# INITIAL VALUE PROBLEM FOR A COUPLED SYSTEM OF KADOMTSEV-PETVIASHVILI II EQUATIONS IN SOBOLEV SPACES OF NEGATIVE INDICES

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### *Abstract*

*It is proved that the initial value problem for a system of two Kadomtsev-Petviashvili II (KP-II) equations coupled through both dispersive and nonlinear terms is locally well posed in the anisotropic Soboloev spaces* $H^{s_1,s_2}(\mathbb{R}^2) \times H^{s_1,s_2}(\mathbb{R}^2)$ *with* $s_1 > -\frac{1}{3}$ *and* $s_2 \geq 0$*, and globally well posed in* $H^{s_1,0}(\mathbb{R}^2) \times H^{s_1,0}(\mathbb{R}^2)$ *with* $s_1 > -\frac{1}{14}$.


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# 1 Introduction

In this article we consider initial value problem (IVP) for a system of two Kadomtsev-Petviashvili II (KP-II) equations coupled through both dispersive and nonlinear terms. More precisely, the IVP for the system is

$$\begin{cases} \partial_t u + a_1\partial_x^3 u + a_2\partial_x^3 v + b_1\partial_x(uv) + b_2 u\partial_x u + b_3 v\partial_x v + \partial_x^{-1}\partial_y^2 u = 0 \\ \partial_t v + a_3\partial_x^3 u + a_4\partial_x^3 v + b_4\partial_x(uv) + b_5 u\partial_x u + b_6 v\partial_x v + \partial_x^{-1}\partial_y^2 v = 0 \\ u(x,y,0) = u_0(x,y) \\ v(x,y,0) = v_0(x,y) \end{cases} \tag{1.1}$$

where $u = u(x,y,t)$ and $v = v(x,y,t)$ are the unknown functions, $(x,y) \in \mathbb{R}^2$ and $t \in \mathbb{R}$, while $u_0$ and $v_0$ are a given functions. $a_1$, $a_2$, $a_3$, $a_3$, $b_1$, $b_2$, $b_3$, $b_4$, $b_5$ and $b_6$ are real constants with $a_1a_4 - a_2a_3 > 0$ and $a_2a_3 > 0$. System in (1.1) was derived by Grimshaw and Zhu [5] in 1994, as a model to describe the oblique strong interaction of weakly, two dimensional, nonlinear, long internal gravity waves in shallow fluids.

Observe that if we consider the system in (1.1) with $a_2 = a_3 = b_1 = b_3 = b_4 = b_5 = 0$ then we obtain the scalar KP II equation. The KP II appear in mathematical models for the description of long dispersive waves which travel essentially in one direction but have small transverse effects. KP II arises as a universal model in wave propagation and may be viewed as a bidimensional generalizations of the Korteweg-de Vries equation. In that case, local and global well posedness have been intensively studied by several authors in recent year, standing out [2], [6], [12], [10], [13], [11], [7] and [8].

In [1] Bourgain developed a method for study of the IVP for the KP II and other nonlinear evolution equations, in wich the essential part consist of the adequate election of functional spaces whose norms are defined by the Fourier transform in the space-time variables and involve the specific structure of the linear part of the equation. Using this method Bourgain in [2] proved that problem (1.1) is locally well posed for the initial data $u_0$ in the Sobolev spaces $H^s\left(\mathbb{R}^2\right)$ with $s \geq 0$.

Following the ideas developed in [7], in this article we prove that IVP (1.1) is locally well posed in the spaces $H^{s_1,s_2}\left(\mathbb{R}^2\right) \times H^{s_1,s_2}\left(\mathbb{R}^2\right)$ with $s_1 > -\frac{1}{3}$ and $s_2 \geq 0$.

The second objective of this paper is to prove global well posed in Sobolev spaces. Our procedure follows the ideas in [3] and [8], to extend the local result mentioned above to $H^{s_1,0}\left(\mathbb{R}^2\right) \times H^{s_1,0}\left(\mathbb{R}^2\right)$ with $s_1 > -\frac{1}{14}$. The solution in any time interval $[0,T]$ is obtained from the local solutions by means of an iterative process in a finite number of steps.

# 2 Transformation of the System and Main Results

In this section we decoupled the dispersive terms in the system (1.1). Let

$$\overrightarrow{u} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$$

and

$$f(u,v) = \begin{pmatrix} b_1 \partial_x (uv) + b_2 u \partial_x u + b_3 v \partial_x v \\ b_4 \partial_x (uv) + b_5 u \partial_x u + b_6 v \partial_x v \end{pmatrix}.$$

Therefore the system (1.1) can be written as

$$\partial_t \overrightarrow{u} + A \partial_x^3 \overrightarrow{u} + F(u,v) + \partial_x^{-1} \partial_y^2 \overrightarrow{u} = 0. \tag{2.2}$$

Since $a_2 a_3 > 0$, there exists $T = 2\begin{pmatrix} \alpha_+ & \alpha_- \\ a_3 & a_3 \end{pmatrix} \in \mathrm{GL}(2)$ such that $T^{-1}AT = \operatorname{diag}(\alpha_+, \alpha_-)$, where $\alpha_\pm = \frac{a_1 + a_4 \pm \sqrt{(a_1 - a_4)^2 + 4a_2 a_3}}{2}$ are the eigenvalues of $A$. We have that $\alpha_+ > \alpha_- > 0$ because $a_1 a_4 - a_2 a_3 > 0$ and $a_2 a_3 > 0$. Now setting $\overrightarrow{u} = T\overrightarrow{v}$ where $\overrightarrow{v} = (\overline{u}, \overline{v})$, we arrive at the system

$$\partial_t \overrightarrow{v} + T^{-1}AT\partial_x^3 \overrightarrow{v} + F\left(\overrightarrow{v}\right) + \partial_x^{-1}\partial_y^2 \overrightarrow{v} = 0$$

with

$$F\left(\overrightarrow{v}\right) = T^{-1}f\left(T\overrightarrow{v}\right) = \begin{pmatrix} \overline{b}_1\partial_x\left(\overline{u}\overline{v}\right) + \overline{b}_2\overline{u}\partial_x\overline{u} + \overline{b}_3\overline{v}\partial_x\overline{v} \\ \overline{b}_4\partial_x\left(\overline{u}\overline{v}\right) + \overline{b}_5\overline{u}\partial_x\overline{u} + \overline{b}_6\overline{v}\partial_x\overline{v} \end{pmatrix}.$$

and some constants $\overline{b}_1, \ldots, \overline{b}_6$. Therefore, the system (1.1) transforms in

$$\begin{cases} \partial_t\overline{u} + \alpha_+\partial_x^3\overline{u} + \overline{b}_1\partial_x\left(\overline{u}\overline{v}\right) + \overline{b}_2\overline{u}\partial_x\overline{u} + \overline{b}_3\overline{v}\partial_x\overline{v} + \partial_x^{-1}\partial_y^2\overline{u} = 0 \\ \partial_t\overline{v} + \alpha^-\partial_x^3\overline{v} + \overline{b}_4\partial_x\left(\overline{u}\overline{v}\right) + \overline{b}_5\overline{u}\partial_x\overline{u} + \overline{b}_6\overline{v}\partial_x\overline{v} + \partial_x^{-1}\partial_y^2\overline{v} = 0. \end{cases} \tag{2.3}$$

Now we make the change of scale

$$\overline{u}\left(x,y,t\right) = \widetilde{u}\left(\alpha_+^{-1/3}x, \alpha_+^{-1/3}y, t\right) \quad \text{and} \quad \overline{v}\left(x,y,t\right) = \widetilde{v}\left(\alpha_-^{-1/3}x, \alpha_+^{-1/3}y, t\right),$$

then we obtain the IVP

$$\begin{cases} \partial_t\widetilde{u} + \partial_x^3\widetilde{u} + \widetilde{b}_1\partial_x\left(\widetilde{u}\widetilde{v}\right) + \widetilde{b}_2\widetilde{u}\partial_x\widetilde{u} + \widetilde{b}_3\widetilde{v}\partial_x\widetilde{v} + \partial_x^{-1}\partial_y^2\widetilde{u} = 0 \\ \partial_t\widetilde{v} + \partial_x^3\widetilde{v} + \widetilde{b}_4\partial_x\left(\widetilde{u}\widetilde{v}\right) + \widetilde{b}_5\widetilde{u}\partial_x\widetilde{u} + \widetilde{b}_6\widetilde{v}\partial_x\widetilde{v} + \partial_x^{-1}\partial_y^2\widetilde{v} = 0 \\ \widetilde{u}\left(x,y,0\right) = \widetilde{u_0}\left(x,y\right) \\ \widetilde{v}\left(x,y,0\right) = \widetilde{v_0}\left(x,y\right), \end{cases} \tag{2.4}$$

where $\widetilde{b}_1$, $\widetilde{b}_2$, $\widetilde{b}_3$, $\widetilde{b}_4$, $\widetilde{b}_5$ and $\widetilde{b}_6$ are constants.

Note that (2.4) has a structure of two coupled KP II equations only in the nonlinear terms. Since that the IVP (1.1) is equivalent to the IVP (2.4), of the results of well-posedness for (2.4) is easy to obtain the corresponding results for (1.1). For the sake of simplicity, from now onwards we will drop " $\widetilde{\cdot}$ " and use the notation $u$, $v$, $u_0$ and $v_0$ in the system (2.4).

In order to state our theorems in a precise way, we give some definitions and introduce the necessary notation. Our initial data will be in the anisotropic Sobolev space of $L^2$-type $H^{s_1,s_2} := H^{s_1,s_2}\left(\mathbb{R}^2\right)$ defined for $s_1, s_2 \in \mathbb{R}$ as the space of tempered distributions in $\mathbb{R}^2$ with norm

$$\|u\|_{H^{s_1,s_2}} := \left(\int_{\mathbb{R}^2} \langle\xi\rangle^{2s_1} \langle\theta\rangle^{2s_2} \left|\widehat{u}\,(\zeta)\right|^2 d\zeta\right)^{1/2},$$

where $\widehat{\cdot}$ is the Fourier transform in the space variables, $\zeta = (\xi, \theta)$ is the variable in the frecuency space, with $\xi$ and $\theta$ corresponding to the space variables $x$ and $y$, respectively, and $\langle\cdot\rangle = 1 + |\cdot|$.

For $s_1, s_2, b \in \mathbb{R}$, $\sigma > 0$ let us consider the space $X_{s_1,s_2;b,\sigma}$ of tempered distributions in $\mathbb{R}^3$ such that

$$\|u\|_{X_{s_1,s_2;b,\sigma}} := \left(\int_{\mathbb{R}^2} \langle\lambda\rangle^{2b} \langle\eta\rangle^{2\sigma} \langle\xi\rangle^{2s_1} \langle\theta\rangle^{2s_2} \left|\widehat{u}\,(\omega)\right|^2 d\omega\right)^{1/2},$$

where $\widehat{\cdot}$ is the Fourier transform in the space-time variables, $\omega = (\zeta, \tau)$ is the variable in the space with $\xi$ and $\theta$ as before, and $\tau$ corresponding to the time variable $t$, $\lambda = \lambda\,(\omega) := \tau - \xi^3 + \frac{\theta^2}{\xi}$ and $\eta = \eta\,(\omega) := \frac{\lambda(\omega)}{1+|\xi|^3}$.

If $b > \frac{1}{2}$ then $X_{s_1,s_2;b,\sigma}$ is continuously embedded in $C_b\left(\mathbb{R}_t, H^{s_1,s_2}\right)$, the space of continuous bounded functions from the variable $t \in \mathbb{R}$ to $H^{s_1,s_2}$. If $\mathcal{S}$ is the Schwartz's space, then $\mathcal{S} \cap X_{s_1,s_2;b,\sigma}$ is dense in $X_{s_1,s_2;b,\sigma}$.

For $T > 0$ and $b > \frac{1}{2}$, let $X^T_{s_1,s_2;b,\sigma}$ be the set of the restrictions to $[0, T]$ of the functions in $X_{s_1,s_2;b,\sigma}$, with norm defined by

$$\|u\|_{X^T_{s_1,s_2;b,\sigma}} = \inf\left\{\|u^e\|_{X_{s_1,s_2;b,\sigma}} : u^e \in X_{s_1,s_2;b,\sigma} \text{ and } \left.u^e\right|_{[0,T]} = u\right\}.$$

Our concept of solution comes from Duhamel's formula for the IVP (2.4). Formally, $(u, v)$ is a solution of the IVP (2.4) if and only if

$$u\,(t) = W\,(t)\,u_0 - \int_0^t W\,(t - t')\,F\,(u, v, \partial_x u, \partial_x v)\;dt'$$

and

$$v(t) = W(t)\, v_0 - \int_0^t W(t-t')\, G(u, v, \partial_x u, \partial_x v)\; dt',$$

where $\{W(t)\}_{t\in\mathbb{R}}$ is the unitary group on $H^{s_1,s_2}$ associated with the linear problem

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x^{-1}\partial_y^2 u = 0 \\ u(x,y,0) = u_0(x,y), \end{cases}$$

defined by

$$\widehat{W(t)\, u_0}(\xi,\theta) = e^{it\left(i\xi^3 - \frac{\theta^2}{\xi}\right)}\, \widehat{u_0}(\xi,\theta),$$

and

$$F(u, v, \partial_x u, \partial_x v) = b_1 \partial_x (uv) + \frac{b_2}{2}\partial_x u^2 + \frac{b_3}{2}\partial_x v^2$$

$$G(u, v, \partial_x u, \partial_x v) = b_4 \partial_x (uv) + \frac{b_5}{2}\partial_x u^2 + \frac{b_6}{2}\partial_x v^2$$

are respective nonlinearities. More precisely, we have the following definition.

**Definition 2.1.** *We say that $(u,v) \in X^T_{s_1,s_2;b,\sigma} \times X^T_{s_1,s_2;b,\sigma}$ is a solution of the IVP (2.4) in the interval $[0,T]$ if there exists extensions $u^e, v^e \in X_{s_1,s_2;b,\sigma}$ such that*

$$\begin{cases} u(t) = W(t)\, u_0 - \displaystyle\int_0^t W(t-t')\, F(u^e, v^e, \partial_x u^e, \partial_x v^e)\; dt' \\[2ex] v(t) = W(t)\, v_0 - \displaystyle\int_0^t W(t-t')\, G(u^e, v^e, \partial_x u^e, \partial_x v^e)\; dt' \end{cases}$$

*for all $t \in [0,T]$.*

Now we are the position to state the main results of this paper. The first result is concerned about the local well-posedness for the IVP (2.4).

**Theorem 2.2.** *Let $s_1 \in \left]-\frac{1}{3}, 0\right[$, $s_2 \geq 0$ and $(u_0, v_0) \in H^{s_1,s_2} \times H^{s_1,s_2}$. If $b > \frac{1}{2}$ and $\sigma > \frac{1}{6}$ satisfy the hypothesis $b + \sigma \leq 1 + s_1$, $b \leq \frac{1}{3}(2 + s_1)$ and $\sigma \leq \frac{1}{3}(1 + s_1)$, then there exists $T = T\left(\|u_0\|_{H^{s_1,s_2}}, \|v_0\|_{H^{s_1,s_2}}\right)$ and a unique solution $(u, v) \in X^T_{s_1,s_2;b,\sigma} \times X^T_{s_1,s_2;b,\sigma}$ of the IVP (2.4) in the interval $[0, T]$ satisfaying*

$$u, v \in X^T_{s_1,s_2;b,\sigma} \subset C\left([0, T], H^{s_1,s_2}\right).$$

*Furthermore, if $\mathcal{B}_R$ is the open ball of radius $R$ centered at $(0, 0)$ in $H^{s_1,s_2} \times H^{s_1,s_2}$, given $T' \in ]0, T[$, the mapping*

$$\Theta : (u_0, v_0) \in \mathcal{B}_R \mapsto (u, v) \in C\left([0, T], H^{s_1,s_2}\right) \times C\left([0, T], H^{s_1,s_2}\right)$$

*is analytic.*

The next theorem deals with the global well-posedness for the IVP (2.4).

**Theorem 2.3.** *For $s_1 \in \left]-\frac{1}{14}, 0\right[$, $T > 0$ and $(u_0, v_0) \in H^{s_1,0} \times H^{s_1,0}$ con $\left(\partial_x^{-1} u_0, \partial_x^{-1} v_0\right) \in \mathcal{S}' \times \mathcal{S}'$, there exists $N > 0$ such that IVP (2.4) has a solution $(u, v)$ in $X^T_{s_1,0;b,\sigma} \times X^T_{s_1,0;b,\sigma}$.*

# 3 Proof of Theorem 2.2

In this section we prove the theorem 2.2, the local well-posedness result for the IVP (2.4). As the methods of proof used here are all well known, we only give a sketch of proof.

To find a local solution to (2.4) let's consider (for suitable $u$ and $v$) the equivalent system of integral equations,

$$\begin{cases} u(t) = \psi_1(t) W(t) u_0 - \psi_1(t) \displaystyle\int_0^t W(t - t') F(u, v, \partial_x u, \partial_x v)\, dt' \\ \\ v(t) = \psi_1(t) W(t) v_0 - \psi_1(t) \displaystyle\int_0^t W(t - t') G(u, v, \partial_x u, \partial_x v)\, dt' \end{cases} \tag{3.5}$$

where $\psi \in C_0^\infty(\mathbb{R})$, $0 \leq \psi \leq 1$, is a cut-off function given by

$$\psi(t) = \begin{cases} 1 & \text{si } |t| \leq 1 \\ 0 & \text{si } |t| \geq 2 \end{cases}$$

and $\psi_\rho(\cdot) = \psi\left(\frac{\cdot}{\rho}\right)$ for $0 < \rho \leq 1$.

Now, we enunciate some estimates that will be used to prove the local well-posedness result.

**Proposition 3.1.** *If $s_1, s_2 \in \mathbb{R}$, $b > \frac{1}{2}$, $\sigma \geq 0$ and $\rho \in ]0,1]$, there exist a constant $C = C(\rho)$ such that*

$$\|\psi_\rho(\cdot)\, W(\cdot)\, u_0\|_{X_{s_1,s_2;b,\sigma}} \leq C\, \|u_0\|_{H^{s_1,s_2}} \tag{3.6}$$

*and, if we define*

$$P_\rho F(t) := \psi_\rho(t) \int_0^t W(t-t')\, F(t')\, dt'$$

*for $F \in X_{s_1,s_2;b-1,\sigma}$, there is a constant $C = C(\rho)$ such that*

$$\|P_\rho F\|_{X_{s_1,s_2;b,\sigma}} \leq C\, \|F\|_{X_{s_1,s_2;b-1,\sigma}}\,. \tag{3.7}$$

*Proof.* For (3.6) and the case $\sigma = 0$ in (3.7), see Ginibre [4]. For case $\sigma \neq 0$ consider the operator $K_\sigma$ defined for $u \in X_{s_1,s_2;b,\sigma}$ by $\widehat{K_\sigma u}(\omega) = \langle\eta\rangle^\sigma\, \widehat{u}(\omega)$. Then $K_\sigma : X_{s_1,s_2;b,\sigma} \to X_{s_1,s_2;b,0}$ is an isometric isomorphism, indeed

$$\begin{aligned} \|K_\sigma u\|_{X_{s_1,s_2;b,0}} &= \left(\int_{\mathbb{R}^3} \langle\lambda\rangle^{2b} \langle\xi\rangle^{2s_1} \langle\theta\rangle^{2s_2} \left|\langle\eta\rangle^\sigma\, \widehat{u}(\omega)\right|^2 d\omega\right)^{1/2} \\ &= \left(\int_{\mathbb{R}^3} \langle\lambda\rangle^{2b} \langle\eta\rangle^{2\sigma} \langle\xi\rangle^{2s_1} \langle\theta\rangle^{2s_2} \left|\widehat{u}(\omega)\right|^2 d\omega\right)^{1/2} \\ &= \|u\|_{X_{s_1,s_2;b,\sigma}}\,. \end{aligned}$$

Therefore we have

$$\begin{aligned}
\|P_\rho F\|_{X_{s_1,s_2;b,\sigma}} &= \left\|\psi_\rho(\cdot)\int_0^t W(\cdot - t')\, F(t')\, dt'\right\|_{X_{s_1,s_2;b,\sigma}} \\
&= \left\|K_\sigma\psi_\rho(\cdot)\int_0^t W(\cdot - t')\, F(t')\, dt'\right\|_{X_{s_1,s_2;b,0}} \\
&= \left\|\psi_\rho(\cdot)\int_0^t W(\cdot - t')\, K_\sigma F(t')\, dt'\right\|_{X_{s_1,s_2;b,0}}
\end{aligned}$$

and by (3.7) in the case $\sigma = 0$

$$\|P_\rho F\|_{X_{s_1,s_2;b,\sigma}} \leq C\, \|K_\sigma F\|_{X_{s_1,s_2;b-1,0}} = C\, \|F\|_{X_{s_1,s_2;b-1,\sigma}},$$

this way, the estimate (3.7) when $\sigma \neq 0$ it's demonstrated. □

In the study of the nonlinear part of the equation, the bilinear form $\partial_x(uv)$ will play a critical role to prove theorem 2.2. More precisely, we have the following result.

**Proposition 3.2.** *Let $s_1 \in \left]-\frac{1}{3}, 0\right[$ and $s_2 \geq 0$. For $b > \frac{1}{2}$ and $\sigma > \frac{1}{6}$ such that $b+\sigma \leq 1+s_1$, $b \leq \frac{1}{3}(2+s_1)$ and $\sigma \leq \frac{1}{3}(1+s_1)$, the following bilinear estimate holds,*

$$\|\partial_x(uv)\|_{X_{s_1,s_2;b-1,\sigma}} \leq C\, \|u\|_{X_{s_1,s_2;b,\sigma}}\, \|v\|_{X_{s_1,s_2;b,\sigma}} \tag{3.8}$$

*for all $u, v \in X_{s_1,s_2;b,\sigma}$.*

The proof can be found in Isaza and Mejía [7], so we skip the details.

### *Proof of Theorem 2.2*

We consider the following function space where we seek a solution to the IVP (2.4). For given $(u_0, v_0) \in H^{s_1,s_2} \times H^{s_1,s_2}$ and $b > \frac{1}{2}$, $\sigma > \frac{1}{6}$ satisfying the hypotheses of proposition 3.2, let us define

$$\mathcal{E} = \left\{(u,v) \in \mathbf{X}_{s_1,s_2;b,\sigma} : \|(u,v)\|_{\mathbf{X}_{s_1,s_2;b,\sigma}} \leq 2CR\right\}.$$

where $\mathbf{X}_{s_1,s_2;b,\sigma} := X_{s_1,s_2;b,\sigma} \times X_{s_1,s_2;b,\sigma}$ and

$$\|(u,v)\|_{\mathbf{X}_{s_1,s_2;b,\sigma}} = \|u\|_{X_{s_1,s_2;b,\sigma}} + \|v\|_{X_{s_1,s_2;b,\sigma}}.$$

Then $\mathcal{E}$ is a complete metric space.

For $(u,v) \in \mathcal{E}$ let us define $\Gamma_1 \times \Gamma_2 (u,v) = (\Gamma_1 (u,v), \Gamma_2 (u,v))$ where the maps $\Gamma_1$ and $\Gamma_2$ are defined by

$$\begin{cases} \Gamma_1 (u,v)(t) = \psi_1 (t) W(t) u_0 - P_1 F(u,v,\partial_x u, \partial_x v)(t) \\ \\ \Gamma_2 (u,v)(t) = \psi_1 (t) W(t) v_0 - P_1 G(u,v,\partial_x u, \partial_x v)(t). \end{cases} \tag{3.9}$$

The good definition of $\Gamma_1 \times \Gamma_2$ is guaranteed by propositions 3.1 and 3.2. We prove that there exist $C > 0$ and $R > 0$ such that $\Gamma_1 \times \Gamma_2$ maps $\mathcal{E}$ into itself and is a contraction.

Using (3.6) and (3.7) we get from (3.9),

$$\begin{aligned} &\|\Gamma_1 (u,v)\|_{X_{s_1,s_2;b,\sigma}} \\ &\leq \|\psi_1 (\cdot) W(\cdot) u_0\|_{X_{s_1,s_2;b,\sigma}} + \|P_1 F(u,v,\partial_x u, \partial_x v)\|_{X_{s_1,s_2;b,\sigma}} \\ &\leq C \|u_0\|_{H^{s_1,s_2}} + C \|F(u,v,\partial_x u,\partial_x v)\|_{X_{s_1,s_2;b-1,\sigma}} \end{aligned}$$

and from (3.8) we have

$$\begin{aligned} &\|\Gamma_1 (u,v)\|_{X_{s_1,s_2;b,\sigma}} \\ &\leq C \|u_0\|_{H^{s_1,s_2}} + C|b_1| \|\partial_x (uv)\|_{X_{s_1,s_2;b-1,\sigma}} \\ &\quad + \frac{C|b_2|}{2} \left\|\partial_x u^2\right\|_{X_{s_1,s_2;b-1,\sigma}} + \frac{C|b_3|}{2} \left\|\partial_x v^2\right\|_{X_{s_1,s_2;b-1,\sigma}} \\ &\leq C \|u_0\|_{H^{s_1,s_2}} + C|b_1| \|u\|_{X_{s_1,s_2;b,\sigma}} \|v\|_{X_{s_1,s_2;b,\sigma}} \\ &\quad + \frac{C|b_2|}{2} \|u\|^2_{X_{s_1,s_2;b,\sigma}} + \frac{C|b_3|}{2} \|v\|^2_{X_{s_1,s_2;b,\sigma}} \\ &\leq C \|u_0\|_{H^{s_1,s_2}} + C\left(\|u\|_{X_{s_1,s_2;b,\sigma}} + \|v\|_{X_{s_1,s_2;b,\sigma}}\right)^2. \end{aligned}$$

If we choose $R > 0$ such that $8C^2R < 1$ and $\|(u_0, v_0)\|_{\mathbf{X}_{s_1,s_2;b,\sigma}} < \frac{R}{2}$, it follows that

$$\|\Gamma_1(u,v)\|_{X_{s_1,s_2;b,\sigma}} \leq \frac{CR}{2} + 4C^3R^2 \leq CR.$$

In the same way we obtain $\|\Gamma_2(u,v)\|_{X_{s_1,s_2;b,\sigma}} \leq CR$, therefore

$$\|(\Gamma_1(u,v), \Gamma_2(u,v))\|_{\mathbf{X}_{s_1,s_2;b,\sigma}} \leq 2CR \tag{3.10}$$

and $\Gamma_1 \times \Gamma_2(u,v) = (\Gamma_1(u,v), \Gamma_2(u,v)) \in \mathcal{E}$

Now we need to show that $\Gamma_1 \times \Gamma_2$ is a contraction. For this, if $(u,v), (\widetilde{u}, \widetilde{v}) \in \mathcal{E}$,

$$\begin{aligned}
&\|\Gamma_1(u,v) - \Gamma_1(\widetilde{u},\widetilde{v})\|_{X_{s_1,s_2;b,\sigma}} \\
&\quad \leq \|P_1F(u,v,\partial_x u,\partial_x v) - P_1F(\widetilde{u},\widetilde{v},\partial_x\widetilde{u},\partial_x\widetilde{v})\|_{X_{s_1,s_2;b,\sigma}} \\
&\quad \leq \|P_1[F(u,v,\partial_x u,\partial_x v) - F(\widetilde{u},\widetilde{v},\partial_x\widetilde{u},\partial_x\widetilde{v})]\|_{X_{s_1,s_2;b,\sigma}} \\
&\quad \leq C\|F(u,v,\partial_x u,\partial_x v) - F(\widetilde{u},\widetilde{v},\partial_x\widetilde{u},\partial_x\widetilde{v})\|_{X_{s_1,s_2;b-1,\sigma}},
\end{aligned}$$

but

$$\begin{aligned}
&F(u,v,\partial_x u,\partial_x v) - F(\widetilde{u},\widetilde{v},\partial_x\widetilde{u},\partial_x\widetilde{v}) \\
&\quad = b_1\partial_x[u(v-\widetilde{v})] + b_1\partial_x[\widetilde{v}(u-\widetilde{u})] \\
&\qquad + \frac{b_2}{2}\partial_x[(u+\widetilde{u})(u-\widetilde{u})] + \frac{b_2}{2}\partial_x[(v+\widetilde{v})(v-\widetilde{v})],
\end{aligned}$$

then

$$
\begin{aligned}
&\left\|\Gamma_1(u,v)-\Gamma_1(\widetilde{u},\widetilde{v})\right\|_{X_{s_1,s_2;b,\sigma}}\\
&\leq C\Big(\left\|\partial_x\left[u\left(v-\widetilde{v}\right)\right]\right\|_{X_{s_1,s_2;b-1,\sigma}}+\left\|\partial_x\left[\widetilde{v}\left(u-\widetilde{u}\right)\right]\right\|_{X_{s_1,s_2;b-1,\sigma}}\\
&\quad+\left\|\partial_x\left[\left(u+\widetilde{u}\right)\left(u-\widetilde{u}\right)\right]\right\|_{X_{s_1,s_2;b-1,\sigma}}+\left\|\partial_x\left[\left(v+\widetilde{v}\right)\left(v-\widetilde{v}\right)\right]\right\|_{X_{s_1,s_2;b-1,\sigma}}\Big)\\
&\leq C\Big(\|u\|_{X_{s_1,s_2;b,\sigma}}\|v-\widetilde{v}\|_{X_{s_1,s_2;b,\sigma}}+\|\widetilde{v}\|_{X_{s_1,s_2;b,\sigma}}\|u-\widetilde{u}\|_{X_{s_1,s_2;b,\sigma}}\\
&\quad+\|u+\widetilde{u}\|_{X_{s_1,s_2;b,\sigma}}\|u-\widetilde{u}\|_{X_{s_1,s_2;b,\sigma}}+\|v+\widetilde{v}\|_{X_{s_1,s_2;b,\sigma}}\|v-\widetilde{v}\|_{X_{s_1,s_2;b,\sigma}}\Big)\\
&\leq C\Big(\|u+\widetilde{u}\|_{X_{s_1,s_2;b,\sigma}}+\|\widetilde{v}\|_{X_{s_1,s_2;b,\sigma}}\Big)\|u-\widetilde{u}\|_{X_{s_1,s_2;b,\sigma}}\\
&\quad+C\Big(\|u\|_{X_{s_1,s_2;b,\sigma}}+\|v+\widetilde{v}\|_{X_{s_1,s_2;b,\sigma}}\Big)\|v-\widetilde{v}\|_{X_{s_1,s_2;b,\sigma}}\\
&\leq C\Big(\|u\|_{X_{s_1,s_2;b,\sigma}}+\|\widetilde{u}\|_{X_{s_1,s_2;b,\sigma}}+\|\widetilde{v}\|_{X_{s_1,s_2;b,\sigma}}\Big)\|u-\widetilde{u}\|_{X_{s_1,s_2;b,\sigma}}\\
&\quad+C\Big(\|u\|_{X_{s_1,s_2;b,\sigma}}+\|v\|_{X_{s_1,s_2;b,\sigma}}+\|\widetilde{v}\|_{X_{s_1,s_2;b,\sigma}}\Big)\|v-\widetilde{v}\|_{X_{s_1,s_2;b,\sigma}}\\
&\leq C\Big(\|u\|_{X_{s_1,s_2;b,\sigma}}+\|v\|_{X_{s_1,s_2;b,\sigma}}+\|\widetilde{u}\|_{X_{s_1,s_2;b,\sigma}}+\|\widetilde{v}\|_{X_{s_1,s_2;b,\sigma}}\Big)\cdot\\
&\qquad\Big(\|u-\widetilde{u}\|_{X_{s_1,s_2;b,\sigma}}+\|v-\widetilde{v}\|_{X_{s_1,s_2;b,\sigma}}\Big)\\
&\leq 4C^2R\left\|\left(u-\widetilde{u},v-\widetilde{v}\right)\right\|_{\mathbf{X}_{s_1,s_2;b,\sigma}}.
\end{aligned}
$$

In the same way we obtain

$$
\left\|\Gamma_2(u,v)-\Gamma_2(\widetilde{u},\widetilde{v})\right\|_{X_{s_1,s_2;b,\sigma}}\leq 4C^2R\left\|\left(u-\widetilde{u},v-\widetilde{v}\right)\right\|_{\mathbf{X}_{s_1,s_2;b,\sigma}}.
$$

Then

$$
\begin{aligned}
&\left\|\left(\Gamma_1(u,v)-\Gamma_1(\widetilde{u},\widetilde{v}),\Gamma_2(u,v)-\Gamma_2(\widetilde{u},\widetilde{v})\right)\right\|_{\mathbf{X}_{s_1,s_2;b,\sigma}}\\
&\qquad\qquad\leq 8C^2R\left\|\left(u-\widetilde{u},v-\widetilde{v}\right)\right\|_{\mathbf{X}_{s_1,s_2;b,\sigma}},
\end{aligned}
\tag{3.11}
$$

since that $8C^2R<1$, $\Gamma_1\times\Gamma_2$ is a contraction

Then, for fixed $(u_0,v_0)\in\mathcal{E}$, by (3.10), $\Gamma_1\times\Gamma_2$ maps $\mathcal{E}$ into $\mathcal{E}$ and by (3.11) because $8C^2R<1$ is a contraction. By the Banach fixed point theorem there is exactly one fixed point of because $\Gamma_1\times\Gamma_2$ in $\mathcal{E}$.

Now by a well known argument the uniqueness of the solution $(u, v)$ follows also in $\mathbf{X}_{\mathbf{s_1},\mathbf{s_2};\mathbf{b},\sigma}$. Furthermore, it is easy to see that the mapping $\Theta : (u_0, v_0) \in \mathcal{B}_R \mapsto (u, v) \in C([0,T], H^{s_1,s_2}) \times C([0,T], H^{s_1,s_2})$

$$\Lambda_T : \mathbf{X}_{s_1,s_2;b,\sigma} \times \mathcal{B}_R \to \mathbf{X}_{s_1,s_2;b,\sigma}$$

defined by

$$\Lambda_T((u, v), (u_0, v_0)) = (\Psi_{u_0}(u, v), \Psi_{v_0}(u, v))$$

is analytic. Therefore a standard use of the implicit function theorem yields the analyticity of the flow map $\Theta : (u_0, v_0) \in \mathcal{B}_R \mapsto (u, v) \in C([0,T], H^{s_1,s_2}) \times C([0,T], H^{s_1,s_2})$. $\square$

# 4 Proof of Theorem 2.3

This section is devoted to extend the local solution obtained in the previous section to the global one. We suppose $s_1 < 0$ throughout this section. Our aim here is to derive as *almost conservation quantity* and use it to prove theorem 2.3. For this, let use the multiplier operator $I$ define in $H^{s_1,0}$ by

$$\widehat{Iu}(\zeta) = m(\xi)\,\widehat{u}(\zeta)$$

where $m$ is a smooth and monotone function given by

$$m(\xi) = \begin{cases} 1 & \text{if } |\xi| \le N \\ N^{-s_1}|\xi|^{s_1} & \text{if } |\xi| > N, \end{cases}$$

with $N \gg 1$ to be fixed later.

**Proposition 4.1.** *Let $-\frac{1}{3} < s_1 < 0$ and $N \gg 1$, then the operator $I$ maps $H^{s_1,0}$ to $L^2$ and*

$$\|u\|_{H^{s_1,0}} \le \|Iu\|_{L^2} \le CN^{-s_1}\|u\|_{H^{s_1,0}}.$$

*Proof.* We have

$$
\begin{aligned}
&\|Iu\|_{L^2}^2 = \|m\widehat{u}\|_{L^2}^2 \\
&= \int_{\mathbb{R}} \left( \int_{|\xi| \leq N} |m(\xi)\,\widehat{u}(\zeta)|^2 \, d\xi \right) d\theta + \int_{\mathbb{R}} \left( \int_{|\xi| > N} |m(\xi)\,\widehat{u}(\zeta)|^2 \, d\xi \right) d\theta \\
&= \int_{\mathbb{R}} \left( \int_{|\xi| \leq N} |\widehat{u}(\zeta)|^2 \, d\xi \right) d\theta + \int_{\mathbb{R}} \left( \int_{|\xi| > N} |\xi|^{2s_1} |\widehat{u}(\zeta)|^2 \, d\xi \right) d\theta \\
&\leq \int_{\mathbb{R}} \left( \int_{|\xi| > N} \langle\xi\rangle^{2s_1} |\widehat{u}(\zeta)|^2 \, d\xi + N^{-2s_1} \int_{|\xi| > N} \langle\xi\rangle^{2s_1} |\widehat{u}(\zeta)|^2 \, d\xi \right) d\theta \\
&\leq CN^{-2s_1} \|u\|_{H^{s_1,0}}^2 .
\end{aligned}
$$

Then $I$ is a bounded operator from $H^{s_1,0}$ to $L^2$. □

As discussed in the section 2 let us consider the IVP (2.4). After introducing the multiplier operator $I$, we have the following variant of the local well-posedness for the IVP (2.4).

**Proposition 4.2.** *For any $(u_0, v_0) \in H^{s_1,0} \times H^{s_1,0}$, $s_1 > -\frac{1}{3}$, the initial value problem (2.4) is locally well-posed in the Banach space $I^{-1}L^2 \times I^{-1}L^2$ with existence lifetime $\rho$ satisfying*

$$
(\|Iu_0\|_{L^2} + \|Iv_0\|_{L^2})^{-\alpha} \leq C\rho, \quad \alpha > 0 \tag{4.12}
$$

*and moreover*

$$
\begin{cases}
\|\psi_\rho Iu\|_{X_{0,0;b,0}} \leq C \|Iu_0\|_{L^2} \\
\|\psi_\rho Iv\|_{X_{0,0;b,0}} \leq C \|Iv_0\|_{L^2} .
\end{cases} \tag{4.13}
$$

The proof of this proposition is not difficult and follows using Duhamel's formula and $X_{s_1,0;b,\sigma}$ space properties reduces matters to use the following bilinear estimate to obtain the contraction.

**Proposition 4.3.** *Let $s_1 > -\frac{1}{3}$. For any $b > \frac{1}{2}$ and $\sigma > \frac{1}{6}$ with $b+\sigma \leq 1+s_1$ and $b \leq \frac{1}{3}(s_1+2)$, there exist $C$ independent of $N$ such that*

$$\|\partial_x I(uv)\|_{X_{s_1,0;b-1,\sigma}} \leq C\|Iu\|_{X_{s_1,0;b,\sigma}}\|Iv\|_{X_{s_1,0;b,\sigma}}, \tag{4.14}$$

*for all $u, v \in X_{s_1,0;b,\sigma}$.*

The following extra smoothing bilinear estimate it is fundamental to find a almost conserved quantity.

**Proposition 4.4.** *For $s_1 \in \left]-\frac{1}{4}, 0\right[$, let $b > \frac{1}{2}$ and $\sigma > \frac{1}{6}$ be chosen to satisfy the hypotheses of proposition 4.3 and the condition $-2s_1 < 2-3b$, then for $\alpha \in \left]-2s_1, 2-3b\right[$ it follows that*

$$\|\partial_x[Iu\,Iv - I(uv)]\|_{X_{s_1,0;b-1,\sigma}} \leq CN^{-\alpha}\|Iu\|_{X_{s_1,0;b,\sigma}}\|Iv\|_{X_{s_1,0;b,\sigma}}. \tag{4.15}$$

The proofs of estimates (4.14) and (4.15) due to Isaza and Mejía can be find in [8].

Now we proceed to introduce the almost conserved quantity. When we apply the operator $I$ to the system (2.4), and take the inner product in $L^2$ with $Iu$, using integration by parts, we get,

$$\begin{aligned}\frac{d}{dt}\|Iu\|_{L^2}^2 &= 2\left\langle \frac{d}{dt}Iu, Iu\right\rangle_{L^2} \\ &= 2\left\langle -\partial_x^3 Iu - b_1\partial_x I(uv) - \frac{b_2}{2}\partial_x Iu^2 - \frac{b_3}{2}\partial_x Iv^2 - \partial_x^{-1}\partial_y^2 Iu, Iu\right\rangle_{L^2} \\ &= -\left\langle \partial_x\left(2b_1 I(uv) + b_2 Iu^2 + b_3 Iv^2\right), Iu\right\rangle_{L^2}\end{aligned}$$

because it is easy to see that $\left\langle \partial_x^3 Iu, Iu\right\rangle = \left\langle \partial_x^{-1}\partial_y^2 Iu, Iu\right\rangle_{L^2} = 0$. Then, an integration with respect to $t$ in $[0,1]$ yields

$$\|Iu(1)\|_{L^2}^2 = \|Iu_0\|_{L^2}^2 + R_1(1) \tag{4.16}$$

where

$$R_1(1) = -\int_0^1 \left\langle \partial_x \left(2b_1 I(uv) + b_3 I v^2 + b_2 I u^2\right), Iu \right\rangle_{L^2} dt, \tag{4.17}$$

and in the same way

$$\|Iv(1)\|_{L^2}^2 = \|Iv_0\|_{L^2}^2 + R_2(1) \tag{4.18}$$

with

$$R_2(1) = -\int_0^1 \left\langle \partial_x \left(2b_4 I(uv) + b_5 I v^2 + b_6 I u^2\right), Iv \right\rangle_{L^2} dt. \tag{4.19}$$

Let us define $R(1) := R_1(1) + R_2(1)$, so that we have from (4.16) and (4.18),

$$\|Iu(1)\|_{L^2}^2 + \|Iv(1)\|_{L^2}^2 = \|Iu_0\|_{L^2}^2 + \|Iv_0\|_{L^2}^2 + R(1). \tag{4.20}$$

**Proposition 4.5.** *The following cancellations hold,*

$$\left\langle \partial_x (Iu)^2, Iu \right\rangle_{L^2} = 0 \quad \textit{and} \quad \left\langle \partial_x (Iv)^2, Iv \right\rangle_{L^2} = 0 \tag{4.21}$$

*and*

$$b_3 I_1 + 2b_1 I_2 + b_5 I_3 + 2b_4 I_4 = 0 \tag{4.22}$$

*where*

$$I_1 = \int_0^1 \left\langle \partial_x (Iv)^2, Iu \right\rangle_{L^2} dt \qquad I_2 = \int_0^1 \left\langle \partial_x (IuIv), Iu \right\rangle_{L^2} dt$$

$$I_3 = \int_0^1 \left\langle \partial_x (Iu)^2, Iv \right\rangle_{L^2} dt \qquad I_4 = \int_0^1 \left\langle \partial_x (IuIv), Iv \right\rangle_{L^2} dt.$$

*Proof.* The proof of (4.21) is trivial and (4.22) follows by using integration by parts. In fact, since $b_1 = b_5$ and $b_3 = b_4$, we have

$$\begin{aligned} I_2 &= -\int_0^1 \langle IuIv, \partial_x Iu\rangle_{L^2} dt = -\int_0^1 \langle Iv, Iu\partial_x Iu\rangle_{L^2} dt \\ &= -\frac{1}{2}\int_0^1 \left\langle Iv, \partial_x (Iu)^2\right\rangle_{L^2} dt = -\frac{1}{2} I_3 \end{aligned}$$

and

$$\begin{aligned} I_4 &= -\int_0^1 \langle IuIv, \partial_x Iv\rangle_{L^2} dt = -\int_0^1 \langle Iu, Iv\partial_x Iv\rangle_{L^2} dt \\ &= -\frac{1}{2}\int_0^1 \left\langle Iu, \partial_x (Iv)^2\right\rangle_{L^2} dt = -\frac{1}{2} I_1, \end{aligned}$$

then $b_3 I_1 + 2b_1 I_2 + b_5 I_3 + 2b_4 I_4 = b_3 I_1 - b_1 I_3 + b_5 I_3 - b_4 I_1 = 0.$ □

Using proposition 4.5,

$$\begin{aligned} R(1) &= 2b_1 \int_0^1 \langle \partial_x \left[IuIv - I(uv)\right], Iu\rangle_{L^2}\ dt \\ &\quad +2b_4 \int_0^1 \langle \partial_x \left[IuIv - I(uv)\right], Iv\rangle_{L^2}\ dt \\ &\quad +b_3 \int_0^1 \left\langle \partial_x \left[\partial_x (Iv)^2 - Iv^2\right], Iu\right\rangle_{L^2}\ dt \\ &\quad +b_5 \int_0^1 \left\langle \partial_x \left[(Iv)^2 - Iv^2\right], Iv\right\rangle_{L^2}\ dt \\ &\quad +b_2 \int_0^1 \left\langle \partial_x \left[(Iu)^2 - Iu^2\right], Iu\right\rangle_{L^2}\ dt \\ &\quad -b_6 \int_0^1 \left\langle \partial_x \left((Iu)^2 - Iu^2\right), Iv\right\rangle_{L^2}\ dt \\ &= J_1 + J_2 + J_3 + J_4 + J_5 + J_6. \end{aligned} \tag{4.23}$$

Using Plancherel identity and Cauchy-Schwarz inequality as in [9], we have

$$
\begin{aligned}
|J_1| &\leq 2b_1 \int_0^1 \left| \int_{\mathbb{R}^2} \mathcal{F}\left(\partial_x \left[IuIv - I(uv)\right]\right)(\zeta)\, \mathcal{F}(Iu)(\zeta)\, d\zeta \right| dt \\
&\leq 2b_1 \int_{\mathbb{R}^3} \left| \chi(t) \langle\lambda\rangle^{c-1} \mathcal{F}\left(\partial_x \left[IuIv - I(uv)\right]\right)(\zeta) \right| \\
&\qquad\qquad \cdot \left| \chi(t) \langle\lambda\rangle^{1-c} \mathcal{F}(Iu)(\zeta) \right| d\zeta\, dt \\
&\leq 2b_1 \left( \int_{\mathbb{R}^3} \chi(t) \langle\lambda\rangle^{2(c-1)} \left| \mathcal{F}\left(\partial_x \left[IuIv - I(uv)\right]\right)(\zeta) \right|^2 d\zeta\, dt \right)^{1/2} \\
&\qquad\qquad \cdot \left( \int_{\mathbb{R}^3} \chi(t) \langle\lambda\rangle^{2(1-c)} \left| \mathcal{F}(Iu)(\zeta) \right|^2 d\zeta\, dt \right)^{1/2} \\
&\leq 2b_1 \left\| \chi(\cdot_t)\, \partial_x \left[IuIv - I(uv)\right] \right\|_{X_{0,0;c-1,0}} \left\| \chi(\cdot_t)\, Iu \right\|_{X_{0,0;1-c,0}} \\
&\leq C \left\| \partial_x \left[IuIv - I(uv)\right] \right\|_{X^1_{0,0;b-1,0}} \left\| Iu \right\|_{X^1_{0,0;(1-c)^+,0}}
\end{aligned}
$$

where $\chi$ is the characteristic function of the interval $[0,1]$, $1 < c < b < \frac{3}{2}$ and the last inequality we have used the following lemma.

**Lemma 4.6.** *If* $\gamma_1 \in \left]0, \frac{1}{2}\right[$ *and* $\gamma_1 < \gamma_2 < \frac{1}{2}$, *then*

$$
\left\| \chi(\cdot_t)\, u \right\|_{X_{0,0;\gamma_1,0}} \leq C \left\| u \right\|_{X_{0,0;\gamma_2,0}}
$$

*and*

$$
\left\| \chi(\cdot_t)\, u \right\|_{X_{0,0;-\gamma_2,0}} \leq C \left\| u \right\|_{X_{0,0;-\gamma_1,0}}.
$$

Using the and proposition 4.4, we get

$$
\begin{aligned}
|J_1| &\leq CN^{-\alpha} \left\| Iu \right\|_{X^1_{0,0;b,0}} \left\| Iv \right\|_{X^1_{0,0;b,0}} \left\| Iu \right\|_{X^1_{0,0;(1-c)^+,0}} \\
&\leq CN^{-\alpha} \left\| Iu \right\|^2_{X^1_{0,0;b,0}} \left\| Iv \right\|_{X^1_{0,0;b,0}}. \qquad (4.24)
\end{aligned}
$$

In the same way, we have

$$
|J_2| \leq CN^{-\alpha} \left\| Iu \right\|_{X^1_{0,0;b,0}} \left\| Iv \right\|^2_{X^1_{0,0;b,0}}, \qquad (4.25)
$$

$$|J_3| \leq CN^{-\alpha} \|Iu\|_{X^1_{0,0;b,0}} \|Iv\|^2_{X^1_{0,0;b,0}}, \tag{4.26}$$

$$|J_4| \leq CN^{-\alpha} \|Iv\|^3_{X^1_{0,0;b,0}}, \tag{4.27}$$

$$|J_5| \leq CN^{-\alpha} \|Iu\|^3_{X^1_{0,0;b,0}} \tag{4.28}$$

and

$$|J_6| \leq CN^{-\alpha} \|Iu\|^2_{X^1_{0,0;b,0}} \|Iv\|_{X^1_{0,0;b,0}} \tag{4.29}$$

Now, using (4.35) and from (4.24) to (4.29), the identity (4.20) yields the following almost conservation law,

$$\begin{aligned}
&\|Iu(1)\|^2_{L^2} + \|Iv(1)\|^2_{L^2} \\
&\quad \leq \|Iu_0\|^2_{L^2} + \|Iv_0\|^2_{L^2} \\
&\qquad + CN^{-\alpha}\Big(\|Iu\|^3_{X^1_{0,0;b,0}} + \|Iu\|^2_{X^1_{0,0;b,0}} \|Iv\|_{X^1_{0,0;b,0}} \\
&\qquad + \|Iu\|_{X^1_{0,0;b,0}} \|Iv\|^2_{X^1_{0,0;b,0}} + \|Iv\|^3_{X^1_{0,0;b,0}}\Big). 
\end{aligned} \tag{4.30}$$

***Proof of Theorem 2.3***

To prove the theorem it is emough to show that the local solution to the IVP (2.4) can be extended to $[0,T]$ for arbitrary $T>0$. To make the analysis easy, for $\lambda \in ]0,1[$we define $u^\lambda(x,y,t) = \lambda^2 u(\lambda x, \lambda^2 y, \lambda^3 t)$, $v^\lambda(x,y,t) = \lambda^2 v(\lambda x, \lambda^2 y, \lambda^3 t)$, $u_0^\lambda(x,y) = \lambda^2 u_0(\lambda x, \lambda^2 y)$ and $v_0^\lambda(x,y) = \lambda^2 v_0(\lambda x, \lambda^2 y)$. Then, $(u,v)$ solves (2.4) in $[0,T]$ with initial data $(u_0, v_0)$ if and only if $(u^\lambda, v^\lambda)$ solves (2.4) in $\left[0, \frac{T}{\lambda^3}\right]$ with initial data $(u_0^\lambda, v_0^\lambda)$. So we are interested in exending $(u^\lambda, v^\lambda)$ to $\left[0, \frac{T}{\lambda^3}\right]$.

By lemma 4.1 we have

$$\left\|Iu_0^\lambda\right\|_{L^2} \leq CN^{-s_1} \left\|u_0^\lambda\right\|_{H^{s_1,0}} \leq CN^{-s_1}\lambda^{\frac{1}{2}+s_1} \|u_0\|_{H^{s_1,0}} \tag{4.31}$$

and in the same way we have

$$\left\|Iv_0^\lambda\right\|_{L^2} \leq CN^{-s_1}\lambda^{\frac{1}{2}+s_1} \|v_0\|_{H^{s_1,0}}, \tag{4.32}$$

$N = N(T)$ will be selected later, but let us choose $\lambda = \lambda(N)$ right now by requiring that

$$\begin{cases} CN^{-s_1}\lambda^{\frac{1}{2}+s_1}\|u_0\|_{H^{s_1,0}} = \left(\frac{\varepsilon_0}{2}\right)^{1/2} \ll 1 \\ CN^{-s_1}\lambda^{\frac{1}{2}+s_1}\|v_0\|_{H^{s_1,0}} = \left(\frac{\varepsilon_0}{2}\right)^{1/2} \ll 1. \end{cases} \tag{4.33}$$

From (4.33) we get $\lambda \sim N^{\frac{2s_1}{1+2s_1}}$ and using (4.33) in (4.31) and (4.32) we get

$$\left\|Iu_0^\lambda\right\|_{L^2}^2 \le \frac{\varepsilon_0}{2} \ll 1 \quad \text{and} \quad \left\|Iv_0^\lambda\right\|_{L^2} \le \frac{\varepsilon_0}{2} \ll 1, \tag{4.34}$$

therefore, if we choose $\varepsilon_0$ arbitrarily small then from theorem 4.2 we see that the IVP (2.4) is well-posed for all $t \in [0, 1]$.

Now, using the almost conserved quantity (4.30), the inequality (4.34) and theorem 4.2, we get

$$\begin{aligned} \left\|Iu^\lambda(1)\right\|_{L^2}^2 + \left\|Iv^\lambda(1)\right\|_{L^2}^2 &\le \frac{\varepsilon_0}{2} + \frac{\varepsilon_0}{2} + 4CN^{-\alpha}\left[\frac{\varepsilon_0}{2}\left(\frac{\varepsilon_0}{2}\right)^{1/2}\right] \\ &\le \varepsilon_0 + CN^{-\alpha}\varepsilon_0. \end{aligned} \tag{4.35}$$

So, we can repeat this process $C^{-1}N^\alpha$ times before doubling the value of $\left\|Iu^\lambda(1)\right\|_{L^2}^2 + \left\|Iv^\lambda(1)\right\|_{L^2}^2$. By this process we can extend the solution to the time interval $[0, C^{-1}N^\alpha]$ by taking $C^{-1}N^\alpha$ time steps of size $O(1)$. As we are interessted in extending the solution to the time interval $[0, \frac{T}{\lambda^3}]$, let us select $N = N(T)$ such that $C^{-1}N^\alpha \ge \frac{T}{\lambda^3}$. That is,

$$N^\alpha \ge \frac{CT}{\lambda^3} \sim TN^{-\frac{6s_1}{1+2s_1}}.$$

That is possible if

$$\alpha \ge -\frac{6s_1}{1+2s_1}. \tag{4.36}$$

If $s_1$ is such that $-\frac{6s_1}{1+2s_1} < 2 - 3b$, then we can find $\alpha$ which satisfies (4.36) and the hypotheses of proposition 4.4. This last inequality is satisfied by an allowed value of $b > \frac{1}{2}$ if $-\frac{6s_1}{1+2s_1} < \frac{1}{2}$, then we choose $s_1 \in \left]-\frac{1}{14}, 0\right[$. This completes the proof of the theorem. □

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**Resumen**

Dado el problema de valor inicial para un sistema de dos ecuaciones de Kadomtsev-Petviashvili II (KP-II) acopladas en los términos dispersivos y no lneales, es demostrado que está bien colocado localmente en los espacios de Sobolev anisotrópicos $H^{s_1,s_2}\left(\mathbb{R}^2\right)\times H^{s_1,s_2}\left(\mathbb{R}^2\right)$ con $s_1>-\frac{1}{3}$ y $s_2\geq 0$ y bien colocado globalmente en $H^{s_1,s_2}\left(\mathbb{R}^2\right)\times H^{s_1,s_2}\left(\mathbb{R}^2\right)$ con $s_1>-\frac{1}{14}$.

**Palabras clave:** Ecuaciones dispersivas no lineales, buena colocación local y global, espacios de Bourgain, leyes de casi conservación.

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