CHARACTERISTIC CLASSES OF MODULES

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Abstract

In this paper we have developed a general theory of characteristic classes of modules. To a given invariant map defined on a Lie algebra, we associate a cohomology class by using the curvature form of a certain kind of connections. Here we present a very simple proof of the invariance theorem (Theorem 12), which states that equivalent connections give rise to the same characteristic class. We have used those invariant maps of [9] to define Chern classes of projective modules and we have derived their basic properties.

It might be interesting to observe that this theory could be applied to define characteristic classes of bilinear maps. In particular, the Euler classes of [6] can be obtained in this way.

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1 The Characteristic Class of an Invariant Map

Let K be a commutative ring with 1.

1. We recall ([7]) that a differential graded (D.G) algebra $A = (A, d)$ over K is a graded algebra $A = \bigoplus_{p=0}^{\infty} A^P$, with a graded K-module homomorphism $d : A \to A$ of degree 1 such that $d^2 = 0$ and which satisfies the Leibniz formula

$$d(a_1 \cdot a_2) = d\,a_1 \cdot a_2 + (-1)^{deg\ a_1}\, a_1 \cdot d\,a_2$$

for $a_1,\ a_2 \in A$.

In addition to this we shall assume these algebras to satisfy the following requirements:

(1) A is commutative: That is, $a \cdot b = (-1)^{ij}\, b \cdot a$ holds for all $a \in A^i,\ b \in A^j$

(2) $a^2 = 0$ for all $a \in A^1$.

From (1) it follows that A^0 is a commutative K-algebra. From now on A will always denote a DG-algebra over K satisfying both (1) and (2) above, and we shall write $R = A^0$.

2. Let $\Phi : P \times Q \to M$ be a bilinear map of R-modules. Then there exists a unique bilinear map

$$\cdot : A^r \otimes P \times A^s \otimes Q \to A^{r+s} \otimes M$$

which satisfies $(a \otimes x) \cdot (b \otimes y) = ab \otimes \Phi(x, y)$ for all $a \in A^r, b \in A^s$, $x \in P,\ y \in Q$.

We shall often denote this map also with Φ.

3. **Definition**. Let M be an R-module. An A-connection on M is a map $\nabla : M \to A^1 \otimes M$ such that

   (1) ∇ is a K-linear

   (2) $\nabla(ax) = da \otimes x + a \cdot \nabla x, (a \in R, x \in M)$.

   In this case, we can define the K-linear maps

$$\begin{aligned} \nabla &= \nabla^i : A^i \otimes M \to A^{i+1} \otimes M \text{ by} \\ & \nabla^i(a \otimes x) = da \otimes x + (-1)^i a \cdot \nabla x \qquad (a \in A^i, x \in M) \end{aligned}$$

   and the curvature K_∇ of ∇ by $\nabla^1 \otimes \nabla : M \to A^2 \otimes M$

   We observe the following formula

$$\nabla^{i+j}(a \cdot b) = \nabla^i a \cdot b + (-1)^i \quad a \cdot \nabla^j b, \quad a \in A^i, \quad b \in A^i \otimes M$$

   We say that a connection ∇ is integrable ([3], [4]) if $K_\nabla = 0$. In this case, it can easily be verified that the sequence

$$M \xrightarrow{\nabla^0} A^1 \otimes M \xrightarrow{\nabla^1} A^2 \otimes M \to \dots$$

   is a complex, and we define the i th-cohomology group

$$H^i(A \otimes M) = \frac{\text{Ker}\,\nabla^i}{Im\nabla^{i-1}}, \qquad (i \geq 0).$$

4. **Proposition**. Let P be a finitely generated projective R-module. Then we have the following

   (1) There exists an A-connection on P.

   (2) If ∇ is an A-connection on P, then $d_\nabla : L(P) \to A^1 \otimes L(P)$ is an A-connection on $L(P) = Hom_R(P, P)$, where

$$d_\nabla(f) = \nabla \circ f - (I \otimes f) \circ \nabla \qquad (f \in L(P))$$

   and we identify $A^1 \otimes L(P)$ canonically with $Hom_R(P, A^1 \otimes P)$.

Proof.

(1) If P is a free module with a basis $(e_1, \dots, e_n)$ then choosing any n-square matrix $(a_{ij})_\varepsilon M_n(A^1)$ $i, j = 1, \dots, n$ and defining

$$\nabla\left(\sum_{i=1}^{n} r_i e_i\right) = \sum_{i=1}^{n} dr_i \otimes e_i + \sum_{i=1}^{n} r_i \left(\sum_{j=1}^{n} a_{ij} \otimes e_j\right)$$

$(r_i \in R)$, gives a connection on P ([8]).

In the general case, we choose an R-module 0 such that $P \oplus Q$ is free of finite rank. Then, if ∇_1 is a connection on $P \oplus Q$, it is easy to see that $\nabla = (I \otimes \pi) \circ \nabla_1 \circ i$ is a connection on P; here $i : P \to P \oplus Q$ is the natural inclusion and $\pi : P \oplus Q \to P$, the natural projection.

(2) If $f \in L(P)$, then it is easy to check that $d_\nabla(f)_\varepsilon Hom_R(P, A^1 \otimes P)$. Now, if $a \in R, f \in L(P)$, then

$$d_\nabla(af) = \nabla \circ (af) - (I \otimes af) \circ \nabla = da \otimes f + ad_\nabla f,$$

where $da \otimes f$ is the map $x \mapsto da \otimes f(x)$.
This shows that d_∇ is a connection on $L(P)$.

5. **Proposition**. Let P be a finitely generated projective R-module. We have

(1) If ∇ is a connection on P, then K_∇ is an R-homomorphim. That is $K_\nabla \in Hom_R(P, A^2 \otimes P) \approx A^2 \otimes L(P)$.

(2) Bianchi's Identy: $d_\nabla(K_\nabla) = 0$ for any connection ∇ on P.

Proof. These are easy computations ([6]).

6. **Proposition**. Let ∇, ∇_1 be two connections on a f. g. projective R-module P. Let $f \in A^1 \otimes L(P)$ be such that $\nabla_1 = \nabla + f$. Then

(1) $d_{\nabla_1}(g) = d_\nabla(g) + [f,g] \qquad (g \in L(P))$
where $[f,g] = f \circ g - (I \otimes g) \circ f$

(2) $K_{\nabla_1} = K_\nabla + d_\nabla f - (I \otimes f) \circ f$

Proof.

(1) Let $x \in P$. We have

$$\begin{aligned} d_{\nabla_1}(g)(x) &= \nabla_1 g(x) - (I \otimes g)\nabla_1(x) \\ &= (\nabla + f)g(x) - (I \otimes g)(\nabla + f)(x) \\ &= \nabla g(x) - (I \otimes g)\nabla(x) + f \circ g(x) - (I \otimes g) \circ f(x) \\ &= d_\nabla(g)(x) + [f,g](x) \end{aligned}$$

(2) Let $x \in P$ and write $\nabla x = \Sigma a_i \otimes x_i$, $fx = \Sigma b_j \otimes y_j$ with $a_i, b_j \in A^1, x_i, y_j \in P$. Then we have

$$\begin{aligned} K_{\nabla_1}(x) &= \nabla_1^1 \nabla_1(x) = \nabla_1^1(\nabla x + fx) = \nabla_1^1\left(\Sigma a_i \otimes x_i + \Sigma b_j \otimes y_i\right) \\ &= \Sigma da_i \otimes x_i - \Sigma a_i \cdot \nabla_1(x_i) + \Sigma db_j \otimes y_j - \Sigma b_j \nabla_1(y_j) \\ &= (\Sigma da_i \otimes x_i - \Sigma a_i \nabla(x_i)) - \Sigma a_i f(x_i) \\ &\quad +(\Sigma db_j \otimes y_j - \Sigma b_j \nabla(y_j)) - \Sigma b_j f(y_j) \\ &= K_\nabla(x) - (I \otimes f)\nabla(x) + \nabla^1 f(x) - (I \otimes f) \circ f(x) \\ &= K_\nabla(x) + d_\nabla f(x) - (I \otimes f) \circ f(x) \end{aligned}$$

7. Let L be a Lie subalgebra of $L(P)$ under the usual bracket product

$$[f,g] = f \circ g - g \circ f \quad (f, g \in L(P))$$

We assume that P is a $f \cdot g$ projective module and that the canonical map $A^1 \otimes L \to A^1 \otimes L(P)$ is an injection.

Definition. Let $\nabla : P \to A^1 \otimes P$ be a connection on P. We shall say that ∇ is compatible with L in case $d_\nabla : L(P) \to A^1 \otimes L(P)$ maps L into $A^1 \otimes L$.

Thus, d_∇ is a connection on L.

If ∇ is compatible with L and if $f \in A^1 \otimes L$, then it is clear that $\nabla + f$ is also compatible with L.

Two connections ∇, ∇_1 compatible with L are said to be equivalent if $\nabla_1 - \nabla \in A^1 \otimes L$.

8. Let $d : M \to A^1 \otimes M$ be an integrable connection on an R-module M. Let ∇ be a connection compatible with L.

Definition. An n-invariant map for (L, V) with values in M is a map $\Phi : L \times \ldots \times L \to M$ such that

(1) Φ is n-multilinear

(2) For all $f_1, \ldots, f_n, g \in L$ it holds

$$\sum_{i=1}^{n} \Phi(f_1, \ldots, [g, f_i], \ldots, f_n) = 0$$

(3) For all $f_1, \ldots, f_n \in L$ it holds

$$d\Phi(f_1, \ldots, f_n) = \sum_{i=1}^{n} \Phi(f_1, \ldots, d_\nabla f_i, \ldots, f_n)$$

Proposition. Suppose that Φ is an n-invariant map for (L, ∇). Then, for all ∇_1 equivalent with ∇, Φ is invariant for (L, ∇_1)

Proof.

First we shall show that

$$\sum_{i=1}^{n} \Phi(f_1, \ldots, [f, f_i], \ldots, f_n)' = 0 \qquad (*)$$

holds for all $f_1, \ldots, f_n \in L(P), f \in A^1 \otimes L(P)$

By linearity we can assume $f = a \otimes g$, $a \in A^1$, $g \in L(P)$. But then $[f, f_i] = a \otimes [g, f_i]$, so that (*) follows from (2). Now, if ∇_1 is equivalent with $\nabla_1 = \nabla + f$, with $f \in A^1 \otimes L$. By prop. 6(1) we have

$$d_{\nabla_1} f_i = d_\nabla f_i + [f, f_i], i = 1, 2, \ldots, n.$$

Hence

$$\begin{aligned} \sum_{i=1}^{n} \Phi(f_1, \ldots, d_{\nabla_1} f_i, \ldots, f_n) &= \sum_{i=1}^{n} \Phi(f_1, \ldots, d_\nabla f_i, \ldots, f_n) \\ &+ \sum_{i=1}^{n} \Phi(f_1, \ldots, [f, f_i], \ldots, f_n) \\ &= d\Phi(f_1, \ldots, f_n) \end{aligned}$$

by using (1) and (*)

9. **Proposition**. *Let Φ be an n-invariant map for (L, ∇). Let $f_i \in A^{d_i} \otimes L$, $i = 1, \ldots, n$. Then* ([2])

$$d\Phi(f_1, \ldots f_n) = \sum_{i=1}^{n} (-1)^{d_1 + d_2 + \ldots + d_{i-1}} \quad \Phi(f_1, \ldots, d_\nabla f_i, \ldots, f_n).$$

Proof.

By additivity we may assume $f_i = a_i \otimes g_i$, $a_i \in A^{d_i}$, $g_i \in L$, $i = 1, \ldots, n$. Then we have

$$\begin{aligned} d\Phi(f_1, \ldots, f_n) &= \sum_{i=1}^{n} (-1)^{d_1 + \ldots + d_{i-1}} a_1 a_2 \ldots da_i \ldots a_n \otimes \Phi(g_1, \ldots, g_n) \\ &\quad + (-1)^{d_1 + \ldots + d_n} a_1 a_2 \ldots a_n \cdot d\Phi(g_1, \ldots, g_n). \end{aligned}$$

Substituting $d\Phi(g_1,\ldots,g_n) = \sum_{i=1}^{n} \Phi(g_1,\ldots,d_\nabla g_i,\ldots,g_n)$ and using

$$
\begin{aligned}
&a_1 a_2 \ldots da_i \ldots a_n \otimes \Phi(g_1,\ldots,g_n) + (-1)^{d_i+\ldots+d_n}\, a_1\, a_2 \ldots a_n \cdot \\
&\Phi(g_1,\ldots,d_\nabla g_i,\ldots,g_n) \\
&= \Phi(a_1 \otimes g_1,\ldots,da_i \otimes g_i,\ldots,a_n \otimes g_n) \\
&+\Phi(a_1 \otimes g_1,\ldots,(-1)^{d_i}\, a_i \cdot d_\nabla g_i,\ldots,a_n \otimes g_n) \\
&= \Phi(f_1,\ldots,d_\nabla f_i,\ldots,f_n),
\end{aligned}
$$

we obtain the required expression.

10. **Corollary**. *If Φ is an n-invariant map for (L,∇), then*

$$\Phi(K_\nabla,\ldots,K_\nabla) \in A^{2n} \otimes M$$

is a cocycle.

Proof. This follows immediately from the preceding proposition and Bianchi's identity.

11. **Definition**. If Φ is an n-invariant map for (L,∇), we define the characteristic class of Φ by

$$\Phi(L,\nabla,A) = \{\Phi(K_\nabla,\ldots,K_\nabla)\} \in H^{2n}(M \otimes A),$$

here $\{\eta\}$ denotes the cohomology class of a cocycle η

12. **Theorem of Invariance**
Let Φ be an n-invariant map for (L,∇). Then for any connection ∇_1 equivalent with ∇ we have

$$\{\Phi(K_{\nabla_1},\ldots,K_{\nabla_1})\} = \{\Phi(K_\nabla,\ldots,K_\nabla)\} \textit{ in } H^{2n}(A \otimes M).$$

Proof.

It is clear that we may assume $\nabla_1 = \nabla + f$, with $f = a \otimes g,\ a \in A^1, g \in L$. Then, $a^2 = 0$ implies

$$(I \otimes f) \circ f = 0$$

and so prop. 6. (2) gives

$$K_{\nabla_1} = K_\nabla + d_\nabla f$$

Thus $\Phi(K_{\nabla_1}, \ldots, K_{\nabla_1}) = \Sigma\Phi(\alpha_1, \ldots, \alpha_n)$ where $\alpha_j = K_\nabla$ or $d_\nabla f$.

We contend that if $\alpha_j = d_\nabla f$ for some j, then

$$\Phi(\alpha_1, \ldots, \alpha_n) = dt,\ t \in A^{2n-1} \otimes M.$$

In fact, replacing α_j by f we have $\Phi(\alpha_1, \ldots, f, \ldots, \alpha_n) \in A^{2n-1} \otimes M$ and

$$\begin{aligned} d\Phi(\alpha_1, \ldots, f, \ldots, \alpha_n) &= \sum_{i<j} \pm\Phi(\alpha_1, \ldots, d_\nabla\alpha_i, \ldots, f, \ldots, \alpha_n) \\ \pm\Phi(\alpha_1, \ldots, d_\nabla f, \ldots, \alpha_n) &+ \sum_{j<i} \pm\Phi(\alpha_1, \ldots, f, \ldots, d_\nabla\alpha_i, \ldots, \alpha_n) \end{aligned}$$

Next we observe that each term in both sums is zero. For if $\alpha_i = K_\nabla$, then $d_\nabla\alpha_i = 0$ by Bianchi's identity. And, if $\alpha_i = d_\nabla f$, then $d_\nabla d_\nabla f = a \cdot d_\nabla d_\nabla g$ and $f = a \otimes g$ are both factors of the term in question. Since $a^2 = 0$, we also obtain zero. Thus $d\Phi(\alpha_1, \ldots, f, \ldots, \alpha_n) = \pm\Phi(\alpha_1, \ldots, \alpha_j, \ldots \alpha_n)$ if $\alpha_j = d_\nabla f$.

Therefore we have

$$\Phi(K_{\nabla_1}, \ldots, K_{\nabla_1}) = \Phi(K_\nabla, \ldots, K_\nabla) + dt, \quad t \in A^{n-1} \otimes M.$$

This proves the theorem.

13. Let $u : A \to B$ be a DG-algebra homomorphism. Write $R = A^\circ, S = B^\circ$. We shall regard any S-module as an R-module via the map u. Consider two modules M and N over R and S, respectively, with integrable connections. Suppose $v : M \to N$ is an R-homomorphism which commutes with connections, that is, we require the following square

$$\begin{array}{ccc} M & \xrightarrow{d} & A^1 \otimes_R M \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle u\otimes v} \\ N & \xrightarrow{d} & B^1 \otimes_S N \end{array}$$

to be commutative.

It is clear that $u \otimes v : A \otimes M \to B \otimes N$ induces a homomorphism $(u \otimes v)_*$ in cohomology $H^*(A \otimes M) \to H^*(B \otimes N)$

Now, let Φ be an invariant map for (L, V) as in 8, where $L \subset L(P)$, P is a f.g. projective module. Define

(1) $P_u = S \otimes_R P$,

(2) $L_u = S \otimes_R L \subset L(P_u)$ $(*)$

(3) $\nabla_u : P_u \to B' \otimes P_u$ by

$$\nabla_u(s \otimes x) = ds \otimes x + s \cdot \overline{\nabla}(x) \qquad (s \in S, x \in P)$$

where $\overline{\nabla}(x)$ is the image of $\nabla(x)$ under the canonical map $A' \otimes P \to B' \otimes P_u$.

(4) Φ_u is the obvious extension of Φ to L_u with values in $\mathbb{N}$

$(*)$ Here we assume that the map $S \otimes_R L \to S \otimes_R L(P)$ is an injection.

Then ∇_u is a connection on P_u compatible with L_u and Φ_u is an invariant map for (L_u, ∇_u). Moreover $(u \otimes i)(K_\nabla) = K_{\nabla_u}$, where $i : L \to L_u$ is the canonical map and from this it follows that

$$(u \otimes v)_*\{\Phi(L, \nabla, A)\} = \{\Phi_u(L_u, \nabla_u, B)\} \in H^*(B \otimes N)$$

Since for any DG-algebra A with $A^\circ = R$ we have a natural DG-algebra homomorphism $\eta : \Omega^* \to A$, where Ω^*, is the de Rham complex of Kähler differentials of R/K, and for any connection $\nabla : P \to \Omega^1 \otimes P$ compatible with L, we obtain

$$(\eta \otimes I)_*\{\Phi(L, \nabla)\} = \{\Phi, (L, \nabla_\eta A)\}$$

where $\Phi(L, \nabla) = \Phi(L, \nabla, \Omega^*)$.

2 Chern Classes of Projective Modules

1. Let A be a DG-algebra over K as in section 1,1. Let P be a f. g. projective R-module and $d : M \to A' \otimes M$ an integrable connection on M. We take $L = L(P)$. Then all connections on P are equivalent. Therefore, for any invariant map $\Phi : L(P) \times \ldots \times L(P) \to M$, there exists a uniquely determined cohomology class

$$\{\Phi(K_\nabla, \ldots, K_\nabla)\} \in H^*(A \otimes M),$$

which is independent of the choice of ∇.

2. Following [9] we define an n-invariant map
$P_n : L(P) \times \ldots \times L(P) \to R$ by
$P_n(f_1, \ldots, f_n; P) =$ Trace of f
where f is the linear map: $\bigwedge^n P \to \bigwedge^n P$ induced by

$$\Sigma f_{\sigma 1} \otimes \ldots \otimes f_{\sigma n} : \bigotimes^n P \to \bigotimes^n P,$$

where σ runs through all permutations of $\{1, 2, \ldots, n\}$.

In what follows in this section we take $M = R$ and the canonical map $d : R \to A' \otimes R$ as an integrable connection on R.

In [9] can be found the proof of the following

Proposition

(1) P_n is invariant for $(L(P), \nabla)$, for all connections ∇ on P.

(2) P_n is symmetric: $P_n(f_{\sigma 1}, \ldots, f_{\sigma n}) = P_n(f_1, \ldots, f_n)$ for all $f_1, \ldots, f_n \in L(P)$, and permutations σ.

(3) If P and Q are f.g. projective R-modules and $f_1, \ldots, f_m \in L(P)$, $g_1, \ldots, g_n \in L(Q)$, then
$P_{m+n}(f_1, \ldots, f_m, g_1, \ldots, g_n; P \oplus Q) = P_m(f_1, \ldots, f_m) \cdot P_m(g_1, \ldots, g_n)$

3. **Definition.** For any f.g projective R-module and any DG-algebra A over K, we define for $n \geq 1$,

$$\begin{aligned} ch_n(P, A) &= \text{the } n \text{ th-Chern class of } P \text{ with values in } A \\ &= \{P_n(K_\nabla, \ldots, K_\nabla)\} \in H^{2n}(A) \end{aligned}$$

For $n = 0$, we set $ch_0(P, A) = 1$

We shall write $ch_n(P) = ch_n(P, \Omega^*)$, where Ω^* is the Rham complex of Kähler differentials of R/K.

4. **Theorem.**

(1) If $u : A \to B$ is a DG-algebra homomorphism then

$$u_* ch_n(P, A) = ch_n(P_u, B)$$

In particular, for any DG-algebra A with $R = A^o$, we have

$$\eta_* ch_n(P) = ch_n(P, A)$$

where η is the natural DG-algebra homomorphism $\Omega^* \xrightarrow{\eta} A$

(2) Let P y Q be f.g projective R-modules. Then

$$ch_n(P \oplus Q, A) = \sum_{k=0}^{n} \binom{n}{k} ch_k(P, A) \cdot ch_{n-k}(Q, A)$$

(3) If P is a free module then $ch_n(P, A) = 0$, for all $n \geq 1$

(4) If $P \simeq Q$ then $ch_n(P, A) = ch_n(Q, A)$, for all $n \geq 0$.

Proof.

(1) This follows from Section 1, 13.

(2) Choosing connections ∇_1, ∇_2 on P and Q, respectively, we see that

$$\nabla : P \oplus Q \to A' \otimes (P \oplus Q)$$

defined by $\nabla(x + y) = \nabla_1(x) + \nabla_2(x), (x \in P, y \in Q)$ is a connection on $P \oplus Q$. This has $K_\nabla = K_{\nabla_1} \oplus K_{\nabla_2}$, and so using (2) and (3) of the preceding proposition we obtain

$$P_n(K_\nabla, \ldots, K_\nabla; P \oplus Q)$$

$$= \sum_{k=0}^{n} \binom{n}{k} P_k(K_{\nabla_1}, \ldots, K_{\nabla_1}; P) \cdot P_{n-k}(K_{\nabla_2}, \ldots, K_{\nabla_2}; Q),$$

which shows (2)

(3) If P is free we can choose the trivial connection with respect to a basis $(e_1, \ldots, e_n)$

$$\begin{array}{rcl} \nabla : P & \to & A' \otimes P \\ \nabla\left(\sum_{i=1}^{n} r_i e_i\right) & = & \sum_{i=1}^{n} dr_1 \otimes e_i \qquad (r_i \in R) \end{array}$$

This has $K_\nabla = 0$, and hence $P_n(K_\nabla, \ldots, K_\nabla, P) = 0$,

(4) We proceed by induction on n. Choose an R-module T so that $P \oplus T$ is free. Therefore $Q \oplus T$ is also free and by (3) and (2) we have

$$0 = \sum_{k=0}^{n} \binom{n}{k} ch_k(P) \cdot ch_{n-k}(T) = \sum_{k=0}^{n} \binom{n}{k} ch_k(0) \cdot ch_{n-k}(T)$$

and so, assuming $ch_k(P) = ch_k(Q)$ for all $k < n$, gives

$$ch_n(P) = ch_n(Q).$$

5. **Remark**
If we know $n!$ is a unit in K, then we can define $Ch_n(P, A) = \frac{1}{n!} ch_n(P, A)$.
Then formula (2) becomes $Ch_n(P \oplus Q, A) = \sum_{k=0}^{n} Ch_k(P, A) \cdot Ch_{n-k}(Q, A)$.

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Resumen

En el presente trabajo se desarrolla una teoría general de clases caracteríticas de módulos. A una aplicación invariante definida en un Algebra de Lie se asocia una clase de cohomología usando la forma de curvatura de un cierto tipo de conexiones. Se presenta una prueba muy simple del teorema de invariancia (Teorema 12), que establece que conexiones equivalentes dan lugar a la misma clase característica.

Se usan estas aplicaciones invariantes [9] para definir las clases de Chern de módulos proyectivos y se derivan sus propiedades básicas.

Es interesante observar que esta teoría puede ser aplicada para definir clases características de aplicaciones bilineales. En particular, las clases de Euler [6] pueden ser obtenidas de esta manera.

Palabras Clave: Algebra de Lie, módulos proyectivos, clases de Chern, clases de Euler, cohomología, formas de curvatura, conexiones, aplicaciones invariantes.

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