F^i := \left\{ \frac{\partial}{\partial x_j} f_i(W_1(h_1), ..., W_d(h_d)) \right\} D^j W_j(h_j) = \sigma_{ij} F^i \mathbf{1}(\cdot \leq T) (\in \hat{H}).$$

And we define the Malliavin derivative $D := (D^1, ..., D^d)^*$;

$$DF^i := \begin{pmatrix} D^1 F^i \\ \vdots \\ D^d F^i \end{pmatrix} = \begin{pmatrix} \sigma_{i1} F^i \mathbf{1}(\cdot \leq T) \\ \vdots \\ \sigma_{id} F^i \mathbf{1}(\cdot \leq T) \end{pmatrix} \quad (\in H).$$

(6). We want to define $(D^1)^*, ..., (D^d)^*$ and D^*. Let $G, G_1, ..., G_d$ be a random variable.

$$(D^i)^*(G\hat{h}) := GW_i(\hat{h}) - < D^i G, \hat{h} >_{\hat{H}} \quad \text{for} \quad \hat{h} \in \hat{H}, \tag{6.2}$$

$$D^*\left(\begin{pmatrix} G_1 h_1 \\ \vdots \\ G_d h_d \end{pmatrix}\right) := \sum_{i=1}^{d} (D^i)^*(G_i h_i) = \sum_{i=1}^{d} G_i W_i(h_i) - \sum_{i=1}^{d} < D^i G_i, h_i >_{\hat{H}}$$

$$\text{for} \quad h^1, ..., h^d \in \hat{H}.$$

(7). Finally we define Malliavin covariance matrix

$$\gamma_F := (< DF^i, DF^j >_H)_{i,j=1,...,d}.$$

And we denote the inverse matrix by γ_F^{-1}.

Then by Lemma 8.16 in the Appendix, we can express the density at $\mathbf{x}$ as;

$$p_F(\mathbf{x}) = A_d \sum_{i=1}^{d} E\left[\frac{F^i - x_i}{|F - \mathbf{x}|^d} \sum_{j=1}^{d} (-1)^{i+j} \frac{\det(\Sigma_i^j)}{\det(\Sigma)} \left\{\frac{W_j(\mathbf{1}(\cdot \le T))}{F^i} + \frac{\sigma_{ij} T}{F^i}\right\}\right]. \tag{6.3}$$

Our approximation to the density is given by

$$p_F^h(\mathbf{x}) = A_d \sum_{i=1}^{d} E\left[\frac{F^i - x_i}{|F - \mathbf{x}|_h^d} \sum_{j=1}^{d} (-1)^{i+j} \frac{\det(\Sigma_i^j)}{\det(\Sigma)} \left\{\frac{W_j(\mathbf{1}(\cdot \le T))}{F^i} + \frac{\sigma_{ij} T}{F^i}\right\}\right]. \tag{6.4}$$

6.2 Simulation

In figures 7 and 8 we show the result of the simulation of (6.3) and (6.4) for the 2-dimensional case at time 1. That is,

$$\frac{dX_t^1}{X_t^1} = 0.01dt + 0.1dW_t^1 + 0.2dW_t^2 \quad \text{and} \quad \frac{dX_t^2}{X_t^2} = 0.02dt + 0.3dW_t^1 + 0.2dW_t^2.$$

We have used $N = 10,000$ Monte Carlo simulations at each point. The result of (6.3) is in Figure 7 and the approximation of the density (6.4) is in Figure

8, where $h = 0.01$. As it can be seen from Figure 7, there are some points where the estimate is unstable. This is clearly due to the infinite variance of the Malliavin-Thalmaier estimator.

In Figure 8 these points do not exist due to the approximation of $\frac{\partial}{\partial x_i} Q_d$.

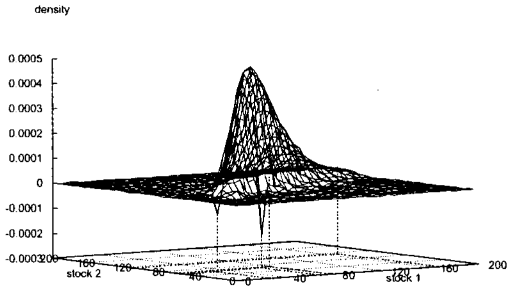


Figure 7: equation (2.5)

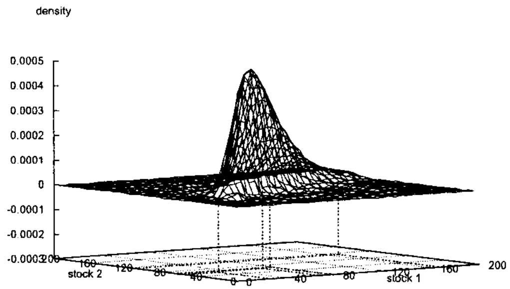


Figure 8: equation (3.1) ($h = 0.01$)

7 Application of the Malliavin-Thalmaier formula to Finance

In this section, we compute Greeks using the Malliavin-Thalmaier Formula. We consider a random vector $F^{\mu} = (F_1^{\mu}, ..., F_d^{\mu})$ $(\mu \in \mathbf{R}^n;\ n \in \mathbf{N})$ which depends on a parameter μ. Suppose that $F^{\mu} \in (\mathbb{D}^{\infty})^d$ is a nondegenerate random vector. And let $f(x_1, ..., x_d)$ be a payoff function in the following class $\mathcal{A}$;[2]

$$\mathcal{A} := \left\{ f : \mathbf{R}^d \to \mathbf{R} : \begin{array}{l} \text{continuous a.e. w.r.t. Lebesgue measure,} \\ \text{and there exist constants } c, a \text{ such that } |f(\mathbf{x})| \leq \frac{c}{(1+|\mathbf{x}|)^a}\ (a > 1). \end{array} \right\}.$$

Note that functions in $\mathcal{A}$ are bounded.
Essentially a greek is defined for $f \in \mathcal{A}$, as the following quantity

$$\frac{\partial}{\partial \mu_j} E\left[f(F_1^{\mu}, ..., F_d^{\mu}) \right]. \quad (j = 1, ..., n)$$

As the study of the second derivative is similar we concentrate on the above quantity and just quote the result for second derivatives. First we give a lemma.

[2] Note that for example in the case of a put option, if we define the payoff function $(K - x)_+$;

$$(K - x)_+ := \begin{cases} K - x & 0 \leq x \leq K \\ 0 & \text{otherwise,} \end{cases}$$

then $(K - x)_+ \in \mathcal{A}$.
In a digital put option case, payoff function is $\mathbf{1}_{[0,K]}(x)$. Therefore it's in $\mathcal{A}$.
Next in a digital call option case, the payoff function $\mathbf{1}_{[K,\infty)}(x)$ doesn't go to 0 as $x \to \infty$. But since stocks don't take negative value, then we can transform as it follows,

$$\mathbf{1}_{[K,\infty)}(x) = 1 - \mathbf{1}_{[0,K)}(x).$$

And now we want to calculate Greeks, that is, derivation of the term 1 is 0. It's enough to calculate the term $\mathbf{1}_{[0,K)}(x)$, which has a compact support.
Finally if we want to compute a Greeks for call option case $(x - K)_+$, then one uses directly g_i and g_i^h after taking the derivative. Although it's known that then a localization is needed.

For $i = 1, \ldots, d$, set

$$\begin{aligned} g_i(\mathbf{y}) &:= \int \cdots \int_{\mathbf{R}^d} f(\mathbf{x}) \frac{\partial}{\partial x_i} Q_d(\mathbf{y} - \mathbf{x}) d\mathbf{x}, \\ g_i^h(\mathbf{y}) &:= \int \cdots \int_{\mathbf{R}^d} f(\mathbf{x}) \frac{\partial}{\partial x_i} Q_d^h(\mathbf{y} - \mathbf{x}) d\mathbf{x}. \end{aligned}$$

And obviously for $x \neq 0$ and $h \to 0$,

$$\frac{\partial}{\partial x_i} Q_d^h(\mathbf{x}) \longrightarrow \frac{\partial}{\partial x_i} Q_d(\mathbf{x})$$

Lemma 7.1 *For $f \in \mathcal{A} \cap L^p(\mathbf{R}^d)$ $(p > 1)$,*

$$g_i^h(\mathbf{y}) \longrightarrow g_i(\mathbf{y}) \quad \text{for all } \mathbf{y} \in \left\{ \mathbf{z} \in \mathbf{R}^d;\ f \text{ is continuous at } \mathbf{z} \right\}.$$

Lemma 7.2 *$f \in \mathcal{A}$ implies that*

$$|g_i(\mathbf{y})| \leq a|\mathbf{y}| + b \quad \text{and} \quad |g_i^h(\mathbf{y})| \leq a|\mathbf{y}| + b,$$

where a and b are constants which depend on d and are independent of h.

The above result follows easily from the assumptions on f. Next we consider convergence in $L^1(\Omega)$.

Lemma 7.3 *Assume that $F^\mu \in (\mathbb{D}^\infty)^d$ is a nondegenerate random vector. And assume that $f \in \mathcal{A}$ is continuous a.e. Then*

$$E\left[g_i^h(F^\mu)\right] \longrightarrow E\left[g_i(F^\mu)\right].$$

Proof. This lemma is trivial from Lemma 7.1 and Lemma 7.2. ∎

We denote expectation with respect to $p_F^h(\mathbf{x})$ by $E^h[\cdot]$. That is,

$$E^h\left[f(F)\right] := \int \cdots \int_{\mathbf{R}^d} f(\mathbf{x}) p_F^h(\mathbf{x}) d\mathbf{x}.$$

Lemma 7.4 *If $f \in \mathcal{A}$, then we have*

$$
\begin{aligned}
E\left[f(F^{\mu})\right] &= \sum_{i=1}^{d} E\left[g_i(F^{\mu})H_{(i)}(F^{\mu};1)\right],\\
E^h\left[f(F^{\mu})\right] &= \sum_{i=1}^{d} E\left[g_i^h(F^{\mu})H_{(i)}(F^{\mu};1)\right].
\end{aligned}
\tag{7.1}
$$

Proof. By the Malliavin-Thalmaier formula,

$$
\begin{aligned}
E\left[f(F^{\mu})\right] &= \int\cdots\int_{\mathbf{R}^d} f(\mathbf{x})E\left[\sum_{i=1}^{d}\frac{\partial}{\partial x_i}Q_d(F^{\mu}-\mathbf{x})H_{(i)}\left(F^{\mu};1\right)\right]d\mathbf{x}\\
&= \sum_{i=1}^{d} E\left[\int\cdots\int_{\mathbf{R}^d} f(\mathbf{x})\frac{\partial}{\partial x_i}Q_d(F^{\mu}-\mathbf{x})d\mathbf{x}H_{(i)}\left(F^{\mu}-\mathbf{x}\right)\right].
\end{aligned}
$$

The second equation (7.1) follows similarly. ■

7.1 First Derivative Case

Now we consider an expression of a first derivative.

Proposition 7.5 *Let $k \in \{1,\ldots,n\}$ be fixed. Suppose that for every $i = 1,\ldots,d$, $H_{(1,\ldots,d,i)}\left(F^{\mu};1\right)$ is differentiable in μ_k and in $L^2(\Omega)$, $\frac{\partial}{\partial \mu_k}H_{(1,\ldots,d,i)}\left(F^{\mu};1\right)$ is in $L^2(\Omega)$, and also $\frac{\partial F_j^{\mu}}{\partial \mu_k} \in L^p(\Omega)$ $(p \geq 4)$ for all $j = 1,\ldots,d$. Then we have*

$$
\frac{\partial}{\partial \mu_k}E^h\left[f(F^{\mu})\right] = \sum_{i=1}^{d}\frac{\partial}{\partial \mu_k}E\left[g_i^h(F^{\mu})H_{(i)}(F^{\mu};1)\right] \longrightarrow
$$

$$
\sum_{i=1}^{d}\frac{\partial}{\partial \mu_k}E\left[g_i(F^{\mu})H_{(i)}(F^{\mu};1)\right] = \frac{\partial}{\partial \mu_k}E\left[f(F^{\mu})\right]
$$

For $i, j = 1,\ldots,d$, put

$$
g_{i,j}^h(\mathbf{y}) := g_{i,j}^h(y_1,\ldots,y_d) := \frac{\partial}{\partial y_j}\int\cdots\int_{\mathbf{R}^d} f(\mathbf{x})\frac{\partial}{\partial x_i}Q_d^h(\mathbf{y}-\mathbf{x})d\mathbf{x}.
$$

Remark 7.6 *Note that if $f \in \mathcal{A}$ then $g^h_{i,i}$ exists for $i = 1, ..., d$.*

Theorem 7.7 *Let $k \in \{1, ..., n\}$ be fixed. Let $f \in \mathcal{A}$. Suppose that for $j = 1, ..., n$, $\frac{\partial F^\mu_j}{\partial \mu_k} \in \mathbb{D}^\infty$. Then*

$$\frac{\partial}{\partial \mu_k} \sum_{i=1}^{d} E\left[g^h_i(F^\mu)H_{(i)}\left(F^\mu; 1\right)\right] = \sum_{i,j=1}^{d} E\left[g^h_{i,i}(F^\mu)H_{(j)}\left(F^\mu; \frac{\partial F^\mu_j}{\partial \mu_k}\right)\right].$$

Moreover if we assume that for all $i = 1, ..., d$, there exists some $g_{i,i}$ such that $g^h_{i,i} \to g_{i,i}$ a.e. and $g^h_{i,i}$ has polynomial growth (independent of h).

$$\frac{\partial}{\partial \mu_k} E^h\left[f(F^\mu)\right] = \sum_{i,j=1}^{d} E\left[g^h_{i,i}(F^\mu)H_{(j)}\left(F^\mu; \frac{\partial F^\mu_j}{\partial \mu_k}\right)\right] \longrightarrow$$
$$\sum_{i,j=1}^{d} E\left[g_{i,i}(F^\mu)H_{(j)}\left(F^\mu; \frac{\partial F^\mu_j}{\partial \mu_k}\right)\right] = \frac{\partial}{\partial \mu_k} E\left[f(F^\mu)\right].$$

Proof. We prove the first part by using integration by parts formula. For $i = 1, ..., d$,

$$\begin{aligned}
\frac{\partial}{\partial \mu_k} E\left[g^h_i(F^\mu)H_{(i)}\left(F^\mu; 1\right)\right] &= \frac{\partial}{\partial \mu_k} E\left[\frac{\partial}{\partial y_i} g^h_i(F^\mu)\right] \\
&= E\left[\sum_{j=1}^{d} \frac{\partial}{\partial y_j}\frac{\partial}{\partial y_i} g^h_i(F^\mu)\frac{\partial F^\mu_j}{\partial \mu_k}\right] \\
&= \sum_{j=1}^{d} E\left[\frac{\partial}{\partial y_i} g^h_i(F^\mu)H_{(j)}\left(F^\mu; \frac{\partial F^\mu_j}{\partial \mu_k}\right)\right].
\end{aligned}$$

where we use $\frac{\partial}{\partial y_i} g^h_i(y)$ ($i = 1, ..., d$) has polynomial growth. Therefore we obtain the first assertion.
The second claim is trivial by the assumptions.

■

In the next section, we consider Greeks in a second derivative case.

Remark 7.8 *The expression in Theorem 7.7 is obviously not unique. In fact we also have*

$$\frac{\partial}{\partial \mu_k}\sum_{i=1}^{d} E\left[g_i^h(F^\mu)H_{(i)}\left(F^\mu;1\right)\right]=\sum_{i,j=1}^{d} E\left[g_{i,j}^h(F^\mu)H_{(i)}\left(F^\mu;\frac{\partial F_j^\mu}{\partial \mu_k}\right)\right].$$

Remark 7.9 *If $g_{i,j}^h$ $(i,j=1,...,d)$ have an explicit representation, then we can calculate Greeks very easily. This is the case for example, in put and call digital options. If we don't have an explicit expression for the multiple integral then one can use any approximation for multiple Lebesgue integrals. Therefore we can calculate Greeks easily.*

Here we consider a similar expression by using the classical expression of a density. And we compare to them (only the first derivative case).

Proposition 7.10 *Let $k \in \{1,...,n\}$ be fixed. Assume that $\int_{-\infty}^{y_d}\cdots\int_{-\infty}^{y_1} f(\mathbf{x})d\mathbf{x}$ has at most polynomial growth. Assume that $H_{(1,...,d)}(F^\mu;1)$ is differentiable in μ_k. And assume that F_i^μ $(i=1,...,d)$ is differentiable in μ_j $(j=1,...,n)$.*

$$\frac{\partial}{\partial \mu_k}E\left[f(F^\mu)\right]=$$
$$E\left[\int_{-\infty}^{F_d^\mu}\cdots\int_{-\infty}^{F_1^\mu} f(\mathbf{x})d\mathbf{x}\left\{\sum_{j=1}^{d}\frac{\partial F_j^\mu}{\partial \mu_k}H_{(1,...,d,j)}(F^\mu;1)+\frac{\partial}{\partial \mu_k}H_{(1,...,d)}(F^\mu;1)\right\}\right].$$

This result can be proved using the integration by parts formula.

7.2 Second Derivative Case

Next we consider a second derivative case. Proofs are similar, so we only quote the results. In this section $C_\mu^2(\mathbf{R}^n)$ denotes the class of functions that are twice differentiable with to the parameter $\mu \in \mathbf{R}$.

Proposition 7.11

- *Let $k,l \in \{1,...,n\}$ be fixed.*

- *Let $F^{\mu} \in (\mathbb{D}^{\infty})^d$ be a nondegenerate random vector, in $C^2(\mathbf{R}^n)$ with $\frac{\partial F^{\mu}_j}{\partial \mu_k}, \frac{\partial F^{\mu}_j}{\partial \mu_l} \in L^p(\Omega)$ $(p \geq 4)$ for all $j = 1, \ldots, d$ and $\frac{\partial^2 F^{\mu}_j}{\partial \mu_l \partial \mu_k} \in L^p(\Omega)$ $(p \geq 4)$ for all $j = 1, \ldots, d$.*

- *Suppose that for every $i = 1, \ldots, d$, $H_{(1,\ldots,d,1,\ldots,d,i)}\left(F^{\mu}; 1\right)$ is in $C^2(\mathbf{R}^n)$ and also $\frac{\partial}{\partial \mu_k} H_{(1,\ldots,d,1,\ldots,d,i)}\left(F^{\mu}; 1\right)$, $\frac{\partial^2}{\partial \mu_l \partial \mu_k} H_{(1,\ldots,d,1,\ldots,d,i)}\left(F^{\mu}; 1\right)$ are in $L^2(\Omega)$.*

Then we have

$$\begin{aligned}
\frac{\partial^2}{\partial \mu_k \partial \mu_l} E^h\left[f(F^{\mu})\right] &= \sum_{i=1}^{d} \frac{\partial^2}{\partial \mu_k \partial \mu_l} E\left[g_i^h(F^{\mu}) H_{(i)}(F^{\mu}; 1)\right] \\
&\longrightarrow \sum_{i=1}^{d} \frac{\partial^2}{\partial \mu_k \partial \mu_l} E\left[g_i(F^{\mu}) H_{(i)}(F^{\mu}; 1)\right] = \frac{\partial^2}{\partial \mu_k \partial \mu_l} E\left[f(F^{\mu})\right]
\end{aligned}$$

We prove this proposition as in the proof of Proposition 7.5. For $i, j, k = 1, \ldots, d$, define

$$g^h_{i,j,k}(\mathbf{y}) := g^h_{i,j,k}(y_1, \ldots, y_d) := \frac{\partial^2}{\partial y_k \partial y_j} \int \cdots \int_{\mathbf{R}^d} f(\mathbf{x}) \frac{\partial}{\partial x_i} Q^h_d(\mathbf{y} - \mathbf{x}) d\mathbf{x}.$$

Theorem 7.12 *Suppose that for $i = 1, \ldots, d$, $l, k = 1, \ldots, n$, $\frac{\partial F^{\mu}_i}{\partial \mu_l}$ and $\frac{\partial^2 F^{\mu}_i}{\partial \mu_k \partial \mu_l}$ are in $\mathbb{D}^{\infty}$. Then*

$$\begin{aligned}
&\sum_{i=1}^{d} \frac{\partial^2}{\partial \mu_k \partial \mu_l} E\left[g_i^h(F^{\mu}) H_{(i)}(F^{\mu}; 1)\right] \\
&= \sum_{i,j=1}^{d} \left\{ \sum_{m=1}^{d} E\left[g^h_{i,i,m}(F^{\mu}) H_{(j)}\left(F^{\mu}; \frac{\partial F^{\mu}_m}{\partial \mu_l} \frac{\partial F^{\mu}_j}{\partial \mu_k}\right)\right] + E\left[g^h_{i,i}(F^{\mu}) H_{(j)}\left(F^{\mu}; \frac{\partial^2 F^{\mu}_j}{\partial \mu_k \partial \mu_l}\right)\right]\right\}.
\end{aligned}$$

Moreover if we assume that for all $i = 1, \ldots, d$, there exists some $g_{i,i}$ such that $g^h_{i,i} \to g_{i,i}$ a.e. as $h \to 0$ and $g^h_{i,i}$ has polynomial growth (independent of h). And also for all $i, j = 1, ., , , d$, $g^h_{i,i,m}$ convergences to some $g_{i,i,m}$ a.e. and has *polynomial growth (independent of h). Then we have*

$$
\begin{aligned}
&\frac{\partial^2}{\partial\mu_l\partial\mu_k}E^h\left[f(F^\mu)\right] \\
&= \sum_{i,j=1}^{d}\left\{\sum_{m=1}^{d}E\left[g^h_{i,i,m}(F^\mu)H_{(j)}\left(F^\mu;\frac{\partial F^\mu_m}{\partial\mu_l}\frac{\partial F^\mu_j}{\partial\mu_k}\right)\right]+E\left[g^h_{i,i}(F^\mu)H_{(j)}\left(F^\mu;\frac{\partial^2 F^\mu_j}{\partial\mu_k\partial\mu_l}\right)\right]\right\} \\
&\longrightarrow \sum_{i,j=1}^{d}\left\{\sum_{m=1}^{d}E\left[g_{i,i,m}(F^\mu)H_{(j)}\left(F^\mu;\frac{\partial F^\mu_m}{\partial\mu_l}\frac{\partial F^\mu_j}{\partial\mu_k}\right)\right]+E\left[g_{i,i}(F^\mu)H_{(j)}\left(F^\mu;\frac{\partial^2 F^\mu_j}{\partial\mu_k\partial\mu_l}\right)\right]\right\} \\
&= \frac{\partial^2}{\partial\mu_l\partial\mu_k}E\left[f(F^\mu)\right].
\end{aligned}
$$

We can prove this theorem by the same way of Theorem 7.7.

Remark 7.13 *We remark that in the above formula, $H_{(i)}$ requires only one Skorohod integral. Even if higher derivatives with respect to μ are considered this fact remains unchanged.*

7.3 Example 1

Now we consider an example. The objective is to calculate Delta in a digital type option where the asset is characterized by the Heston model. First we define the Heston model as follows;

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sqrt{1-\rho^2}\sqrt{v_t}S_t dW^S_t + \rho\sqrt{v_t}S_t dW^v_t, \\
dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW^v_t,
\end{aligned}
$$

where their initial values for the stock price process S and the volatility process v are s_0 and v_0, respectively.

And our option price is written as follows

$$
E\left[e^{-rT}\mathbf{1}(K_S \le S_T)\mathbf{1}(v_T \le K_v)\right],
$$

where r expresses a constant interest rate. Without loss of generality, we assume that $r = 0$. K_S and K_v are strike prices of stock and volatility respectively.

Remark 7.14 *The Heston model leads to an incomplete market. Therefore there are many equivalent martingale measures. We do not discuss that problem here.*

Then the Delta of above option is;

$$\begin{aligned} &\frac{\partial}{\partial s_0} E\Big[\mathbf{1}(K_S \leq S_T)\mathbf{1}(v_T \leq K_v)\Big] \\ &= E\left[\mathbf{1}(S_T \geq K_S)\mathbf{1}(v_T \leq K_v)\frac{W_T^S}{\sqrt{1-\rho^2} s_0 \int_0^T \sqrt{v_u}du}\right]. \end{aligned} \tag{7.2}$$

Remark 7.15 *In the Heston model, one has to prove the Malliavin differentiability of v. This result can be found in Alos, Ewald [1]. In fact, the volatility process v_t is not in $\mathbb{D}^\infty$. But since s_0 depends on only S_t and v_t is independent of W_t^S, the above calculation works well. In fact, this is also one case where g_i can be computed explicitly which is completely independent of the assumed model.*

We simulate above Delta by using the following parameters; $s_0 = 100$, $\mu = 0.1$, $v_0 = 0.08$, $\kappa = 2$, $\theta = 0.08$, $\sigma = 0.2$, $K_S = 100$, $K_v = 0.08$, $\rho = 0.2$, $t = 1$. And a number of time step $n = 50$. When pricing, then a number of Monte Carlo simulation $N = 10,000,000$ times. The gradient of Figure 9 is about 0.008, and the delta by equation (7.2) is close to 0.008. (Figure 10 and Figure 11).

Remark 7.16 *We have chosen the above parameter so as to ensure the existence and uniqueness of the equation defining v and so that is strictly positive with probability one. For more details, see Section 6.2.2. in Lamberton, Lapeyre [5].*

7.4 Example 2

Here we give an example in higher dimensional case. We consider d-dimensional linear stochastic differential equation case $\mathbf{S}_t = (S_t^1, ..., S_t^d)$ de-

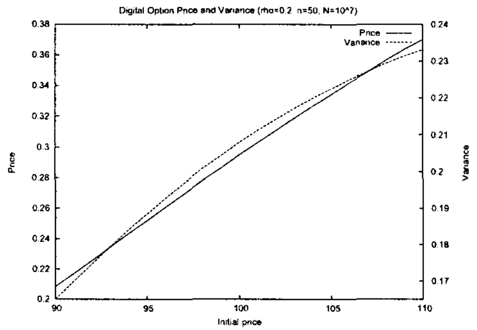


Figure 9: Initial price - Option price

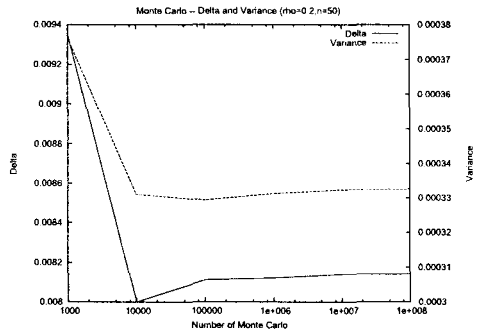


Figure 10: MC - Delta

fined by

$$\frac{dS_t^i}{S_t^i} = \lambda^i(S_t^1, ..., S_t^{i-1}, S_t^{i+1}, ..., S_t^d)dt + \sum_{j=1}^{d} \sigma_j^i(S_t^1, ..., S_t^{i-1}, S_t^{i+1}, ..., S_t^d)dW_t^j,$$

where $i = 1, ..., d$, and $\{W_t^1\}, ..., \{W_t^d\}$ are d-independent Brownian motions. For $i, j = 1, ..., d$, $\mu^i, \sigma_j^i : \mathbf{R}^{d-1} \to \mathbf{R}$ are $C^\infty(\mathbf{R}^{d-1})$ measurable functions and σ_j^i are bounded.

MC	Delta	Variance
10^3	0.00954274	0.000392088
10^4	0.0081553	0.000345127
10^5	0.0082723	0.00034328
10^6	0.00828167	0.000345285
10^7	0.00830052	0.000346419
10^8	0.00830217	0.000346655

Figure 11: Delta & Variance

And we consider the following option; for $f \in \mathcal{A}$,

$$e^{-rT}E\left[f(S_T^1, ..., S_T^d)\right].$$

Then from Theorem 7.7, the delta of this option is; $k = 1, ..., d$

$$\begin{aligned}
&\frac{\partial}{\partial s_k} e^{-rT} E\left[f(S_T^1, ..., S_T^d)\right] \\
&= e^{-rT} \sum_{i,j=1}^{d} E\left[\frac{\partial}{\partial y_i} \int \cdots \int_{\mathbf{R}^d} f(x_1, ..., x_d) \frac{S_T^i - x_i}{|\mathbf{S}_T - \mathbf{x}|^d} d\mathbf{x} H_{(j)}\left(\mathbf{S}_T; \frac{\partial S_T^j}{\partial s_k}\right)\right]
\end{aligned}$$

Again, we remark that in various cases the above multiple Lebesgue integral can be computed explicitly and the simulation of the above quantity requires only one Skorohod integral which in many cases can be written explicitly.

8 Appendix

Here we quote various technical lemmas use throughout the text. For proofs, we refer the reader to the full paper that will appear elsewhere.

8.1 Proof of Corollary 2.4

Lemma 8.1 *For $x_i \geq 0$ $(i = 1, ..., d)$, the following inequalities hold;*
(i). For $d = 2$ and $1 < \kappa < 2 - \frac{2}{p}$ we have

$$\int_0^{x_2}\int_0^{x_1} |Q_2(\mathbf{y})|^{\frac{p}{p-1}} dy_1 dy_2 \leq 2\pi s_2^{-\frac{p}{p-1}} \left\{ \frac{p-1}{(2-\kappa)p-2} + |\mathbf{x}|^{\frac{2p-1}{p-1}} \right\}.$$

(ii). For $p > \frac{d}{2} \geq \frac{3}{2}$,

$$\int_0^{x_d} \cdots \int_0^{x_1} |Q_d(\mathbf{y})|^{\frac{p}{p-1}} dy_1 ... dy_d \leq (2\pi)^{d-1} s_d^{-\frac{p}{p-1}} \frac{p-1}{2p-d} |\mathbf{x}|.$$

(iii). For $p > d \geq 2$,

$$\int_0^{x_d} \cdots \int_0^{x_1} \left| \frac{\partial}{\partial y_i} Q_d(\mathbf{y}) \right|^{\frac{p}{p-1}} dy_1 ... dy_d \leq (2\pi)^{d-1} A_d^{\frac{p}{p-1}} \frac{p-1}{p-d} |\mathbf{x}|^{\frac{p-d}{p-1}} \quad (i = 1, ..., d).$$

8.2 Lemmas used in the proof of Theorem 3.4

Lemma 8.2 *For $i = 1, ..., d$,*

$$A_d \left(\lim_{\varepsilon \to 0} \Phi^F_{i,\varepsilon}(\mathbf{x}) \right) \int_0^1 \frac{(r^2+h)^{\frac{d}{2}} - r^d}{(r^2+h)^{\frac{d}{2}}} dr$$
$$\times \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdots \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Theta_i c_1^{d-2} \cdots c_{d-2} d\theta_1 ... d\theta_{d-1} = 0.$$

Lemma 8.3 *For $i, j = 1, ..., d$,*

$$A_d \left(\lim_{\varepsilon \to 0} \frac{\partial}{\partial y_j} \Phi^F_{i,\varepsilon}(\mathbf{x}) \right) \int_0^1 r \frac{(r^2+h)^{\frac{d}{2}} - r^d}{(r^2+h)^{\frac{d}{2}}} dr$$
$$\times \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdots \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Theta_i \Theta_j c_1^{d-2} \cdots c_{d-2} d\theta_1 \cdots d\theta_{d-1} = \begin{cases} C^x_{1,i} h \ln \frac{1}{h} + C^x_{2,i} h + o(h) & (i=j) \\ 0 & (i \neq j) \end{cases},$$

where

$$C^x_{1,i} := \frac{d}{4}A_d\left(\lim_{\varepsilon\to 0}\frac{\partial}{\partial y_i}\Phi^F_{i,\varepsilon}(\mathbf{x})\right)\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cdots\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\Theta_i^2 c_1^{d-2}\cdots c_{d-2}d\theta_1\cdots d\theta_{d-1},$$

$$C^x_{2,i} := A_d\left(\lim_{\varepsilon\to 0}\frac{\partial}{\partial y_i}\Phi^F_{i,\varepsilon}(\mathbf{x})\right)\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cdots\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\Theta_i^2 c_1^{d-2}\cdots c_{d-2}d\theta_1\cdots d\theta_{d-1}$$
$$\times\left[\int_0^1 u\frac{(u^2+1)^{\frac{d}{2}}-u^d}{(u^2+1)^{\frac{d}{2}}}du+\frac{1}{4}\left(\ln(2)-\ln\left(2^d+2^{\frac{d}{2}}\right)\right)+M_d^0\right],$$

and M_d^0 *is a constant (defined in the proof).*

Lemma 8.4 *For* $i,j,k=1,...,d,$

$$\frac{A_d}{2}\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cdots\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1 r^2\frac{(r^2+h)^{\frac{d}{2}}-r^d}{(r^2+h)^{\frac{d}{2}}}\Theta_i\Theta_j\Theta_k c_1^{d-2}\cdots c_{d-2}$$
$$\times\int_0^1\frac{\partial^2}{\partial y_k\partial y_j}\Phi^F_{i,\varepsilon}(\mathbf{x}+\gamma r\Theta)d\gamma dr d\theta_1\cdots d\theta_{d-1}$$
$$= C^x_{3,i,j,k}h+o(h), \tag{8.1}$$

where

$$C^x_{3,i,j,k} := \frac{dA_d}{4}\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cdots\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1\Theta_i\Theta_j\Theta_k c_1^{d-2}\cdots c_{d-2}$$
$$\times\int_0^1\left(\lim_{\varepsilon\to 0}\frac{\partial^2}{\partial y_k\partial y_j}\Phi^F_{i,\varepsilon}(\mathbf{x}+\gamma r\Theta)\right)d\gamma dr d\theta_1\cdots d\theta_{d-1}.$$

Lemma 8.5 *For i=1,...,d,*

$$A_d\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cdots\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_1^\infty\frac{(r^2+h)^{\frac{d}{2}}-r^d}{(r^2+h)^{\frac{d}{2}}}\Theta_i c_1^{d-2}\cdots c_{d-2}\left(\lim_{\varepsilon\to 0}\Phi^F_{i,\varepsilon}(r\Theta+\mathbf{x})\right)$$
$$dr d\theta_1...d\theta_{d-1} = C^x_{4,i}h+o(h), \tag{8.2}$$

where

$$C^x_{4,i}$$
$$:= \tfrac{d}{2}A_d\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cdots\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_1^\infty\tfrac{1}{r^2}\Theta_i c_1^{d-2}\cdots c_{d-2}\left(\lim_{\varepsilon\to 0}\Phi^F_{i,\varepsilon}(r\Theta+\mathbf{x})\right)dr d\theta_1...d\theta_{d-1}.$$

8.3 Lemmas used in the proof of Theorem 4.1

We give some lemmas for Section 4.1.

Lemma 8.6 *For $i, j = 1, 2$,*

$$
\begin{aligned}
&A_2^2\left(\lim_{\varepsilon\to 0}\hat{\Phi}^F_{i,j,\varepsilon}(\mathbf{x})\right)\int_0^{2|\mathbf{x}|+1}\frac{r^3}{(r^2+h)^2}dr\int_0^{2\pi}\Theta_i\Theta_j d\theta \\
&= \begin{cases} C^x_{5,i}\ln\dfrac{1}{h} + O(1) & (i = j) \\ 0 & (i \neq j) \end{cases}.
\end{aligned}
\tag{8.3}
$$

where

$$C^x_{5,i} = \frac{\pi}{2}A_2^2\left(\lim_{\varepsilon\to 0}\hat{\Phi}^F_{i,i,\varepsilon}(\mathbf{x})\right).$$

Lemma 8.7 *For $i, j, k = 1, 2$,*

$$A_2^2\int_0^{2\pi}\int_0^{2|\mathbf{x}|+1}\frac{r^4\Theta_i\Theta_j\Theta_k}{(r^2+h)^2}\left\{\lim_{\varepsilon\to 0}\left(\int_0^1\frac{\partial}{\partial y_k}\hat{\Phi}^F_{i,j,\varepsilon}(\mathbf{x}+\gamma r\Theta)d\gamma\right)\right\}drd\theta \le C_6,$$

where C_6 is a constant independent on $\mathbf{x}$.

Proof. By Lemma 3.2, $\lim_{\varepsilon\to 0}(\int_0^1 \frac{\partial}{\partial y_k}\hat{\Phi}^F_{i,j,\varepsilon}(\mathbf{x}+\gamma r\Theta)d\gamma)$ is uniformly bounded. Therefore the result follows. ■

Lemma 8.8 *Let F be a nondegenerate random vector and $G \in \mathbb{D}^\infty$. For $p \ge 1$, then there exists some constant C such that*

$$\lim_{\varepsilon\to 0} E\Big[\eta_\varepsilon(F-\mathbf{x})G\Big] \le \frac{C}{|\mathbf{x}|^p} \quad (\mathbf{x} \in \mathbf{R}^d).$$

Proof. Using the IBP formula, where $a_i \in \{1, \ldots, d\}$, $i = 1, \ldots, n$

$$
\begin{aligned}
&\prod_{i=1}^n x_{a_i}\lim_{\varepsilon\to 0}E\Big[\eta^F_\varepsilon(\mathbf{x})G\Big] \\
&= E\left[\left(\frac{\partial}{\partial x_1}\cdots\frac{\partial}{\partial x_d}\prod_{i=1}^d \mathbf{1}_{[0,\infty)}(F_i - x_i)\right)\prod_{i=1}^n F_{a_i}G\right] \\
&= E\left[\left(\prod_{i=1}^d \mathbf{1}_{[0,\infty)}(F_i - x_i)\right)H_{(1,\ldots,d)}\Big(F;\prod_{i=1}^n F_{a_i}G\Big)\right] \le C < \infty.
\end{aligned}
$$

■

Lemma 8.9 *For $i, j = 1, 2$,*

$$A_2^2 \int_0^{2\pi} \int_{2|\mathbf{x}|+1}^{\infty} r \frac{r^2 \Theta_i \Theta_j}{(r^2+h)^2} \left(\lim_{\varepsilon \to 0} \hat{\Phi}^F_{i,j,\varepsilon}(r\Theta + \mathbf{x}) \right) dr d\theta \le C_7.$$

Proof. From Lemma 8.8,

$$\begin{aligned}
&\left| A_2^2 \int_0^{2\pi} \int_{2|\mathbf{x}|+1}^{\infty} r \frac{r^2 \Theta_i \Theta_j}{(r^2+h)^2} \left(\lim_{\varepsilon \to 0} \hat{\Phi}^F_{i,j,\varepsilon}(r\Theta + \mathbf{x}) \right) dr d\theta \right| \\
&\le A_2^2 \int_0^{2\pi} \int_{2|\mathbf{x}|+1}^{\infty} \frac{r^3}{(r^2+h)^2} \frac{C}{|r\Theta + \mathbf{x}|^2} dr d\theta \\
&\le \frac{C}{2|\mathbf{x}|+1}.
\end{aligned}$$

■

8.4 Lemmas used in the proof of Theorem 4.2

We give some lemmas for Section 4.2.

Lemma 8.10 *Set $I(n, m) = \int \sin^n x \cos^m x dx$ $(n + m \neq 0)$. Then*

$$\begin{aligned}
I(n, m) &= -\frac{\sin^{n-1} x \cos^{m+1} x}{m+n} + \frac{n-1}{m+n} I(n-2, m) \\
&= \frac{\sin^{n+1} x \cos^{m-1} x}{n+m} + \frac{m-1}{m+n} I(n, m-2).
\end{aligned}$$

Proof. This is proven using the integration by parts formula.

■

Lemma 8.11 *For $i, j = 1, \ldots, d$,*

$$\begin{aligned}
&A_d^2 \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdots \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 \frac{r^{d+1} \Theta_i \Theta_j}{(r^2+h)^d} c_1^{d-2} \cdots c_{d-2} \left\{ \lim_{\varepsilon \to 0} \hat{\Phi}^F_{i,j,\varepsilon}(\mathbf{x}) \right\} dr d\theta_1 \ldots d\theta_{d-1} \\
&= \begin{cases} C^x_{8,i} \dfrac{1}{h^{\frac{d}{2}-1}} + o\Big(\dfrac{1}{h^{\frac{d}{2}-1}}\Big) & (i = j) \\ 0 & (i \neq j), \end{cases} \qquad (8.4)
\end{aligned}$$

where

$$C_{8,i}^x = \begin{cases} \frac{3\pi}{16} A_d^2 \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Theta_i^2 c_1 \left\{ \lim_{\varepsilon\to 0} \hat{\Phi}_{i,i,\varepsilon}^F(\mathbf{x}) \right\} d\theta_1 d\theta_2 & (d=3) \\ \left(\frac{1}{d-2} \prod_{k=0}^{\frac{d}{2}-1} \frac{2+2k}{d+2k} \right) A_d^2 \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdots \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Theta_i^2 c_1^{d-2} \cdots c_{d-2} & (d \geq 4 : \text{ even}) \\ \left\{ \lim_{\varepsilon\to 0} \hat{\Phi}_{i,i,\varepsilon}^F(\mathbf{x}) \right\} d\theta_1 \ldots d\theta_{d-1} & \\ \frac{\pi}{4} \left(\prod_{k=0}^{\frac{d-7}{2}} \frac{3+2k}{2+2k} \right) \left(\prod_{k=0}^{\frac{d-1}{2}} \frac{1+2k}{2m+2k} \right) A_d^2 \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdots \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Theta_i^2 c_1^{d-2} \cdots c_{d-2} & \\ \left\{ \lim_{\varepsilon\to 0} \hat{\Phi}_{i,i,\varepsilon}^F(\mathbf{x}) \right\} d\theta_1 \ldots d\theta_{d-1} & (d \geq 5 : \text{ odd}), \end{cases}$$

where if $d = 5$, then we define $\prod_{k=0}^{\frac{d-7}{2}} \frac{3+2k}{2+2k} = 1$.

Lemma 8.12 *For $i, j, k = 1, ..., d$,*

$$A_d^2 \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdots \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 \frac{r^{d+2}\Theta_i \Theta_j \Theta_k}{(r^2+h)^d} c_1^{d-2} \cdots c_{d-2}$$
$$\times \left\{ \lim_{\varepsilon\to 0} \int_0^1 \frac{\partial}{\partial y_k} \hat{\Phi}_{i,j,\varepsilon}^F(\mathbf{x} + \gamma r \Theta) d\gamma \right\} dr d\theta_1 \ldots d\theta_{d-1} = O\left(\frac{1}{h^{\frac{d-3}{2}}} \right).$$

Proof. By Lemma 3.2, $\lim_{\varepsilon\to 0}(\int_0^1 \frac{\partial}{\partial y_k} \hat{\Phi}_{i,j,\varepsilon}^F(\mathbf{x} + \gamma r \Theta) d\gamma)$ is bounded. Therefore the result follows.

■

Lemma 8.13 *For $i, j = 1, ..., d$, there exists some constant C such that*

$$A_d^2 \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdots \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_1^{\infty} \frac{r^{d+1}\Theta_i \Theta_j}{(r^2+h)^d} c_1^{d-2} \cdots c_{d-2} \left(\lim_{\varepsilon\to 0} \hat{\Phi}_{i,j,\varepsilon}^F(r\Theta + \mathbf{x}) \right) dr d\theta_1 \ldots d\theta_{d-1}$$
$$\leq C.$$

Proof. By Lemma 3.2, $\lim_{\varepsilon\to 0} \hat{\Phi}_{i,j,\varepsilon}^F(r\Theta + \mathbf{x})$ is bounded. Then we can easily check.

■

8.5 Lemmas used in the proof of Theorem 5.1

In this section, we give some lemmas used to prove the central limit theorem.

Lemma 8.14 *For any $d \geq 2$ and $0 < p < \frac{1}{2}$, we have*

$$N \times |\mathcal{R}| \leq o\left(h^p\right).$$

Lemma 8.15

$$E\left[\left(\zeta^{n,N,h}\right)^2\right] = \begin{cases} C_3^x \ln \frac{1}{h} + O(1) & (d = 2) \\ C_4^x \frac{1}{h^{\frac{d}{2}-1}} + o\left(h^{\frac{2-d}{2}}\right) & (d \geq 3). \end{cases}$$

Proof. In the case of $d = 2$, the result follows from Theorem 3.4 and Theorem 4.1. In the case $d \geq 3$ it follows from Theorem 3.4 and Theorem 4.2.

■

8.6 Lemma for Section 6.1

Here we obtain the weights $H_{(i)}$ in the classical setting.

Lemma 8.16 *Let F be a nondegenerate random vector then the density of $F = X_t$, solution of equation (6.1), can be expressed as*

$$p_F(\mathbf{x}) = A_d \sum_{i=1}^{d} E\left[\frac{F^i - x_i}{|F - \mathbf{x}|^d} \sum_{k=1}^{d} (-1)^{i+k} \frac{\det(\Sigma_i^k)}{\det(\Sigma)} \left\{\frac{W_k(\mathbf{1}(\cdot \leq T))}{F^i} + \frac{\sigma_{ik} T}{F^i}\right\}\right]. \quad (8.5)$$

Proof. For $i, l = 1, \ldots, d$, we have

$$D^i \frac{\partial}{\partial x_l} Q_d(F - \mathbf{x}) = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j \partial x_l} Q_d(F - \mathbf{x}) D^i F^j.$$

We solve this simultaneous equation by using Cramer's formula and obtain $\frac{\partial^2}{\partial x_j \partial x_l} Q_d(F - \mathbf{x})$;

$$\begin{aligned} &\frac{\partial^2}{\partial x_j \partial x_l} Q_d(F - \mathbf{x}) \\ &= \frac{1}{F^j \det(\Sigma)} \mathbf{1}(\cdot \leq T) \sum_{k=1}^{d} (-1)^{k+j} \det\left(\Sigma_j^k\right) D^k \left(\frac{\partial}{\partial x_l} Q_d(F - \mathbf{x})\right) \ a.s., \end{aligned}$$

where

$$\Sigma := \begin{pmatrix} \sigma_{11} & \cdots & \sigma_{1d} \\ \vdots & \ddots & \vdots \\ \sigma_{d1} & \cdots & \sigma_{dd} \end{pmatrix}.$$

Σ_j^k is a $(d-1) \times (d-1)$ matrix obtained from Σ by deleting row k and column j.

Then by a duality argument;

$$\begin{aligned} &E\left[\frac{\partial^2}{\partial x_l^2} Q_d(F - \mathbf{x})\right] \\ &= \sum_{k=1}^{d} (-1)^{k+l} \frac{\det(\Sigma_l^k)}{\det(\Sigma)} E\left[\frac{\partial}{\partial x_l} Q_d(F - \mathbf{x}) (D^k)^* \left(\frac{\mathbf{1}(\cdot \leq T)}{F^l}\right)\right]. \end{aligned} \tag{8.6}$$

The result follows from (6.2). ∎

Remark 8.17 *In the above proof, we need to introduce a local property. Since the function $\frac{\partial}{\partial x_i} Q_d(\mathbf{x})$ doesn't satisfy the Lipschitz condition (See Proposition 1.2.4., p.29 in Nualart [7]), the chain rule doesn't work well. But now F has a continuous density. Then $P(F = \mathbf{x}) = 0$.*

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Resumen

La fórmula de Malliavin-Thalmaier se introdujo para la simulación de funciones de densidad de probabilidad multidimensionales. Cuando la fórmula de integración por partes se aplica directamente en simulaciones computacionales, mostramos que es inestable. Proponemos una aproximación a la fórmula de Malliavin-Thalmaier. En este trabajo hallamos el orden del sesgo y la varianza del error de aproximación y obtenemos una fórmula explícita de Malliavin-Thalmaier para el cálculo de las Griegas en finanzas. Los pesos obtenidos están libres del problema de la multidimensionalidad.

MSC 2000: 60H07; 60H35; 60J60; 62G07; 65C05; 60F05

Palabras Clave: Cálculo de Malliavin, Ingeniería Financiera, Griegas, Análisis de Sensibilidad, Estimación de Densidad.

Comentarios Finales

Este artículo es una versión resumida de la presentación que tuve el honor de impartir durante el Congreso Internacional de Matemáticas PUCP realizado entre el 14 al 17 de agosto del 2007. Fui estudiante de la carrera de Estadística durante los años 1981-1985. Como expresé durante la mesa redonda del dia 16 de agosto, la educación recibida durante aquellos años fue fundamental para poder establecerme en mi mundo profesional. Hasta cierto punto, lo más importante de aquellos años, más allá del conocimiento específico, fue la habituación al método matemático de rigurosidad. Esto se demuestra efectivamente con el hecho de que la gran mayoría de graduados de la PUCP en el área de matemáticas han trabajado en áreas relacionadas al álgebra.

Sin embargo, no quiero dejar de lado el hecho que más allá de intentar cubrir muchas áreas lo más importante siempre fue cubrir pocas y con intensidad. No quiero tampoco decir que la educación recibida fue perfecta. Pero si quiero decir que me dio lo suficiente para poder finalmente sobrevivir en el díficil mundo de la investigación.

Siempre hay espacio para mejorar y deseo que los pasos tomados sean en esta dirección. Finalmente deseo agradecer a los organizadores de este Congreso así como a todos mis profesores de aquellos años por la educación recibida.

Esta nota se ha escrito tomando en cuenta el aspecto divulgador por encima de la exactitud matemática. Los detalles matemáticos de las pruebas aparecerán en otro artículo.

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