FRACTIONALLY INTEGRATED PROCESSES OF ORNSTEIN-UHLENBECK TYPE

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November, 2007

Abstract

An estimation methodology to deal with fractionally integrated processes of Ornstein-Uhlenbeck type is proposed. The methodology is based on the continuous Whittle contrast. A simulation study is performed by driving this process with a symmetric CGMY background Lévy process.

Keywords: *Fractional Lévy processes. Long range dependence. Frecuency domain estimation.*

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1 Processes of Ornstein-Uhlenbeck Type

The main contribution of this paper is to propose an estimation methodology to deal with a long-range dependence process of Ornstein-Uhlenbeck type.

Given $\lambda > 0$ and a Lévy process $\tilde{\mathbf{Z}} = \{\tilde{Z}(t)\}$ with generating triplet $(\sigma^0, \gamma^0, \nu^0)$, $\mathbf{X} = \{X(t)\}$ is said to be a process of Ornstein-Uhlenbeck type generated by $(\sigma^0, \gamma^0, \nu^0, \lambda)$, if it is càdlàg and satisfies the stochastic differential equation

$$\begin{cases} dX(t) = -\lambda X(t)dt + d\tilde{Z}(\lambda t) \\ X(0) = X_0 \end{cases} \tag{1}$$

where X_0 is a random variable independent of $\mathbf{Z}$. We will refer to $\mathbf{Z}$ as the background driving Lévy process (BDLP) of $\mathbf{X}$.

The unique solution of (1) is proved to be

$$X(t) = e^{-\lambda t}X_0 + \int_0^t e^{-\lambda(t-s)}d\tilde{Z}(\lambda s). \tag{2}$$

This satifies the recursive equation

$$X(t+\Delta) = \exp(-\lambda\Delta)\left(X(t) + \exp(-\lambda t)\int_t^{t+\Delta} \exp(\lambda s)d\tilde{Z}(\lambda s)\right). \tag{3}$$

Equation (2) generalizes the classical solution given by Ornstein and Uhlenbeck (1930) to the Langevin equation (1908) in the modelling of the position of a particle under a frictional force. These authors considered $\tilde{\mathbf{Z}}$ as a standard Brownian motion.

We mention two important properties of a process of Ornstein-Uhlenbeck type $\mathbf{X}$. First, the auto-correlation of this process

$$\rho(\Delta) = Corr(X(t), X(t+\Delta) = \exp(-\lambda|\Delta|), \tag{4}$$

decays exponentially. Finally, **X** is stationary under some mild conditions. More precisely, Sato [1999] proves the next.

Proposition 1.1 *Let* **X** *be a process of Ornstein-Uhlenbeck type generated by* $(\sigma_0, \gamma_0, \nu_0, \lambda)$ *such that*

$$\int_{|x|>1} \log(|x|) d\nu_0(x) < \infty, \tag{5}$$

then **X** *has a unique self-decomposable* [1] *stationary distribution* μ.

Conversely, for any $\lambda > 0$ *and any self-decomposable distribution D, there exists a unique triplet* $(\sigma_0, \gamma_0, \nu_0)$ *satisfying* (5) *and a process of Ornstein-Uhlenbeck type* **X** *generated by* $(\sigma_0, \gamma_0, \nu_0, \lambda)$ *such that D is the stationary distribution of* **X**.

2 Long-range Dependence

The subject of long range dependence has sparkled considerable interest over the last few years. A good survey of this topic is Doukhan et al. [2002]. The main result of this work will be to provide a estimation methodology to deal with a class of long-range dependence processes.

We adopt the following definition of long-range dependence.

Definition 2.1 *Let* $\mathbf{X} = \{X(t)\}$ *be a stationary process with autocovariance function* γ. **X** *is said to be a long-range dependence process or to have long*

[1] A random variable X (or its distribution) is said to be self-decomposable if for any $0 < a < 1$, $\exists$ a random variable Y_a, independent of X, such that

$$X \overset{\mathcal{D}}{=} aX + Y_a,$$

where $\overset{\mathcal{D}}{=}$ means equality in distribution.

memory if there exists $0 < d < \frac{1}{2}$ *and a constant* $c_\gamma > 0$ *such that*

$$\lim_{h\to\infty} \frac{\gamma(h)}{h^{2d-1}} = c_\gamma. \tag{6}$$

Otherwise the process is said to be a short-range dependence process or to have short memory. An example of a short-range dependence process is the process of Ornstein-Uhlenbeck type given in the previous section. Observe that independently of the BDLP, or the stationary model, any process of this type shares the same short-memory autocorrelation function. In this chapter we study how to cope with such limitation by introducing a long-range dependence process of Ornstein-Uhlenbeck type. We handle then the inference problem for this process by means of spectral techniques. This approach is needed to overcome the difficulties associated to the complex distribution of this process.

Proposition 1.1 guarantees not only the stationarity of $\mathbf{X}$, but also the existence of a Lévy process $\check{\mathbf{Z}} = \{\check{Z}(t)\}$, independent of $\tilde{\mathbf{Z}}$ but with the same distribution, such that

$$X_0 \stackrel{\mathcal{D}}{=} \int_0^\infty \exp(-\lambda s)d\check{Z}(\lambda s) = \int_0^\infty \exp(-s)d\check{Z}(s).$$

This yields the representation in law

$$\begin{aligned} X(t) &= \int_0^\infty \exp(-\lambda(t+s))d\check{Z}(\lambda s) + \int_0^t \exp(-\lambda(t-s))d\tilde{Z}(\lambda s) \\ &= \int_{-\infty}^0 \exp(-\lambda(t-s))d\check{Z}(-\lambda s) + \int_0^t \exp(-\lambda(t-s))d\tilde{Z}(\lambda s) \\ &= \int_{-\infty}^t g_0(t-s)dZ(\lambda s), \end{aligned} \tag{7}$$

where $\mathbf{Z} = \{Z(t)\}_{t\in\mathbb{R}}$ is now a two-sided Lévy process given by

$$Z(t) = \tilde{Z}(t)\mathbf{1}_{[0,\infty[}(t) - \check{Z}(-t-)\mathbf{1}_{]-\infty,0]}(t)$$

and

$$g_0(x) = \begin{cases} \exp(-\lambda x), & \text{if } x > 0 \\ 0\ , & \text{otherwise.} \end{cases}$$

3 The Continuous Moving Average Process

Albeit with a different timing in the driving Lévy process, the next definition provides a natural generalization of (7).

Definition 3.1 *A stochastic process* $\mathbf{X} = \{X(t)\}$ *is said to be a continuous time moving average process, or shortly a MA process, if*

$$X(t) = \int_{-\infty}^{t} g(t-s)dZ(s), \tag{8}$$

for some measurable function $g : \mathbb{R} \to \mathbb{R}$ *and some two-sided Lévy process* $\mathbf{Z} = \{Z(t)\}$. *We will call g and* $\mathbf{Z}$*, respectively, the kernel and the BDLP of* $\mathbf{X}$.

Similarly to (7), any MA process can be written as

$$X(t) = \int_0^{\infty} g(t+s)dZ(s) + \int_0^{t} g(t-s)dZ(s),$$

where although we use the same notation, the BDLP processes on the right hand side are understood to be independent copies of a common Lévy process.

Proposition 3.1 *Let* $\mathbf{X}$ *be the process in* (8) *and let g and* $\mathbf{Z}$ *be, respectively, the kernel and BDLP of* $\mathbf{X}$*, where* (σ^2, γ, ν) *denotes the generating triplet of* $\mathbf{Z}$. *Then* $\mathbf{X}$ *is a well-defined, infinitely divisible and strictly stationary process if, and only if, the following three integrals below are finite* *a)* $\int_{-\infty}^{\infty} g^2(s)ds.$

b) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (|g(s)x|^2 \wedge 1)\nu(dx)ds.$

c) $\gamma_g = \int_{-\infty}^{\infty}\left(\gamma g(s) + \int_{-\infty}^{\infty} g(s)x(\mathbf{1}_{\bar{B}(0,1)}(g(s)x) - \mathbf{1}_{\bar{B}(0,1)}(x))\nu(dx)\right)ds.$

Moreover, for any real numbers $t_1 \leq t_2 \leq \ldots \leq t_n$*, the cumulant function of* $(X(t_1), X(t_2), \ldots, X(t_n))$ *is given by*

$$C(\vartheta_1,\ldots,\vartheta_n) = \int_0^{\infty} C_{Z(1)}\Big(\sum_{k=1}^{n}\vartheta_k g(t_k+s)\Big)ds + \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} C_{Z(1)}\Big(\sum_{k=j}^{n}\vartheta_k g(t_k-s)\Big)ds, \tag{9}$$

where we use the convention $t_0 = 0.$

Proof: See Valdivieso [2007] ■

Based on the cumulant function (9) one can easily derive the autocovariance function of the MA process. This is given by

$$\gamma(h) = E[Z(1)^2]\int_0^{\infty} g(|h|+s)g(s)ds. \tag{10}$$

From now on we will restrict our study to a 'causal' MA process **X** with zero-mean BDLP **Z** of finite variance. A MA process **X** is said to be causal if its kernel function satisfies

$$g(x) = 0, \quad \forall x < 0.$$

In other words, a causal MA process depends only on the past of the BDLP **Z**.

4 The Fractionally Integrated Ornstein- Uhlenbeck Process

To induce long memory, Brockwell and Marquardt [2005] have proposed to convolute the kernel of a short-range dependence process of Ornstein-Uhlenbeck type with the slow decaying function

$$\eta(x) = \frac{x^{d-1}}{\Gamma(d)} \mathbf{1}_{]0,\infty[}(x),$$

where $0 < d < \frac{1}{2}$. The fractionally integrated Ornstein-Uhlenbeck Lévy process $\mathbf{X_d} = \{X_d(t)\}_{t\in\mathbb{R}}$ is then defined as

$$X_d(t) = \int_{-\infty}^{t} g_d(t-s)dZ(s),$$

where

$$\begin{aligned} g_d(x) &= \int_{-\infty}^{x} g_0(x-s)\eta(s)ds \\ &= \int_{0}^{x} \exp(-\lambda(x-s))\frac{s^{d-1}}{\Gamma(d)}ds \\ &= (-\lambda)^{-d}\exp(-\lambda x)\Gamma(-\lambda x, d) \end{aligned}$$

and

$$\Gamma(a,d) = \int_{0}^{a} \frac{\exp(-s)s^{d-1}}{\Gamma(d)}ds$$

denotes the incomplete Gamma function. Hereafter we will refer to $\mathbf{X_d}$ as a FIOUL process.

Observe that the kernel function g_d is, by the restriction $0 < d < \frac{1}{2}$, square integrable. This easily follows from the inequality

$$|g_d(x)| \le Kx^{d-1},$$

for some $K > 0$ and all $x > 0$.

The next proposition presents the autocovariance function of a FIOUL process and its Fourier transform. This transform, called the spectral density of the process, will play a central role in our estimation methodology.

Proposition 4.1 *The variance, autocovariance and spectral density functions of the FIOUL process are given, respectively, by*

$$\gamma_d(0) = \frac{E[Z(1)^2]}{2\lambda^{2d+1}cos(\pi d)},$$

$$\gamma_d(h) = \gamma_d(0)\left(cosh(\lambda h) - \frac{1}{2}\exp(\lambda h)\Gamma(\lambda h, 2d) + \right.$$
$$\left. + \frac{1}{2}\exp(-\lambda h)(-1)^{-2d}\Gamma(-\lambda h, 2d)\right) \tag{11}$$

and

$$f_d(\vartheta) = \frac{E[Z(1)^2]}{2\pi|\vartheta|^{2d}(\lambda^2 + \vartheta^2)}. \tag{12}$$

Proof: By the convolution theorem, the Fourier transform of g_d is given by

$$\hat{g}_d(\vartheta) = \mathcal{F}[g_d](\vartheta) = 2\pi\mathcal{F}[g_0](\vartheta)\mathcal{F}[\eta](\vartheta)$$
$$= 2\pi\frac{1}{2\pi(\lambda + i\vartheta)}\frac{1}{2\pi(i\vartheta)^d} = \frac{1}{2\pi(i\vartheta)^d(\lambda + i\vartheta)}.$$

Hence, by the inversion theorem

$$g_d(x) = \mathcal{F}^{-1}[\hat{g}_d](x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(i\vartheta x)\frac{1}{(i\vartheta)^d(\lambda + i\vartheta)}d\vartheta.$$

Since $g_d(x) = 0$ for any $x < 0$, the autocovariance function (10) can be expressed as

$$\gamma_d(h) = E[Z(1)^2]\int_{-\infty}^{\infty}g_d(s+h)g_d(s)ds.$$

Then the spectral density of the FIOUL process is given by

$$\begin{aligned} f_d(\vartheta) &= \frac{E[Z(1)^2]}{2\pi}\int_{-\infty}^{\infty}\exp(-i\vartheta h)\Big(\int_{-\infty}^{\infty} g_d(s+h)g_d(s)ds\Big)dh \\ &= \frac{E[Z(1)^2]}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\exp(-i\vartheta(u-s))g_d(u)du\right)g_d(s)ds \\ &= 2\pi E[Z(1)^2]\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-i\vartheta u)g_d(u)du\right) \\ &\times \left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(i\vartheta s)g_d(s)ds\right) \\ &= 2\pi E[Z(1)^2]\hat{g}_d(\vartheta)\hat{g}_d(-\vartheta) \\ &= \frac{E[Z(1)^2]}{2\pi|\vartheta|^{2d}(\lambda^2+\vartheta^2)}. \end{aligned}$$

On the other side, let γ_0 and f_0 be the autocovariance and spectral density functions of the MA process (8) with kernel g_0; i.e,

$$\gamma_0(h) = E[Z(1)^2]\frac{\exp(-\lambda|h|)}{2\lambda} \quad \text{and} \quad f_0(\vartheta) = |\vartheta|^{2d} f_d(\vartheta).$$

Hence, by the convolution theorem

$$\gamma_d(h) = \mathcal{F}^{-1}[f_d](h) = \mathcal{F}^{-1}[\mathcal{F}[\gamma_0]\mathcal{F}[\varrho]](h) = \int_{-\infty}^{\infty}\gamma_0(h-s)\varrho(s)ds,$$

where

$$\varrho(s) = \int_{-\infty}^{\infty}\frac{\exp(i\vartheta s)}{|\vartheta|^{2d}}d\vartheta = 2sin(\pi d)\Gamma(1-2d)|s|^{2d-1}.$$

This yields

$$\begin{aligned} \gamma_d(h) &= \gamma_d(0)(cosh(\lambda h) - \frac{1}{2}\exp(\lambda h)\Gamma(\lambda h, 2d) \\ &+ \frac{1}{2}\exp(-\lambda h)(-1)^{-2d}\Gamma(-\lambda h, 2d)), \end{aligned}$$

where $\gamma_d(0) = \frac{E[Z(1)^2]}{2\lambda^{2d+1}cos(\pi d)}$ represents the variance of the FIOUL process. ■

Remark 4.1 *The kernel and autocovariance function of a FIOUL process can be asymptotically approximated by*

$$g_d(x) \sim \frac{x^{d-1}}{\lambda\Gamma(d)} \quad as \ x \to \infty$$

and

$$\gamma_d(h) \sim \frac{E[Z(1)^2]h^{2d-1}\Gamma(1-2d)}{\lambda^2\Gamma(d)\Gamma(1-d)} \quad as \ h \to \infty.$$

Observe that the last expression guarantees the long memory of the FIOUL process.

5 The Whittle Approach

In contrast to a process of Ornstein-Uhlenbeck type, a FIOUL process $\mathbf{X_d} = \{X_d(t)\}$ is not Markovian and has conjoint cumulant function of the form (9). Then the likelihood inference approach requires the inversion of the logarithm of (9). No analytical solution exists for this and any numerical solution is limited by the curse of dimensionality.

Instead of working on the time domain, one interesting alternative is working on the frequency domain. This will exploit the closed form of the spectral density in proposition 4.1. To be more precise, suppose for instance, as in the seminal work of Whittle [1962], that $\{X_j\}_{j=1,2,\dots,n}$ is a discrete stationary Gaussian process with auto-covariance function γ and finite variance $\gamma(0)$. A well known result in spectral theory states that

$$\gamma(n) = \int_{]-\pi,\pi]} \exp(in\vartheta) f(\vartheta) d\vartheta, \tag{13}$$

where f is the so-called spectral density of the series. The classical estimator of f is the periodogram

$$I_n(\vartheta) = \frac{1}{2\pi n}|\sum_{j=1}^{n} \exp(-i\vartheta j)X_j|^2, \quad -\pi < \vartheta \leq \pi,$$

which can be proved to be unbiased but not consistent. Brockwell and Davis [1986] have proved that if we take the partition $\{\mp\vartheta_k = \mp\frac{2\pi k}{n}\}_{k=1,2,\dots,\frac{n}{2}}$, then the random variables $I_n(\vartheta_k)$ are asymptotically independent and exponentially distributed with mean $f(\vartheta_k)$. Then, for n sufficiently large, we can build the "log-likelihood function" (not on $\{X_j\}$, but on $\{I_n(\vartheta_k)\}_{k=1,2,\dots,m}$ with $m = [\frac{n-1}{2}]$)

$$\begin{aligned} K(\theta) &= \sum_{k=1}^{m}(log(\frac{1}{f(\vartheta_k)}) - \frac{I_n(\vartheta_k)}{f(\vartheta_k)}) \\ &= -\sum_{k=1}^{m}(log(f(\vartheta_k)) + \frac{I_n(\vartheta_k)}{f(\vartheta_k)}) \end{aligned}$$

and define the Whittle estimator as

$$\arg\min_{\theta} \sum_{k=1}^{m}(log(f(\vartheta_k)) + \frac{I_n(\vartheta_k)}{f(\vartheta_k)}).$$

One surprising fact is that this estimator also works with a large class of continuous processes which are not necessarily Gaussian. Gao [2004] for instance, has adapted the Whittle methodology to estimate the continuous fractional stochastic volatility model proposed by Comte and Renault [1996]. Recently, Leonenko and Sakhno [2006] have extended the Whittle approach to deal with Lévy random fields.

5.1 High Order Spectral Theory

Before studying the extension of the Whittle estimator, it is relevant to introduce some theory of high order spectral analysis. A good account of this can be found in Brillinger [1981].

We define the joint cumulant of a p-dimensional random vector $(X_1, X_2, \dots, X_p)$ as

$$cum(X_1, X_2, \dots, X_p) = \sum (-1)^{r-1}(r-1)! E[\prod_{j\in V_1} X_j] \dots E[\prod_{j\in V_r} X_j],$$

where the summation extends over all partitions $(V_1, V_2, \dots, V_r)$ of $\{1, 2, \dots, p\}$ and the vector above has the required absolute moments. Note that this term corresponds to the $i^p \vartheta_1 \vartheta_2 \dots \vartheta_p$ coefficient in the Taylor expansion of the cumulant function C of this vector; i.e,

$$cum(X_1, X_2, \dots, X_p) = \frac{1}{i^p} \frac{\partial^p C(\vartheta_1, \dots, \vartheta_p)}{\partial \vartheta_1 \dots \partial \vartheta_p}|_{(\vartheta_1, \dots, \vartheta_p)=0}.$$

An application of the Faà di Bruno's formula yields the next relation between $p-$th order moments and cumulants

$$E[X_1 X_2 \dots X_p] = \sum \prod_{j=1}^{r} cum(\{X_k\}_{k\in V_j}),$$

where the summation extends over all partitions $(V_1, V_2, \dots, V_r)$ of $\{1, 2, \dots, p\}$.

Leonov and Shiryaev [1959] have developed the following useful formula to calculate cumulants of products of random variables

$$cum(\prod_{i=1}^{n_1} X_i, \prod_{i=n_1+1}^{n_2} X_i, \dots, \prod_{i=n_{q-1}+1}^{n_q} X_i) = \sum \prod_{j=1}^{p} cum(\{X_k\}_{k\in V_j}),$$

where $1 \leq n_1 < n_2 < \dots < n_q$ and the summation extends over all indecomposable partitions $(V_1, V_2, \dots, V_p)$ of the table $\mathcal{T} = \{R_1, R_2, \dots, R_q\}$ with rows $R_1 = \{1, 2, \dots, n_1\}, \dots, R_q = \{n_{q-1}, \dots, n_q\}$; that is, over those partitions of the table $\mathcal{T}$ in which there exits no sets $V_{i_1}, \dots V_{i_n}$ $(n < p)$ such that for some rows $R_{j_1}, \dots, R_{j_m}$ $(m < q)$ of $\mathcal{T}$ the following equality holds

$$R_{j_1} \bigcup \dots \bigcup R_{j_m} = V_{i_1} \bigcup \dots \bigcup V_{i_n}.$$

5.2 The Aliasing Problem

Let $\mathbf{X} = \{X(t)\}_{t\in\mathbb{R}}$ be a zero-mean continuous and strictly stationary process with spectral density f and autocovariance function γ such that $\gamma(0) < \infty$. It is very well known that these functions satisfy the relation

$$\gamma(h) = \int_{-\infty}^{\infty} \exp(ih\vartheta) f(\vartheta) d\vartheta, \tag{14}$$

which says that γ is the inverse Fourier transform of f.

Expressions (13) and (14) reveal some differences between discrete and continuous processes. This phenomenon is known as the aliasing problem. In most practical circumstances, observations on $\mathbf{X} = \{X(t)\}_{t\in\mathbb{R}}$ are made at discrete intervals of time, even though the underlying process is continuous. This focusses the attention on the discrete observed process $\mathbf{Y} = \{Y_j\}$, where $Y_j = X(j\Delta)$ and $j \in \mathbb{Z}$. Due to the differences between the autocovariances functions of $\mathbf{X}$ and $\mathbf{Y}$ one may ask about the relation between their corresponding spectral densities. To elucidate this, we can start writing the autocovariance function of $\mathbf{Y}$ at lag $n \in \mathbb{Z}$ as

$$\begin{aligned}
\gamma_Y(n) &= Cov(Y_j, Y_{j+n}) = Cov(X(j\Delta), X((j+n)\Delta)) \\
&= \int_{-\infty}^{\infty} \exp(in\Delta\vartheta) f(\vartheta) d\vartheta \\
&= \sum_{h=-\infty}^{\infty} \int_{]\frac{(2h-1)\pi}{\Delta}, \frac{(2h+1)\pi}{\Delta}]} \exp(in\Delta\vartheta) f(\vartheta) d\vartheta \\
&= \sum_{h=-\infty}^{\infty} \int_{]-\pi,\pi]} \exp(in\vartheta) \frac{1}{\Delta} f(\frac{\vartheta + 2h\pi}{\Delta}) d\vartheta \\
&= \int_{]-\pi,\pi]} \exp(in\vartheta) (\frac{1}{\Delta} \sum_{h=-\infty}^{\infty} f(\frac{\vartheta + 2h\pi}{\Delta})) d\vartheta.
\end{aligned}$$

Then, the spectral density of the process $\mathbf{Y}$ is given by

$$f_Y(\vartheta) = \frac{1}{\Delta}\sum_{h=-\infty}^{\infty} f(\frac{\vartheta - 2h\pi}{\Delta}), \quad -\pi < \vartheta \leq \pi. \tag{15}$$

5.3 The Minimum Contrast Whittle Estimator

Given a strictly stationary zero-mean stochastic process $\mathbf{X} = \{X(t)\}$, we introduce the p−th order cumulant function of this process by

$$c_p(t_1, t_2, \ldots, t_{p-1}) = cum(X(t), X(t + t_1), \ldots, X(t + t_{p-1})),$$

whenever the right hand side exists. By the stationarity property, this function does not depend on t. Similarly to the (second order) spectral density f of $\mathbf{X}$, we define the p−th order spectral density, f_p, of $\mathbf{X}$ as the Fourier transform of the p-th order cumulant function of $\mathbf{X}$. In other words, f_p satisfies

$$c_p(t_1, \ldots, t_{p-1}) = \int_{\mathbb{R}^{p-1}} \exp(i\sum_{j=1}^{p-1} \vartheta_j t_j) f_p(\vartheta_1, \ldots, \vartheta_{p-1}) d\vartheta_1 \ldots d\vartheta_{p-1}$$

and is explicitly given by

$$\frac{1}{(2\pi)^{p-1}} \int_{\mathbb{R}^{p-1}} \exp(-i\sum_{j=1}^{p-1} \vartheta_j t_j) c_p(t_1, \ldots, t_{p-1})\, dt_1 \ldots dt_{p-1}.$$

If $\mathbf{X}$ is discretely observed, the $p - th$ order cumulant of $\mathbf{Y} = \{X(j\Delta)\}$ turns out to be for any integers $k_1, \ldots, k_p$:

$$c_p(k_1, \ldots, k_{p-1}) = \int_{]-\pi,\pi]^{p-1}} \exp(i\sum_{j=1}^{p-1} \vartheta_j k_j) f_p(\vartheta_1, \ldots, \vartheta_{p-1}) d\vartheta_1 \ldots d\vartheta_{p-1}.$$

This yields the $p - th$ order spectral density of **Y**

$$f_p(\vartheta_1, \ldots, \vartheta_{p-1}) = \frac{1}{(2\pi)^{p-1}} \sum_{k_1 \in \mathbb{Z}} \cdots \sum_{k_{p-1} \in \mathbb{Z}} \exp(-i \sum_{j=1}^{p-1} \vartheta_j k_j) c_p(k_1, \ldots, k_{p-1}).$$

Following Leonenko and Sakhno [2006] we will work in this thesis with the Whittle minimum contrast estimator

$$\hat{\theta}_\tau = \arg\min_{\theta \in \Theta} U_\tau(\theta), \tag{16}$$

being $\Theta \subset \mathbb{R}^m$ a compact set,

$$U_\tau(\theta) = \frac{1}{4\pi} \int_\Lambda (log(f(\vartheta;\theta)) + \frac{I_\tau(\vartheta)}{f(\vartheta;\theta)})\omega(\vartheta)d\vartheta,$$

ω an absolutely integrable symmetric weight function, $f(\vartheta;\theta)$ the spectral density (12) or (15) and

$$I_\tau(\vartheta) = \frac{1}{2\pi\tau} |d_\tau(\vartheta)|^2$$

the periodogram of the second order with

$$\Lambda = \mathbb{R}, \;\; \tau = T \;\; \text{and} \;\; d_\tau(\vartheta) = \int_0^\tau \exp(-i\vartheta s) X(s) ds,$$

if **X** is continuously observed; or

$$\Lambda =]-\pi, \pi], \;\; \tau = n \;\; \text{and} \;\; d_\tau(\vartheta) = \sum_{j=1}^{\tau} \exp(-i\vartheta j) Y_j,$$

if **X** is discretely observed via the process $\mathbf{Y} = \{Y_j\} = \{X(j\Delta)\}_{j=1,2,\ldots,n}$ with $T = n\Delta$.

We mention, however, that Leonenko and Sakhno [2006] were mainly concerned with the continuous case and they have analyzed (16) only for the case $\Delta = 1$.

To study the consistency of $\hat{\theta}_T$ we will need of the following proposition.

Proposition 5.1 *Let φ_1 and φ_2 be two real symmetric non-random functions and consider the second and fourth-order spectral densities of the processes* **X** *and* **Y**. *Suppose that the following two conditions hold*

(1) $G(u;\varphi_1) = \int_\Lambda f(\vartheta - u)\varphi_1(\vartheta)d\vartheta$ *is bounded and continuous at* $u = 0$.

(2) $G_4(u_1,u_2,u_3;\varphi_1,\varphi_2) = 2\int_\Lambda f(\vartheta+u_1)f(-\vartheta+u_3)\varphi_1(\vartheta)\varphi_2(u_1+u_2+\vartheta)\,d\vartheta + \int_{\Lambda^2} f_4(\vartheta_1+u_1,-\vartheta_1+u_2,\vartheta_2+u_3)\varphi_1(\vartheta_1)\varphi_2(\vartheta_2)d\vartheta_1 d\vartheta_2$ *is bounded for any* (u_1,u_2,u_3) *and it is continuous at* $(0,0,0)$.

Then

$$J_T(\varphi_1) = \int_\Lambda I_T(\vartheta)\varphi_1(\vartheta)d\vartheta \xrightarrow{P} J(\varphi_1) = \int_\Lambda f(\vartheta)\varphi_1(\vartheta)d\vartheta \quad as \quad T \to \infty.$$

Proof: Let $\alpha_1,\ldots,\alpha_p$ be arbitrary real numbers. Since cumulants are multilineal, we have for the continuous case

$$cum(d_T(\alpha_1),\ldots,d_T(\alpha_p))$$

$$= \int_{[0,T]^p} \exp(-i\sum_{j=1}^{p}\alpha_j t_j)c_p(t_1-t_p,\ldots,t_{p-1}-t_p)dt_1\ldots dt_p$$

$$= \int_{\Lambda^{p-1}} f_p(\vartheta_1,\ldots,\vartheta_{p-1}) \times \int_{[0,T]^p}\prod_{j=1}^{p-1}\exp(it_j(\vartheta_j-\alpha_j))$$

$$\exp(it_p(-\sum_{j=1}^{p-1}\vartheta_j-\alpha_p))dt_1\ldots dt_p d\vartheta_1\ldots d\vartheta_{p-1}$$

$$= \int_{\Lambda^{p-1}} f_p(\vartheta_1,\ldots,\vartheta_{p-1})\prod_{j=1}^{p-1}\Phi_1^T(\vartheta_j-\alpha_j)\Phi_1^T(-\sum_{j=1}^{p-1}\vartheta_j-\alpha_p)d\vartheta_1\ldots d\vartheta_{p-1},$$

where

$$\Phi_1^T(\lambda) = \int_0^T \exp(it\lambda)dt = \frac{2\sin(\frac{T\lambda}{2})}{\lambda}\exp(i\frac{T\lambda}{2}) \quad \text{and} \quad \Lambda = \mathbb{R}.$$

The same applies if $\mathbf{X}$ is discretely observed. In this case we just need to set $\Lambda =]-\pi, \pi]$, $\tau = n$ and consider

$$\Phi_1^n(\lambda) = \Delta\sum_{k=1}^{n}\exp(ik\lambda) = \frac{\sin(\frac{n\lambda}{2})}{\sin(\frac{\lambda}{2})}\exp(i\frac{(n+1)\lambda}{2})$$

in place of $\Phi_1^T(\lambda)$. Note that if $\sum_{j=1}^{p}\lambda_j = 0$, then $\frac{1}{(2\pi)^{p-1}\tau}\prod_{j=1}^{p}\Phi_1^\tau(\lambda_j)$ equals to the multidimensional kernel of Féjer type

$$\Phi_p^T(\lambda_1, \lambda_2, \ldots, \lambda_{p-1}) = \frac{1}{(2\pi)^{p-1}T}\int_{[0,T]^p}\exp(i\sum_{j=1}^{p}\lambda_j t_j)dt_1\ldots dt_p$$

$$= \frac{1}{(2\pi)^{p-1}T}\prod_{j=1}^{p}\frac{2\sin(\frac{T\lambda_j}{2})}{\lambda_j}\exp(i\frac{T\lambda_j}{2}) = \frac{1}{(2\pi)^{p-1}T}\prod_{j=1}^{p}\frac{2\sin(\frac{T\lambda_j}{2})}{\lambda_j}$$

or

$$\Phi_p^n(\lambda_1, \lambda_2, \ldots, \lambda_{p-1}) = \frac{1}{(2\pi)^{p-1}n}\sum_{k_1=1}^{n}\cdots\sum_{k_p=1}^{n}\exp(i\sum_{j=1}^{p}\lambda_j k_j)$$

$$= \frac{1}{(2\pi)^{p-1}n}\prod_{j=1}^{p}\frac{\sin(\frac{n\lambda_j}{2})}{\sin(\frac{\lambda_j}{2})}\exp(i\frac{(n+1)\lambda_j}{2}) = \frac{1}{(2\pi)^{p-1}n}\prod_{j=1}^{p}\frac{\sin(\frac{n\lambda_j}{2})}{\sin(\frac{\lambda_j\Delta}{2})}$$

if, respectively, $\mathbf{X}$ is continuously or discretely observed. Provided that G is bounded and continuous in the first argument at 0, we have in each case

$$\lim_{T\to\infty}\int_\Lambda \Phi_2^T(u)G(u;\varphi_1)du = \lim_{T\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty}(\frac{\sin x}{x})^2 G(\frac{2x}{T};\varphi_1)dx = G(0;\varphi_1)$$

or

$$\lim_{n\to\infty}\int_{\Lambda}\Phi_2^n(u)G(u;\varphi_1)du = \lim_{n\to\infty}\frac{1}{\pi}\int_{-\frac{\pi n}{2}}^{\frac{\pi n}{2}}(\frac{\sin x}{x})^2\frac{1}{(\frac{n}{x}\sin(\frac{x}{n}))^2}G(\frac{2x}{n};\varphi_1)dx$$

$$= G(0;\varphi_1)\frac{1}{\pi}\int_{-\infty}^{\infty}(\frac{\sin x}{x})^2dx = G(0;\varphi_1)$$

Similarly, if G_4 satisfies the conditions on the theorem, then we have for the continuous case

$$\lim_{T\to\infty}\int_{\Lambda^3}\Phi_4^T(u_1,u_2,u_3)G_4(u_1,u_2,u_3;\varphi_1,\varphi_2)du_1du_2du_3$$

$$= \lim_{T\to\infty}\frac{1}{\pi^3}\int_{\mathbb{R}^3}(\prod_{j=1}^{3}\frac{\sin x_j}{x_j})\frac{\sin(x_1+x_2+x_3)}{x_1+x_2+x_3}G_4(\frac{2x_1}{T},\frac{2x_2}{T},\frac{2x_3}{T};\varphi_1,\varphi_2)\, dx_1dx_2dx_3$$

$$= G_4(0,0,0;\varphi_1,\varphi_2)\frac{1}{\pi^3}\int_{\mathbb{R}^3}(\prod_{j=1}^{3}\frac{\sin x_j}{x_j})\frac{\sin(x_1+x_2+x_3)}{x_1+x_2+x_3}dx_1dx_2dx_3.$$

In order to evaluate

$$H = \int_{\mathbb{R}^3}(\prod_{j=1}^{3}\frac{\sin x_j}{x_j})\frac{\sin(x_1+x_2+x_3)}{x_1+x_2+x_3}dx_1dx_2dx_3$$

we rewrite it as

$$H = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sin x_1}{x_1}\frac{\sin x_2}{x_2}g(x_1+x_2)dx_1dx_2,$$

with

$$g(x) = \int_{-\infty}^{\infty}\frac{\sin u\sin(u+x)}{u(u+x)}du.$$

Because g is the convolution of the $sinc(x) = \frac{\sin x}{x}$ function with itself, g can be easily evaluated with the help of the convolution theorem. Indeed, the Fourier transform of g equals

$$\mathcal{F}[g](\vartheta) = 2\pi(\mathcal{F}[sinc](\vartheta))^2 = \begin{cases} \frac{\pi}{2}, & \text{if } |\vartheta| < 1 \\ 0, & \text{otherwise} \end{cases}$$

and so

$$g(x) = \mathcal{F}^{-1}[\mathcal{F}[g]](x) = \pi \frac{\sin x}{x}.$$

Plugging this expression into H, we obtain

$$H = \pi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\prod_{j=1}^{2} \frac{\sin x_j}{x_j}) \frac{\sin(x_1 + x_2)}{x_1 + x_2} dx_1 dx_2$$

$$= \pi \int_{-\infty}^{\infty} \frac{\sin x_1}{x_1} g(x_1) dx_1 = \pi^2 \int_{-\infty}^{\infty} (\frac{\sin x}{x})^2 dx = \pi^3$$

and consequently

$$\lim_{T\to\infty} \int_{\Lambda^3} \Phi_4^T(u_1, u_2, u_3) G_4(u_1, u_2, u_3; \varphi_1, \varphi_2) du_1 du_2 du_3 = G_4(0, 0, 0; \varphi_1, \varphi_2).$$

The discrete counterpart relation

$$\lim_{n\to\infty} \int_{\Lambda^3} \Phi_4^n(u_1, u_2, u_3) G_4(u_1, u_2, u_3; \varphi_1, \varphi_2) du_1 du_2 du_3 = G_4(0, 0, 0; \varphi_1, \varphi_2)$$

can be easily verified as before.

Under the condition $\sum_{j=1}^{p} \alpha_j = 0$, the d_T cumulant reduces to

$$cum(d_T(\alpha_1), \ldots, d_T(\alpha_p))$$

$$= (2\pi)^{p-1}\tau \int_{\Lambda^{p-1}} f_p(\vartheta_1,\ldots,\vartheta_{p-1})\Phi_p^\tau(\vartheta_1-\alpha_1,\ldots,\vartheta_{p-1}-\alpha_{p-1})\, d\vartheta_1\ldots d\vartheta_{p-1}$$

$$= (2\pi)^{p-1}\tau \int_{\Lambda^{p-1}} \Phi_p^\tau(u_1,\ldots,u_{p-1})f_p(u_1+\alpha_1,\ldots,u_{p-1}+\alpha_{p-1})\, du_1\ldots du_{p-1}. \qquad (17)$$

In particular, for $p = 2$, (17) yields

$$E[I_T(\vartheta)] = \frac{cum(d_\tau(\vartheta), d_\tau(-\vartheta))}{2\pi\tau} = \int_\Lambda \Phi_2^\tau(u)f(u+\vartheta)du$$

and so

$$\begin{aligned} E[J_\tau(\varphi_1)] &= E[\int_\Lambda I_\tau(\vartheta)\varphi_1(\vartheta)d\vartheta] \\ &= \int_\Lambda \varphi_1(\vartheta)(\int_\Lambda \Phi_2^\tau(u)f(u+\vartheta)du)d\vartheta \\ &= \int_\Lambda \Phi_2^\tau(u)(\int_\Lambda f(u+\vartheta)\varphi_1(\vartheta)d\vartheta)du \\ &= \int_\Lambda \Phi_2^\tau(u)G(u;\varphi_1)du \to \int_\Lambda f(\vartheta)\varphi_1(\vartheta)d\vartheta \ \text{ as } \ \tau\to\infty. \end{aligned}$$

In other words,

$$\int_\Lambda (E[I_\tau(\vartheta)] - f(\vartheta))\varphi_1(\vartheta)d\vartheta \to 0 \ \text{ as } \ \tau\to\infty.$$

On the other hand,

$$Cov(J_\tau(\varphi_1), J_\tau(\varphi_2))$$

$$= \frac{1}{(2\pi\tau)^2}\int_{\Lambda^2} cum(d_\tau(\vartheta_1)d_\tau(-\vartheta_1), d_\tau(\vartheta_2)d_\tau(-\vartheta_2))\varphi_1(\vartheta_1)\varphi_2(\vartheta_2)d\vartheta_1 d\vartheta_2.$$

By the Leonov-Shiryaev formula and (17), the integrand function above equals

$$
\begin{aligned}
& cum(d_\tau(\vartheta_1), d_\tau(-\vartheta_1), d_\tau(\vartheta_2), d_\tau(-\vartheta_2)) \\
& + cum(d_\tau(\vartheta_1), d_\tau(\vartheta_2)) cum(d_\tau(-\vartheta_1), d_\tau(-\vartheta_2)) \\
& + cum(d_\tau(\vartheta_1), d_\tau(-\vartheta_2)) cum(d_\tau(-\vartheta_1), d_\tau(\vartheta_2)) \\
= & (2\pi)^3 \tau \int_{\Lambda^3} \Phi_4^\tau(u_1, u_2, u_3) f_4(u_1 + \vartheta_1, u_2 - \vartheta_1, u_3 + \vartheta_2) du_1 du_2 du_3 \\
& +(2\pi)^3 \tau \int_{\Lambda^2} \Phi_4^\tau(s_1 - \vartheta_1, -s_1 - \vartheta_2, s_2 + \vartheta_1) f(s_1) f(s_2) ds_1 ds_2 \\
& +(2\pi)^3 \tau \int_{\Lambda^2} \Phi_4^\tau(s_1 - \vartheta_1, -s_1 + \vartheta_2, s_2 + \vartheta_1) f(s_1) f(s_2) ds_1 ds_2.
\end{aligned}
$$

Hence

$$
\begin{aligned}
Cov(J_\tau(\varphi_1), J_\tau(\varphi_2)) = \frac{2\pi}{\tau} \int_{\Lambda^3} & \Phi_4^\tau(u_1, u_2, u_3) G_4(u_1, u_2, u_3; \varphi_1, \varphi_2) \\
& du_1 du_2 du_3 \\
= O(\tau^{-1}) \text{ as } \tau \to \infty.
\end{aligned}
$$

Furthermore, since

$$
E[(\int_\Lambda (I_\tau(\vartheta) - E[I_\tau(\vartheta)])\varphi_1(\vartheta) d\vartheta)^2] = Cov(J_\tau(\varphi_1), J_\tau(\varphi_1)),
$$

the Markov inequality yields

$$
\int_\Lambda (I_\tau(\vartheta) - E[I_\tau(\vartheta)])\varphi_1(\vartheta) d\vartheta \xrightarrow{P} 0 \quad \text{as} \quad \tau \to \infty.
$$

and we have in the limit

$$
J_\tau(\varphi_1) - J(\varphi_1) = \int_\Lambda (I_\tau(\vartheta) - f(\vartheta))\varphi_1(\vartheta) d\vartheta
$$

$$
= \int_\Lambda (I_\tau(\vartheta) - E[I_\tau(\vartheta)])\varphi_1(\vartheta) d\vartheta + \int_\Lambda (E[I_\tau(\vartheta)] - f(\vartheta))\varphi_1(\vartheta) d\vartheta \xrightarrow{P} 0.
$$

■

Theorem 5.2 *Let* $\mathbf{X} = \{X(t)\}_{t\in[0,T]}$ *be a strictly stationary zero-mean Lévy process with second and fourth finite moments and corresponding spectral densities* $f(\vartheta;\theta)$ *and* $f_4(\vartheta_1,\vartheta_2,\vartheta_3;\theta)$. *Let* θ_0 *be the true parameter value and define* $\varphi(\vartheta;\theta) = \frac{1}{f(\vartheta;\theta)}\omega(\vartheta)$. *If the following conditions apply:*

*(***I***)* $f(\vartheta;\theta_1) \neq f(\vartheta;\theta_2)$ *for* $\theta_1 \neq \theta_2$*, almost everywhere in* $\mathbb{R}$.

*(***II***) Condition* (1) *in proposition 5.1 holds for* $\varphi_1(\vartheta) = \varphi(\vartheta;\theta)$ *and* $f(\vartheta) = f(\vartheta;\theta_0)$.

*(***III***) Condition* (2) *in proposition 5.1 holds for* $\varphi_1(\vartheta) = \varphi_2(\vartheta) = \varphi(\vartheta;\theta)$, $f(\vartheta) = f(\vartheta;\theta_0)$ *and* $f_4(\vartheta_1,\vartheta_2,\vartheta_3) = f_4(\vartheta_1,\vartheta_2,\vartheta_3;\theta_0)$.

*(***IV***) There exists a function* $v : \mathbb{R} \to \mathbb{R}$ *such that the function* $h(\vartheta;\theta) = \frac{1}{f(\vartheta;\theta)}v(\vartheta)$ *is uniformly continuous in* $\mathbb{R}\times\Theta$ *and (1) and (2) in proposition 5.1 hold for* $\varphi_1(\vartheta) = \varphi_2(\vartheta) = \frac{\omega(\vartheta)}{v(\vartheta)}$.

Then (16) *defines a consistent estimator of* θ_0*, that is,* $\hat{\theta}_\tau \xrightarrow{P} \theta_0$ *as* $\tau \to \infty$.

<u>Proof:</u> To simplify, all limits below will be understood to be taken as $\tau \to \infty$. Conditions **(II)**, **(III)** and proposition 5.1 imply that

$$\begin{aligned} U_\tau(\theta) &= \frac{1}{4\pi}\int_\Lambda (log(f(\vartheta;\theta)) + \frac{I_\tau(\vartheta)}{f(\vartheta;\theta)})\omega(\vartheta)d\vartheta \\ &\xrightarrow{P} U(\theta) = \frac{1}{4\pi}\int_\Lambda (log(f(\vartheta;\theta)) + \frac{f(\vartheta;\theta_0)}{f(\vartheta;\theta)})\omega(\vartheta)d\vartheta. \end{aligned} \tag{18}$$

Hence

$$U_\tau(\theta) - U_\tau(\theta_0) \xrightarrow{P} K(\theta_0;\theta),$$

where

$$K(\theta_0;\theta) = U(\theta) - U(\theta_0) = \frac{1}{4\pi}\int_\Lambda \left(\frac{f(\vartheta;\theta_0)}{f(\vartheta;\theta)} - 1 - \log(\frac{f(\vartheta;\theta_0)}{f(\vartheta;\theta)})\right)\omega(\vartheta)d\vartheta$$

is a non-negative function by virtue of the inequality

$$x - 1 \geq \log(x), \forall x > 0.$$

Furthermore, by condition (**I**), this inequality is strictly if $\theta \neq \theta_0$. Thus, given $\epsilon > 0$ there exists $\eta > 0$ such that for $|\theta - \theta_0| \geq \epsilon$:

$$U(\theta_0) < U(\theta) - \eta.$$

We claim that if (18) holds uniformly in $\theta \in \Theta$, then $\hat{\theta}_\tau$ will be a consistent estimator of θ_0. Indeed, the uniform convergence implies that

$$\begin{aligned} U(\hat{\theta}_\tau) - U(\theta_0) &= U(\hat{\theta}_\tau) - U_\tau(\theta_0) + U_\tau(\theta_0) - U(\theta_0) \\ &\leq U(\hat{\theta}_\tau) - U_\tau(\hat{\theta}_\tau) + U_\tau(\theta_0) - U(\theta_0) \xrightarrow{P} 0 \end{aligned}$$

and so

$$P(|\hat{\theta}_\tau - \theta_0| \geq \epsilon) \leq P(U(\hat{\theta}_\tau) - U(\theta_0) > \eta) \to 0.$$

To justify the uniform convergence, we invoke condition (**IV**) and proposition 5.1 to write

$$\int_\Lambda I_\tau(\vartheta)\frac{\omega(\vartheta)}{\nu(\vartheta)}d\vartheta = O_P(1).$$

Given $\epsilon > 0$, let $\eta(\epsilon)$ denotes the modulus of continuity of the uniformly continuous function h. Then for any $\theta_1, \theta_2 \in \Theta$ such that $|\theta_1 - \theta_2| \leq \epsilon$:

$$|U_\tau(\theta_1) - U_\tau(\theta_2)|$$

$$\begin{aligned} &= \frac{1}{4\pi}\left|\int_\Lambda \left(\log\left(\frac{h(\vartheta;\theta_2)}{h(\vartheta;\theta_1)}\right) + \frac{I_\tau(\vartheta)}{\nu(\vartheta)}h(\vartheta;\theta_1) - h(\vartheta;\theta_2)\right)\omega(\vartheta)d\vartheta\right| \\ &\leq \frac{1}{4\pi}\left|\int_\Lambda \left(\frac{h(\vartheta;\theta_2)}{h(\vartheta;\theta_1)} - 1\right)\omega(\vartheta)d\vartheta\right| + \frac{\eta(\epsilon)}{4\pi}\left|\int_\Lambda I_\tau(\vartheta)\frac{\omega(\vartheta)}{\nu(\vartheta)}d\vartheta\right| \\ &\leq \frac{1}{4\pi}\eta(\epsilon)\left(\left|\int_\Lambda \frac{f(\vartheta;\theta_1)\omega(\vartheta)}{\nu(\vartheta)}d\vartheta\right| + \left|\int_\Lambda I_\tau(\vartheta)\frac{\omega(\vartheta)}{\nu(\vartheta)}d\vartheta\right|\right) \end{aligned}$$

and the proof is complete. ∎

6 Estimation of the FIOUL Process

As seen in the previous section the minimum contrast estimator (16) is quite general and can be applied to any strictly stationary zero-mean process satisfying the conditions in theorem 5.2. In this section we study the possibility to apply this methodology to the FIOUL process $\mathbf{X_d} = \{X_d(t)\}$. As shown in proposition 4.1, this process has a symmetric and strictly positive spectral density

$$f(\vartheta;\theta) = \frac{\sigma^2}{2\pi|\vartheta|^{2d}(\lambda^2+\vartheta^2)},$$

where $\theta = (\lambda, d, \sigma)$ and $\sigma^2 = E[Z(1)^2]$ is the variance of the BDLP $\mathbf{Z} = \{Z(t)\}$ of $\mathbf{X_d}$. We remark that σ^2 may depend in turn on the inner parameters of the BDLP.

Similarly to f, we can evaluate the fourth order spectral density, f_4, of $\mathbf{X_d}$. This is given by the Fourier inverse transform of the fourth order cumulant function

$$\begin{aligned} c_4(t_1,t_2,t_3) &= cum(X_d(0), X_d(t_1), X_d(t_2), X_d(t_3)) \\ &= \frac{\partial^4 C(\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4)}{\partial\vartheta_1\partial\vartheta_2\partial\vartheta_3\partial\vartheta_4}\Big|_{(\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4)=(0,0,0,0)}, \end{aligned}$$

where $C(\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4)$ is the cumulant function (9) with $g = g_d$ at the points $0 \le t_1 \le t_2 \le t_3$. This yields

$$c_4(t_1,t_2,t_3) = C^{(4)}_{Z(1)}(0)\int_0^\infty g_d(s)g_d(s+t_1)g_d(s+t_2)g_d(s+t_3)ds$$

and the fourth order spectral density

$$\begin{aligned} &f_4(\vartheta_1,\vartheta_2,\vartheta_3;\theta) \\ &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\exp(-i(\vartheta_1 t_1+\vartheta_2 t_2+\vartheta_3 t_3))c_4(t_1,t_2,t_3)dt_1dt_2dt_3 \end{aligned}$$

$$
\begin{aligned}
&= 2\pi C^{(4)}_{Z(1)}(0)\left(\frac{1}{2\pi}\int_0^\infty \exp(-i\vartheta_1 u)g_d(u)du\right) \\
&\times \left(\frac{1}{2\pi}\int_0^\infty \exp(-i\vartheta_2 u)g_d(u)du\right)\left(\frac{1}{2\pi}\int_0^\infty \exp(-i\vartheta_3 u)g_d(u)du\right) \\
&\times \left(\frac{1}{2\pi}\int_0^\infty \exp(is(\vartheta_1+\vartheta_2+\vartheta_3))g_d(s)ds\right) \\
&= 2\pi C^{(4)}_{Z(1)}(0)\hat{g}_d(\vartheta_1)\hat{g}_d(\vartheta_2)\hat{g}_d(\vartheta_3)\hat{g}_d(-(\vartheta_1+\vartheta_2+\vartheta_3)) \\
&= \frac{C^{(4)}_{Z(1)}(0)}{(2\pi)^3(-1)^d(\vartheta_1\vartheta_2\vartheta_3(\vartheta_1+\vartheta_2+\vartheta_3))^d\zeta(\lambda,\vartheta_1,\vartheta_2,\vartheta_3)},
\end{aligned}
$$

where

$$
C^{(4)}_{Z(1)}(0) = E[Z(1)^4] - 3E[Z(1)^2]^2,
$$

$$
\zeta(\lambda,\vartheta_1,\vartheta_2,\vartheta_3) = \prod_{k=1}^{3}(\lambda + i\vartheta_k)(\lambda - i(\vartheta_1+\vartheta_2+\vartheta_3))
$$

and $\hat{g}_d$ is the Fourier transform of the kernel g_d that was derived in proposition 4.1.

We examine now the conditions in theorem 5.2 to ensure the consistency of (16). As in Gao [2004], we will work hereafter with the weight function

$$
\omega(\vartheta) = \frac{1}{1+\vartheta^2}
$$

and assume that $\mathbf{X}$ is continuously observed. See however, remark 6.1.

Condition **(I)**:

This is an identifiability condition that restrict the estimation of the whole set of parameters to the estimation of the vector parameter $\theta = (\lambda, d, \sigma)$, where $\sigma^2 = E[Z(1)^2] = \sigma^2(\theta_1)$. Unfortunately, the minimum contrast estimator (16) does not provide information about the BDLP vector parameter θ_1.

Condition **(II)**:

By the monote convergence theorem, this condition is satisfied if we prove that

$$\frac{f(\vartheta;\theta_0)\omega(\vartheta)}{f(\vartheta;\theta)} \in L^1, \forall\theta \in \Theta.$$

Let

$$I = \int_{-\infty}^{\infty} \frac{f(\vartheta;\theta_0)\omega(\vartheta)}{f(\vartheta;\theta)} d\vartheta = 2\frac{\sigma_0^2}{\sigma^2} \int_0^{\infty} \frac{\vartheta^a(\lambda^2+\vartheta^2)}{(\lambda_0^2+\vartheta^2)(1+\vartheta^2)} d\vartheta,$$

where $a = 2(d-d_0)$. Because $d \in]0, \frac{1}{2}[$, $|a| < 1$. Making the change of variable $x = \vartheta^2$ and using the identity (see Gradshteyn and Ryzhik [1994], pag. 337)

$$\int_0^{\infty} \frac{x^{\mu-1}(x+\beta)}{(x+\gamma)(x+\delta)} dx = \pi \csc(\mu\pi)\left((\frac{\gamma-\beta}{\gamma-\delta})\gamma^{\mu-1} + (\frac{\delta-\beta}{\delta-\gamma})\delta^{\mu-1} \right),$$

with $0 < \mu < 1$, we obtain

$$I = \frac{\sigma_0^2}{\sigma^2} \int_0^{\infty} \frac{x^{\frac{a+1}{2}-1}(x+\lambda^2)}{(x+\lambda_0^2)(x+1)} dx = \frac{\sigma_0^2\pi}{\sigma^2\cos(\frac{a\pi}{2})}(\frac{\lambda^2(\lambda_0^{a-1}-1)+1-\lambda_0^{a+1}}{1-\lambda_0^2}) < \infty$$

This is true even if $\lambda_0 = 1$, since the last function tends to

$$\frac{\sigma_0^2\pi}{2\sigma^2 cos(\frac{a\pi}{2})}(a+1-\lambda^2(a-1)) \text{ as } \lambda_0 \to 1.$$

Condition **(III)**:

Since

$$\begin{aligned} G_4(0,0,0;\varphi,\varphi) &= 2\int_{-\infty}^{\infty} f(\vartheta;\theta_0)^2\varphi(\vartheta)^2 d\vartheta \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_4(\vartheta_1,-\vartheta_1,\vartheta_2;\theta_0)\varphi(\vartheta_1)\varphi(\vartheta_2)d\vartheta_1 d\vartheta_2, \end{aligned}$$

where

$$\begin{aligned} f_4(\vartheta_1, -\vartheta_1, \vartheta_2; \theta_0) &= \frac{C^{(4)}_{Z(1)}(0)}{(2\pi)^3 |\vartheta_1|^{2d_0} |\vartheta_2|^{2d_0} (\lambda_0^2 + \vartheta_1^2)(\lambda_0^2 + \vartheta_2^2)} \\ &= \frac{E_{\theta_0}[Z(1)^4] - 3E_{\theta_0}[Z(1)^2]^2}{2\pi E_{\theta_0}[Z(1)^2]^2} f(\vartheta_1; \theta_0) f(\vartheta_2; \theta_0), \end{aligned}$$

this condition will be satisfied if we prove that

$$\frac{f(\vartheta; \theta_0)\omega(\vartheta)}{f(\vartheta; \theta)} \in L^2, \ \forall \theta \in \Theta.$$

We will show this by restricting the vector parameter $\theta = (\lambda, d, \sigma)$ to satisfy the condition $|d - d_0| \leq \frac{1}{4}$.

Let

$$I = \int_{-\infty}^{\infty} (\frac{f(\vartheta; \theta_0)\omega(\vartheta)}{f(\vartheta; \theta)})^2 d\vartheta = 2\frac{\sigma_0^4}{\sigma^4} \int_0^{\infty} \frac{\vartheta^a (\lambda^2 + \vartheta^2)^2}{(\lambda_0^2 + \vartheta^2)^2 (1 + \vartheta^2)^2} d\vartheta,$$

where $a = 4(d - d0)$, satisfy $|a| < 1$. By Gradshteyn and Ryzhik [1994], pag. 335, we have the identity

$$\int_0^{\infty} \frac{x^{r-1}}{(1+x)^2 (x + \beta)^2} dx = B(4 - r, r) {}_2F_1(2, 4 - r, 4, 1 - \beta), \quad \text{for} \quad 0 < r < 4.$$

Here B denotes the Beta function and ${}_2F_1$ the Gaussian hypergeometric function.

An application of this formula and the change of variable $x = \vartheta^2$ yield

$$I = \frac{\sigma_0^4}{\sigma^4} \int_0^{\infty} \frac{x^{\frac{a+1}{2}-1} (\lambda^2 + x)^2}{(1+x)^2 (x + \lambda_0^2)^2} dx = \frac{\lambda^4 \sigma_0^4}{\sigma^4} \int_0^{\infty} \frac{x^{\frac{a+1}{2}-1}}{(1+x)^2 (x + \lambda_0^2)^2} dx$$

$$+ \frac{2\lambda^2 \sigma_0^4 \lambda^4}{\sigma^4} \int_0^{\infty} \frac{x^{\frac{a+3}{2}-1}}{(1+x)^2 (x + \lambda_0^2)^2} dx + \frac{\sigma_0^4 \lambda^4}{\sigma^4} \int_0^{\infty} \frac{x^{\frac{a+5}{2}-1}}{(1+x)^2 (x + \lambda_0^2)^2} dx$$

$$= \frac{\lambda^4 \sigma_0^4}{\sigma^4} B(\frac{7-a}{2}, \frac{a+1}{2}) {}_2F_1(2, \frac{7-a}{2}, 4, 1 - \lambda_0^2)$$

$$+\frac{2\lambda^2\sigma_0^4\lambda^4}{\sigma^4}B(\frac{5-a}{2},\frac{a+3}{2})_2F_1(2,\frac{5-a}{2},4,1-\lambda_0^2)$$

$$+\frac{\sigma_0^4\lambda^4}{\sigma^4}B(\frac{3-a}{2},\frac{a+5}{2})_2F_1(2,\frac{3-a}{2},4,1-\lambda_0^2)) < \infty,$$

provided that $|\lambda_0| \neq 1$. However, if $|\lambda_0| = 1$, the identity

$$\int_0^\infty \frac{x^{\mu-1}}{(p+qx^\nu)^{n+1}}dx = \frac{1}{\nu p^{n+1}}(\frac{p}{q})^{\frac{\mu}{\nu}}\frac{\Gamma(\frac{\mu}{\nu})\Gamma(1+n-\frac{\mu}{\nu})}{\Gamma(1+n)},$$

where $0 < \frac{\mu}{\nu} < n+1$, $p \neq 0$ and $q \neq 0$ (see Gradshteyn and Ryzhik [1994], pag. 341) implies that

$$I = \frac{2\sigma_0^4}{\sigma^4}\int_0^\infty \frac{\vartheta^a(\lambda^2+\vartheta^2)^2}{(1+\vartheta^2)^4}d\vartheta$$

$$= \frac{2\sigma_0^4}{\sigma^4}\left(\lambda^4\int_0^\infty \frac{\vartheta^a}{(1+\vartheta^2)^4}d\vartheta + 2\lambda^2\int_0^\infty \frac{\vartheta^{a+2}}{(1+\vartheta^2)^4}d\vartheta + \int_0^\infty \frac{\vartheta^{a+4}}{(1+\vartheta^2)^4}d\vartheta\right)$$

$$= \frac{\sigma_0^4}{6\sigma^4}\left(\lambda^4\Gamma(\frac{a+1}{2})\Gamma(\frac{7-a}{2}) + 2\lambda^2\Gamma(\frac{a+3}{2})\Gamma(\frac{5-a}{2}) + \Gamma(\frac{a+5}{2})\Gamma(\frac{3-a}{2})\right)$$

is also finite.

Condition **(IV)**:

As a simple choice we could take $v(\vartheta) = 1$. Then depending on the variance of the BDLP, the uniform continuity of

$$h(\vartheta;\theta) = \frac{1}{f(\vartheta;\theta)} = \frac{2\pi|\vartheta|^{2d}(\lambda^2+\vartheta^2)}{\sigma^2(\theta_1)},$$

seems easy to be guaranteed. Moreover, by the previous analysis, it is easy to verify that conditions (1) and (2) in proposition 5.1 hold for $\varphi_1(\vartheta) = \varphi_2(\vartheta) = \omega(\vartheta)$.

Remark 6.1 *The previous analysis also holds if* $\mathbf{X_d}$ *is discretely observed. We show for instance condition* (**II**). *Let* f_Y *be the spectral density* (15) *and let* $f = f_d$ *be the spectral density* (12) *of* $\mathbf{X_d}$. *Then*

$$\frac{1}{\Delta} f(\frac{\vartheta}{\Delta}; \theta) \leq f_Y(\vartheta; \theta)$$

and

$$I = \int_{]-\pi,\pi]} \frac{f_Y(\vartheta; \theta_0)\omega(\vartheta)}{f_Y(\vartheta; \theta)} d\vartheta \leq \Delta \int_{]-\pi,\pi]} \frac{f_Y(\vartheta; \theta_0)\omega(\vartheta)}{f(\frac{\vartheta}{\Delta}; \theta)} d\vartheta.$$

Because the function $\frac{\omega(\vartheta)}{f(\frac{\vartheta}{\Delta})}$ *is continuous at* ϑ *on the interval* $[-\pi, \pi]$, *we can find a finite constant* C *such that*

$$I \leq C\Delta \int_{]-\pi,\pi]} f_Y(\vartheta; \theta_0) d\vartheta = C\Delta\gamma(0) < \infty.$$

Note that the same result holds without the weight function; i.e, by taking $\omega(\vartheta) = 1$. *This also holds for the other conditions.*

In order to implement the Whittle methodology, let us assume that $\mathbf{X_d}$ is discretely observed via the process $\mathbf{Y} = \{X_d(j\Delta)\} = \{Y_j\}$. Then, an approximation of the objective function

$$\begin{aligned} U_n(\vartheta) &= \frac{1}{4\pi} \int_{-\pi}^{\pi} (\log(f_Y(\vartheta; \theta) + \frac{I_n(\vartheta)}{f_Y(\vartheta)})\omega(\vartheta) d\vartheta \\ &= \frac{1}{2\pi} \int_{0}^{\pi} (\log(f_Y(\vartheta; \theta) + \frac{I_n(\vartheta)}{f_Y(\vartheta)})\omega(\vartheta) d\vartheta \end{aligned}$$

is obtained by first considering the partition

$$\{\vartheta_k = \frac{2\pi k}{n}\},$$

with k running from 1 to $m = [\frac{n-1}{2}]$, and then using the right-hand side rectangular rule

$$U_n(\theta) \simeq \frac{1}{n} \sum_{k=1}^{m} (log(f_Y(\vartheta_k)) + \frac{I_n(\vartheta_k)}{f_Y(\vartheta_k)})\omega(\vartheta_k), \tag{19}$$

where the weight function ω can be omitted. Recall that the spectral density f_Y in (12) is an infinite sum. Therefore it needs to be truncated to a sufficiently large integer value M. This gives the approximation

$$f_Y(\vartheta) = \frac{1}{\Delta} \sum_{h=-M}^{M} f(\frac{\vartheta - 2h\pi}{\Delta}), \quad -\pi < \vartheta \leq \pi.$$

On the other hand, we observe that the periodogram points

$$I_n(\vartheta_k) = \frac{1}{2\pi n} |\sum_{j=1}^{n} \exp(-i\vartheta_k j) Y_j|^2$$

can be readily calculated by an application of the discrete fast Fourier transform.

One nice feature of the present model is that the minimization of (19) can be simplified by rewriting the objective function as

$$U_n(\theta) = \frac{1}{n} \left(m \log(\frac{\sigma^2}{2\pi\Delta}) + \sum_{k=1}^{m} log(g(\vartheta_k)) + \frac{2\pi\Delta}{\sigma^2} \sum_{k=1}^{m} \frac{I_n(\vartheta_k)}{g(\vartheta_k)} \right), \qquad (20)$$

where

$$g(\vartheta) \equiv g(\vartheta; \lambda, d) = \frac{2\pi\Delta f_Y(\vartheta)}{\sigma^2} = \sum_{h=-M}^{M} \frac{1}{|\frac{\vartheta - 2h\pi}{\Delta}|^{2d} (\lambda^2 + (\frac{\vartheta - 2h\pi}{\Delta})^2)}.$$

Then, differentiating (20) with respect to σ^2 and equating to zero gives

$$\sigma^2 = \frac{2\pi\Delta}{m} \sum_{k=1}^{m} \frac{I_n(\vartheta_k)}{g(\vartheta_k)}. \qquad (21)$$

Substituting (21) in (20) yields

$$U_n(\theta) = \frac{1}{n} (m \log(\sum_{k=1}^{m} \frac{I_n(\vartheta_k)}{g(\vartheta_k)}) + \sum_{k=1}^{m} log(g(\vartheta_k)) + m).$$

As a consequence, the Whittle estimator can be defined as $(\hat{\lambda}, \hat{d}, \hat{\sigma})$, where

$$(\hat{\lambda}, \hat{d}) = \arg\min_{(\lambda,d)\in\Theta} \sum_{k=1}^{m} log(g(\vartheta_k; \lambda, d)) + m\log(\sum_{k=1}^{m} \frac{I_n(\vartheta_k)}{g(\vartheta_k; \lambda, d)}),$$

$$\hat{\sigma} = \sqrt{\frac{2\pi\Delta}{m} \sum_{k=1}^{m} \frac{I_n(\vartheta_k)}{g(\vartheta_k; \hat{\lambda}, \hat{d})}}$$

and Θ is a compact set in $[0, \infty[\times[0, 0.5]$.

7 Simulation of a FIOUL Process with Symmetric CGMY BDLP

We will consider in this work a FIOUL process $\mathbf{X_d}$ with BDLP $\mathbf{Z} = \{Z(t)\}$ given by a symmetric $CGMY(C, M, M, \alpha)$-Lévy process with fixed parameter $\alpha \in]0, 1[$. This process, which refers to the Car, Geman, Madam and Yor model in Carr et al. [2002], can be seen in turn as a particular tempered $\alpha-$stable process. The family of tempered stable processes has been extensively studied by Rosiński [2007] and Cohen and Rosiński [2007]. Following these authors, we present the next series representation for $\mathbf{Z} = \{Z(t)\}_{t\in[0,T]}$:

$$Z(t) = A_\epsilon W(t) + M^\epsilon(t),$$

where $\epsilon > 0$ is small, $A_\epsilon = \epsilon^{1-\frac{\alpha}{2}} \sqrt{2C(\frac{\alpha}{2-\alpha})}$, $\mathbf{W} = \{W(t)\}$ is a standard Brownian motion,

$$M^\epsilon(t) = \sum_{\{j/\Gamma_j \leq \frac{2CT}{\alpha\epsilon^\alpha}\}} \mathbf{1}_{[\tau_j,\infty[}(t) \min\{\left(\frac{2CT}{\alpha\Gamma_j}\right)^{\frac{1}{\alpha}}, \frac{e_j u_j^{\frac{1}{\alpha}}}{M}\}(-1)^{B_j},$$

is a compound Poisson process, $\{B_j\}$ is an i.i.d. sequence of Bernoulli random variable with parameter 0.5, $\{u_j\}$ is an i.i.d. sequence of uniform random variables in the interval $]0, 1[$, $\{e_j\}$ is an i.i.d. sequence of exponential random variables with parameter 1, $\Gamma_1 < \Gamma_2 < \ldots$ are the arrival times of a Poisson process with intensity parameter 1 and $\{\tau_i\}$ is an i.i.d. sequence of uniform random variables in $[0, T]$. All sequences are assumed independent of each other. Details are given in Valdivieso [2007].

As a result, a direct integration of the previous process gives the next simulation scheme to the FIOUL process $\mathbf{X_d} = \{X_d(t)\}$:

$$\begin{aligned} X_d(t) &= \int_{-\infty}^{t} g_d(t-s)dZ(s) = A_\epsilon B(t) + \int_0^\infty g_d(t+s)dM^\epsilon(s) \\ &+ \int_0^t g_d(t-s)dM^\epsilon(s) \equiv A_\epsilon B(t) + Y_1(t) + Y_2(t), \end{aligned}$$

where $\mathbf{B} = \{B(t) = \int_{-\infty}^{t} g_d(t-s)dW(s)\}$ is a stationary Gaussian process with autocovariance function $\gamma(h) = \int_0^\infty g_d(|h|+s)g_d(s)ds$, $\mathbf{Y}_2 = \{Y_2(t)\}$ is given by:

$$Y_2(t) = \int_0^t g_d(t-s)dM^\epsilon(s)$$

$$= \sum_{\{j/\Gamma_j < \frac{2CT}{\alpha\epsilon^\alpha}\}} g_d(t-\tau_j)\mathbf{1}_{[\tau_j,\infty[}(t) \min\{(\frac{2CT}{\alpha\Gamma_j})^{\frac{1}{\alpha}}, \frac{e_j u_j^{\frac{1}{\alpha}}}{M}\}(-1)^{B_j}.$$

and $\mathbf{Y}_1$ by

$$Y_1(t) = \int_0^K g_d(t+s)dM^\epsilon(s)$$

$$= \sum_{\{j/\tilde{\Gamma}_j < \frac{2CK}{\alpha\epsilon^\alpha}\}} g_d(t+\tilde{\tau}_j) \min\{(\frac{2CK}{\alpha\tilde{\Gamma}_j})^{\frac{1}{\alpha}}, \frac{\tilde{e}_j \tilde{u}_j^{\frac{1}{\alpha}}}{M}\}(-1)^{\tilde{B}_j},$$

where the super-tilde means that all these random sequences are generated independently from the ones of $\mathbf{Y}_2$, $\{\tilde{\tau}_j\}$ is an i.i.d. sequence of uniform random variables on the interval $[0, K]$ and K is a truncation constant that can be determined, for instance, by fixing a small value e and taking $K = (\lambda\Gamma(d)e)^{\frac{1}{d-1}}$.

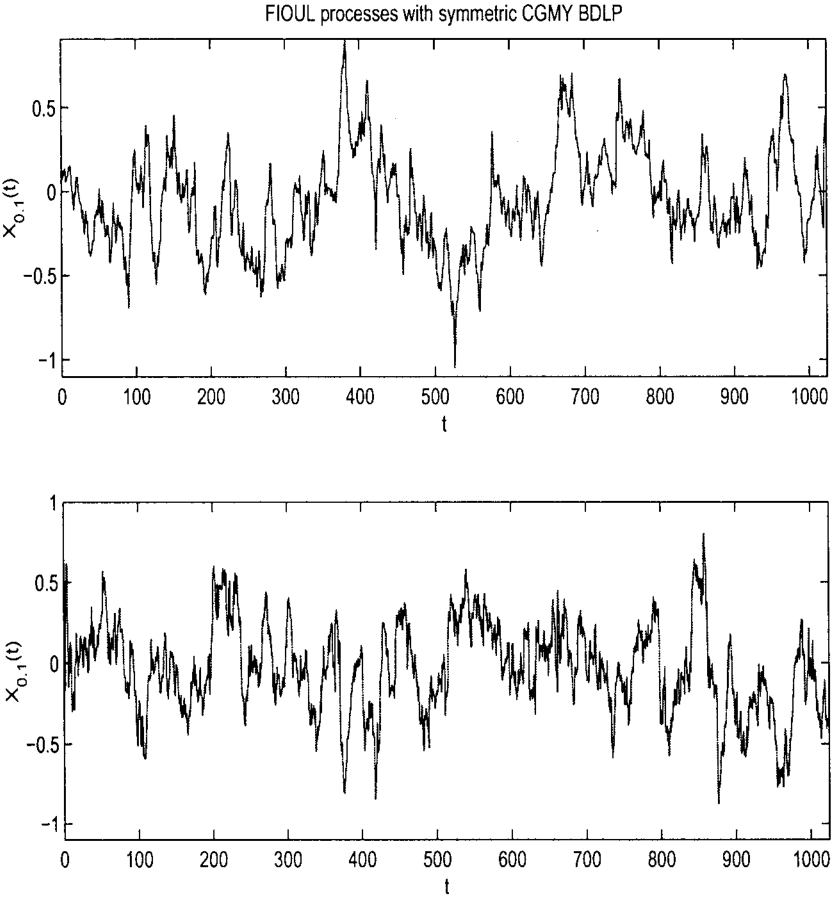

Figure 1: Sample paths of a FIOUL process with CGMY(0.1784,10,10,0.5) BDLP, $\lambda = 0.1$ and $d = 0.1$. The upper process was simulated with $\Delta = 1$ and the lower process with $\Delta = 0.25$.

8 Simulation Results

We present in this section a simulation study of the performance of the Whittle estimator for a FIOUL process **X** with symmetric CGMM BDLP Lévy process. We have chosen for this a FIOUL process with damping parameter $\lambda = 0.1$, memory parameter $d = 0.1$ and $CGMY(0.1784,\ 10, 10, 0.5)$ BDLP, which has a standard deviation $\sigma = 0.1$. To analyze the contribution of the sampling frequency, we have selected observations of **X** at equidistant points with $\Delta = 1$ and $\Delta = 0.25$. The simulation of **X** was conducted via the direct integration scheme in Section 7 with $\epsilon = 0.001$ and $T = 1024$. Figure 1 displays two of these simulated paths and Tables 1 and 2 show the estimation results based on 100 simulations of **X**. We indicate there the mean, median, standard deviation, root mean square error and running time in seconds of these 100 estimations. All the procedures were ran on a PC Pentium IV with 2.4 Mhz using a Matlab software. As seen, the Whittle estimator performs not only very well, but also fast. The more accurate results were obtained for the deviation parameter σ. We observe also more efficient results for $\Delta = 0.25$. The mean running time in this case is below one minute, while for $\Delta = 1$ is slightly above this time. We must indicate, however, that for $\Delta = 1$ more than 70 % of the estimations took around 15 seconds. The high variance observed in Table 1 is due to the fact that in cases of no convergence we switched from the fmincon optimization algorithm in Matlab to an hybrid differential evolution algorithm. One way to reduce these times is by improving the poor performance observed in the initial estimators. These were calculated with the log-periodogram or the R/S regression technique (see for details Doukhan et al. [2002]).

Table 1: Whittle estimations for the vector parameter (λ, d, σ) in a FIOUL process with $CGMY(0.1784, 10, 10, 0.5)$ symmetric BDLP and $\Delta = 1$.

	$\lambda = 0.1$	$d = 0.1$	$\sigma = 0.1$	Time
Mean	0.10542	0.10574	0.10003	70.729
Median	0.10153	0.10618	0.1002	14.617
Std. Dev.	0.026434	0.033915	0.0044479	89.953
RMSE	0.026855	0.03423	0.0044257	

Table 2: Whittle estimations for the vector parameter (λ, d, σ) in a FIOUL process with $CGMY(0.1784, 10, 10, 0.5)$ symmetric BDLP and $\Delta = 0.25$.

	$\lambda = 0.1$	$d = 0.1$	$\sigma = 0.1$	Time
Mean	0.1029	0.099302	0.10117	55.348
Median	0.10105	0.10001	0.10146	53.195
Std. Dev.	0.016649	0.013404	0.0046814	8.6153
RMSE	0.016817	0.013355	0.0048031	

References

[1] D.R. Brillinger, *Time Series:Data Analysis and Theory*. Holden, San Francisco, (1981).

[2] P. Brockwell and R. Davis, *Time Series: Theory and Methods*. Springer-Verlag, New York, (1986).

[3] P. Brockwell and T. Marquardt, *Lévy-driven and fractionally integrated ARMA processes with continuous time parameter*. Statistica Sinica, (2005) 15:477–494.

[4] J. Carr, H. Geman, D. Madam, and M. Yor, *The Fine Structure of Assets Returns: An Empirical Investigation.* The Journal of Business, (2002) 75: 305–332.

[5] S. Cohen and J. Rosiński, *Gaussian approximation of multivariate Lévy processes with applications to simulation of tempered stable processes.* Bernoulli, (2007). To appear.

[6] F. Comte and E. Renault, *Long memory continuous time models.* Journal of Econometrics, (1996) 73:101–149.

[7] P. Doukhan, G. Oppenheim, and M.S. Taqqu, *Theory and Applications of Long Range Dependence.* Birkhauser, (2002).

[8] J. Gao, *Modelling long-range-dependent Gaussian processes with application in continuous-time financial models.* Journal of Applied Probability, (2004) 41:467–482.

[9] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products.* Academic Press, (1994).

[10] N. N. Leonenko and L. M. Sakhno, On the Whittle estimators for some classes of continuous parameter random processes and fields. *Statistics and Probability Letters*, (2006) 76:781–795.

[11] V. V. Leonov and A. N. Shiryaev, *On a method of calculation of semi-invariants.* Theory Probability Applications, (1959) 4:319–329.

[12] J. Rosiński, *Tempering stable processes.* Stochastic Processes and their Applications. (2007) 117:677–707.

[13] K. Sato, *Lévy Processes and Infinitely Divisible Distributions.* Cambridge University Press, Cambridge, (1999).

[14] L. Valdivieso, *Likelihood Inference in Processes of Ornstein-Uhlenbeck Type*. PhD thesis, Departement Wiskunde, Katholieke Universiteit Leuven, October (2007).

[15] P. Whittle, *Gaussian estimation in stationary time series*. Bulletin of the International Statistical Institute, (1962) 39:105–129.

Resumen

Se propone una metodología para la estimación de un proceso fraccionado integrado de tipo Ornstein-Uhlenbeck. La metodología se basa en el contraste continuo de Whittle. Se presenta un estudio se simulación en el cual este proceso es conducido por un proceso CGMY de Lévy simétrico.

Palabras Clave: Procesos de Lévy fraccionados. Dependencia a largo plazo. Estimación en el dominio de las frecuencias.

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