ON BINOMIAL AND TRINOMIAL OPERATOR REPRESENTATIONS OF CERTAIN POLYNOMIALS

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Abstract

A new technique is evolved to give operator representation of certain polynomials.

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1 Introduction

Here finite series representations of binomial and trinomial partial differential operators have been used to establish operational representation of various known polynomials. The technique used and the results obtained are believed to be new.

2 Definitions, Notations and Results Used

In deriving the operational representations of various polynomials, use has been made of the fact that

$$D^{\mu}x^{\lambda} = \frac{\Gamma(1+\lambda)}{\Gamma(1+\lambda-\mu)}x^{\lambda-\mu},\ D \equiv \frac{d}{dx}, \tag{2.1}$$

where λ and μ, $\lambda \geq \mu$ are arbitrary real numbers.

In particular, use has been made of the following results:

$$D^r e^{-x} = (-1)^r e^{-x} \tag{2.2}$$

$$D^r x^{-\alpha} = (\alpha)_r(-1)^r x^{-\alpha-r}, \alpha \text{ is not an integer} \tag{2.3}$$

$$D^r x^{-\alpha-n} = (\alpha+n)_r(-1)^r\ x^{-\alpha-n-r} \tag{2.4}$$

$$D^{n-r}x^{\alpha-1+n} = \frac{(\alpha)_n}{(\alpha)_r}x^{\alpha-1+r} \tag{2.5}$$

$$D^{n-r}x^{-\alpha} = \frac{(\alpha)_n(-1)^n}{(1-\alpha-n)_r}x^{-\alpha-n+r}, \alpha \text{ is not an integer} \tag{2.6}$$

where n and r denote positive integers and

$$(a)_n = a(a+1)\cdots(a+n-1); (a)_0 = 1$$

We also need the definitions of the following polynomials in terms of hypergeometric function and also their notations (see [5], [7]).

Legendre Polynomials

It is denoted by the symbol $P_n(x)$ and is defined as

$$P_n(x) = {}_2F_1\left[-n, n+1; 1; \frac{1-x}{2}\right] \tag{2.7}$$

Laguerre Polynomials

It is denoted by the symbol $L_n^{(\alpha)}(x)$ and is defined as

$$L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!}\, {}_1F_1[-n; 1+\alpha; x] \tag{2.8}$$

Jacobi Polynomials

It is denoted by the symbol $P_n^{(\alpha,\beta)}(x)$ and is defined as

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\, {}_2F_1\left[-n; 1+\alpha+\beta+n; 1+\alpha; \frac{1-x}{2}\right] \tag{2.9}$$

Ultraspherical Polynomials

The special case $\beta = \alpha$ of the Jacobi polynomial is called ultraspherical polynomial and is denoted by $P_n^{(\alpha,\alpha)}(x)$. It is thus defined as

$$P_n^{(\alpha,\alpha)}(x) = \frac{(1+\alpha)_n}{n!}\, {}_2F_1\left[-n; 1+2\alpha+n; 1+\alpha; \frac{1-x}{2}\right] \tag{2.10}$$

Gagenbauer Polynomials

The Gagenbauer polynomial $C_n^{\nu}(x)$ is generalization of Legendre polynomial and is defined as

$$C_n^{(\nu)}(x) = \frac{(2\nu)_n}{n!}\, {}_2F_1\left[-n, 2\nu+n; \nu+\frac{1}{2}; \frac{1-x}{2}\right] \tag{2.11}$$

Bateman's $\mathbf{Z_n(x)}$

Bateman's polynomial $Z_n(x)$ is defined as

$$Z_n(x) = {}_2F_2[-n, n+1; 1, 1; x]$$

Bateman's generalization of $\mathbf{Z_n(x)}$

Bateman moved from $Z_n(x)$ to the more general polynomial

$$ {}_2F_2\left[-n, 2\nu+n; \nu+\frac{1}{2}, 1+b; t\right] $$

It may be remarked here that the above polynomial is Gagenbauer type generalization of $Z_n(x)$. We will therefore adopt the symbol $Z_n^\nu(b,t)$. Thus, we have

$$ Z_n^\nu(b,t) = {}_2F_2\left[-n, 2\nu+n; \nu+\frac{1}{2}, 1+b; t\right] \tag{2.12} $$

A Jacobi type generalization of $Z_n(x)$ may be denoted bye the symbol $Z_n^{(\alpha,\beta)}(b,x)$ and is defined as

$$ Z_n^{(\alpha,\beta)}(b,x) = {}_2F_2\left[-n, 1+\alpha+\beta+n; 1+\alpha, 1+b; x\right] \tag{2.13} $$

Rice Polynomials

Rice polynomial $H_n(\xi,p,v)$ is defined by

$$ H_n(\xi,p,v) = {}_3F_2\left[-n, n+1, \xi; 1, p; \nu\right] \tag{2.14} $$

A Jacobi type generalization of Rice's $H_n(\xi,p,v)$ is due to Khandekar [4] who denoted his generalized polynomial by the symbol $H_n^{(\alpha,\beta)}(\xi,p,v)$ and is defined it by

$$ H_n^{(\alpha,\beta)}(\xi,p,v) = \frac{(1+\alpha)_n}{n!}\, {}_3F_2\left[-n, 1+\alpha+\beta+n, \xi; 1+\alpha, p; v\right] \tag{2.15} $$

Sister Celine's Polynomials

Fasenmyer, Sister M. Celine denoted her polynomial by the symbol

$$ f_n\left[\begin{array}{ccc} a_1, \cdots, a_p & ; & \\ & & x \\ b_1, \cdots, b_q & ; & \end{array}\right] $$

as defined it by

$$f_n\left[\begin{array}{ll} a_1,\cdots,a_p & ; \\ & x \\ b_1,\cdots,b_q & ; \end{array}\right]$$

$$= {}_{p+2}F_{q+2}\left[\begin{array}{ll} -n,n+1,a_1,\cdots,a_p & ; \\ & x \\ 1,\frac{1}{2},b_1,\cdots,b_q & ; \end{array}\right] \tag{2.16}$$

Bessel Polynomials

Simple Bessel polynomial $y_n(x)$ is defined by

$$y_n(x) ={}_2 F_0\left[-n,n+1;\_\,;-\frac{x}{2}\right] \tag{2.17}$$

and the generalized Bessel polynomials $Y_n(a,b,x)$ and is defined as

$$y_n(a,b,x) = {}_2F_0\left[-n,a-1+n;\_\,;-\frac{x}{b}\right] \tag{2.18}$$

Tchebicheff Polynomials

The Tchebicheff polynomials $T_n(x)$ and $U_n(x)$ of the first and second kinds, respectively are special ultraspherical polynomials. In details

$$T_n(x) = \frac{n!}{(\frac{1}{2})_n}P_n^{(-\frac{1}{2},-\frac{1}{2})}(x) \tag{2.19}$$

$$U_n(x) = \frac{n+1!}{(\frac{3}{2})_n}P_n^{(\frac{1}{2},\frac{1}{2})}(x) \tag{2.20}$$

In terms of hypergeometric function their definition will be as follows:

$$T_n(x) = {}_2F_1\left[-n,n;\frac{1}{2};\frac{1-x}{2}\right] \tag{2.21}$$

$$U_n(x) = (n+1){}_2F_1\left[-n,n+2;\frac{3}{2};\frac{1-x}{2}\right] \tag{2.22}$$

Kaunhauser Polynomials

Kaunhauser's polynomials are denoted by the symbol $Z_n^\alpha(x;k)$ and is defined by

$$Z_n^\alpha(x;k) = \frac{(1+\alpha)_{kn}}{n!} \; {}_1F_k \left[-n; \Delta(k,\alpha+1); (\frac{x}{k})^k\right] \tag{2.23}$$

where $\Delta(k,\alpha)$ stands for the set of k parameters $\frac{\alpha}{k}, \frac{\alpha+1}{k}, \cdots, \frac{\alpha+k-1}{k}$.

Lagrange Polynomials

Lagrange's polynomials $g_n^{(\alpha,\beta)}(x,y)$ are defined by

$$g_n^{(\alpha,\beta)}(x,y) = \frac{(\alpha)_n x^n}{n!} \; {}_2F_1 \left[-n; \beta; 1-\alpha-n; \frac{y}{x}\right] \tag{2.24}$$

Similarly, Lagrange polynomials of three variables may be denoted by the symbol $g_n^{(\alpha,\beta,\gamma)}(x,y,z)$ and be defined as

$$g_n^{(\alpha,\beta,\gamma)}(x,y,z) = \frac{(\alpha)_n}{n!} x^n \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s}(\beta)_r(\gamma)_s}{r!s!(1-\alpha-n)_{r+s}} \left(\frac{y}{x}\right)^r \left(\frac{z}{x}\right)^s \tag{2.25}$$

or alternatively as

$$g_n^{(\alpha,\beta,\gamma)}(x,y,z) = \frac{(\alpha)_n}{n!} x^n F_1 \left[-n; \beta, \gamma; 1-\alpha-n; \frac{y}{x} \frac{z}{x}\right] \tag{2.26}$$

where F_1 is an Appell function defined by

$$F_1(a,b,b',c;x,y) = \sum_{n,k=0}^{\infty} \frac{(a)_{n+k}(b)_k(b')_n x^k y^n}{k!n!(c)_{n+k}} \tag{2.27}$$

Laguerre Polynomials of two and three variables

S. F. Rageb [6], in 1991, defined Laguerre polynomials of two variables $L_n^{(\alpha,\beta)}(x,y)$ as follows:

$$L_n^{(\alpha,\beta)}(x,y) = \frac{(\alpha+1)_n(\beta+1)_n}{(n!)^2} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s}}{(\alpha+1)_s(\beta+1)_r} \frac{y^r}{r!} \frac{x^s}{s!} \tag{2.28}$$

Similarly, Laguerre polynomials of three variables may be denoted $L_n^{(\alpha,\beta,\gamma)}(x,y,z)$ and be defined as

$$L_n^{(\alpha,\beta,\gamma)}(x,y,z) = \frac{(\alpha+1)_n(\beta+1)_n(\gamma+1)_n}{(n!)^3}$$

$$\sum_{r=0}^{n}\sum_{s=0}^{n-r}\sum_{k=0}^{n-r-s} \frac{(-n)_{r+s+k}x^k y^s z^r}{r!s!k!(1+\alpha)_k(1+\beta)_r(1+\gamma)_r} \tag{2.29}$$

3 Operational Representations

If $D_x \equiv \dfrac{\partial}{\partial x}$ and $D_y \equiv \dfrac{\partial}{\partial y}$, we can write the binomial expansion for $(D_x + D_y)^n$ as

$$(D_x + D_y)^n \equiv \sum_{r=0}^{n} {}^nC_r D_x^{n-r} D_y^r \tag{3.1}$$

where ${}^nC_r = \dfrac{n!}{r!(n-r)!}$. If we write the finite series on the right of (3.1) in reverse order we can also write (3.1) as

$$(D_x + D_y)^n \equiv \sum_{r=0}^{n} {}^nC_r D_x^r D_y^{n-r} \tag{3.2}$$

If $F(x,y)$ is a function of x and y, we obtain the following from (3.1) and (3.2):

$$(D_x + D_y)^n F(x,y) = \sum_{r=0}^{n} \frac{(-n)_r(-1)^r}{r!} D_x^{n-r} D_y^r F(x,y) \tag{3.3}$$

and

$$(D_x + D_y)^n F(x,y) = \sum_{r=0}^{n} \frac{(-n)_r(-1)^r}{r!} D_x^r D_y^{n-r} F(x,y) \tag{3.4}$$

In particular, if $F(x,y) = f(x)g(y)$, we can write (3.3) and (3.4) in the form

$$(D_x + D_y)^n\{f(x)g(y)\} = \sum_{r=0}^{n} \frac{(-n)_r(-1)^r}{r!} D_x^{n-r} f(x) D_y^r g(y) \qquad (3.5)$$

and

$$(D_x + D_y)^n\{f(x)g(y)\} = \sum_{r=0}^{n} \frac{(-n)_r(-1)^r}{r!} D_x^{r} f(x) D_y^{n-r} g(y) \qquad (3.6)$$

Now by taking special values of $f(x)$ and $g(y)$ in (3.5), we obtain the following binomial partial differential operator representations of some of the well known polynomials:

$$(D_x + D_y)^n\{x^n y^{-1-n}\} = n!\, y^{-1-n} P_n\left(1 - \frac{2x}{y}\right) \qquad (3.7)$$

$$(D_x + D_y)^n\{x^{\alpha+n} e^{-y}\} = n!\, x^{\alpha} e^{-y} L_n^{(\alpha)}(x) \qquad (3.8)$$

$$(D_x + D_y)^n\{x^{\alpha+n} y^{-1-\alpha-\beta-n}\}$$
$$= n!\, x^{\alpha} y^{-1-\alpha-\beta-n} P_n^{(\alpha,\beta)}\left(1 - \frac{2x}{y}\right) \qquad (3.9)$$

$$(D_x + D_y)^n\{x^{\alpha+n} y^{-1-2\alpha-n}\}$$
$$= n!\; x^{\alpha} y^{-1-2\alpha-n} P_n^{(\alpha,\alpha)}\left(1 - \frac{2x}{y}\right) \qquad (3.10)$$

$$(D_x + D_y)^n\{x^{\nu-\frac{1}{2}+n} y^{-2\nu-n}\}$$
$$= \frac{n!\left(\nu + \frac{1}{2}\right)_n}{(2\nu)_n} x^{\nu-\frac{1}{2}} y^{-2\nu-n} C_n^{\nu}\left(1 - \frac{2x}{y}\right) \qquad (3.11)$$

$$(D_x + D_y)^n\{x^{n-\frac{1}{2}}y^{-n}\} = \left(\frac{1}{2}\right)_n x^{-\frac{1}{2}}y^{-n}T_n\left(1-\frac{2x}{y}\right) \tag{3.12}$$

$$(D_x + D_y)^n\{x^{n+\frac{1}{2}}y^{-2-n}\}$$
$$= \frac{n!\left(\frac{3}{2}\right)_n}{(n+1)_n}x^{\frac{1}{2}}y^{-2-n}U_n\left(1-\frac{2x}{y}\right) \tag{3.13}$$

$$(D_xD_y + D_z)^n\{x^ny^nz^{-1-n}\} = (n!)^2z^{-1-n}Z_n\left(\frac{xy}{z}\right) \tag{3.14}$$

$$(D_xD_y + D_z)^n\{x^{\nu-\frac{1}{2}+n}y^{b+n}z^{-2\nu-n}\}$$
$$= \left(\nu+\frac{1}{2}\right)_n (b+1)_nx^{\nu-\frac{1}{2}}y^bz^{-2\nu-n}Z_n^\nu\left(b,\frac{xy}{z}\right) \tag{3.15}$$

$$(D_xD_y + D_z)^n\{x^{\alpha+n}y^{b+n}z^{-1-\alpha-\beta-n}\}$$
$$= (1+\alpha)_n(1+b)_nx^\alpha y^bz^{-1-\alpha-\beta-n}Z_n^{(\alpha,\beta)}\left(b,\frac{xy}{z}\right) \tag{3.16}$$

$$(D_wD_x + D_yD_z)^n\{w^nx^{p-1+n}y^{-1-n}z^{-\xi}\}$$
$$= n!(p)_nx^{p-1}y^{-1-n}z^{-\xi}H_n\left(\xi,p,-\frac{wx}{yz}\right) \tag{3.17}$$

$$(D_vD_w + D_xD_yD_z)^n\{v^nw^{p-1+n}x^{-1-n}y^{-\xi}e^{-z}\}$$
$$= n!(p)_nw^{p-1}x^{-1-n}y^{-\xi}e^{-z}H_n\left(\xi,p,-\frac{vw}{xy}\right) \tag{3.18}$$

$$(D_w D_x + D_y D_z)^n \{w^{\alpha+n} x^{p-1+n} y^{-1-\alpha-\beta-n} z^{-\xi}\}$$
$$= n!(p)_n w^\alpha x^{p-n} y^{-1-\alpha-\beta-n} z^{-\xi} H_n^{(\alpha,\beta)}\left(\xi, p, -\frac{wx}{yz}\right) \quad (3.19)$$

$$(D_v D_w + D_x D_y D_z)^n \{v^{\alpha+n} w^{p-1+n} x^{-1-\alpha-\beta-n} y^{-\xi} e^{-z}\}$$
$$= n!(p)_n v^\alpha w^{p-1} x^{-1-\alpha-\beta-n} y^{-\xi} e^{-z} H_n^{(\alpha,\beta)}\left(\xi, p, \frac{vw}{xy}\right) \quad (3.20)$$

$$(D_{\frac{2}{x}} + D_y)^n \left\{\left(\frac{x}{2}\right)^{n+1} e^{-y}\right\}$$
$$= \left(\frac{x}{2}\right)^{n+1} e^{-y} y_n(x)(-1)^n, \text{ using (3.2)} \quad (3.21)$$

$$(D_{\frac{b}{x}} + D_y)^n \left\{\left(\frac{x}{b}\right)^{a-1+n} e^{-y}\right\}.$$
$$= \left(\frac{x}{b}\right)^{a-1+n} e^{-y} y_n(a,b,x)(-1)^n, \text{ using (3.2)} \quad (3.22)$$

$$(D_{x^{-1}} + D_{y^{-1}})^n \left\{x^\alpha y^\beta\right\} = n!(-1)^n x^{\alpha+n} y^\beta g_n^{(\alpha,\beta)}(x,y) \quad (3.23)$$

$$(D_{\frac{x_1}{k}} D_{\frac{x_2}{k}} \ldots D_{\frac{x_k}{k}} + D_y)^n \left(\frac{x_1}{k}\right)^{\frac{\alpha+1}{k}+n-1} \left(\frac{x_2}{k}\right)^{\frac{\alpha+2}{k}+n-1}$$
$$\ldots \left(\frac{x_k}{k}\right)^{\frac{\alpha+k}{k}+n-1} e^{-y} = \frac{n!}{k^{kn}} \left(\frac{x_1}{k}\right)^{\frac{\alpha+1}{k}-1} \left(\frac{x_2}{k}\right)^{\frac{\alpha+2}{k}-1}$$
$$\ldots \left(\frac{x_k}{k}\right)^{\frac{\alpha+k}{k}-1} Z_n^\alpha((x_1 x_2 \ldots x_k)^{\frac{1}{k}}; k) e^{-y} \quad (3.24)$$

4 Trinomial Operator Representations

If $D_x \equiv \frac{\partial}{\partial x}, D_y \equiv \frac{\partial}{\partial y}$ and $D_z \equiv \frac{\partial}{\partial z}$, we can write the trinomial expansion for $(D_x + D_y + D_z)^n$ as

$$(D_x + D_y + D_z)^n \equiv \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s}(-1)^{r+s}}{r!s!} D_x^{n-r-s} D_y^r D_z^s \tag{4.1}$$

Operating (4.1) on $F(x, y, z)$, we get

$$\begin{aligned}&(D_x + D_y + D_z)^n F(x, y, z)\\ &= \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s}(-1)^{r+s}}{r!s!} D_x^{n-r-s} D_y^r D_z^s F(x, y, z)\end{aligned} \tag{4.2}$$

In particular, if $F(x, y, z) = f(x)g(y)h(z)$, then (4.2) gives

$$\begin{aligned}&(D_x + D_y + D_z)^n \{f(x)g(y)h(z)\}\\ &= \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s}(-1)^{r+s}}{r!s!} D_x^{n-r-s} f(x) D_y^r g(y) D_z^s h(z)\end{aligned} \tag{4.3}$$

Similarly, we have

$$\begin{aligned}&(D_x D_y + D_x D_z + D_y D_z)^n \{f(x)g(y)h(z)\}\\ &= \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s}(-1)^{r+s}}{r!s!} D_x^{n-s} f(x) D_y^{n-r} g(y) D_z^{r+s} h(z)\end{aligned} \tag{4.4}$$

Choosing special values of $f(x), g(y)$ and $h(z)$ in (4.3) and (4.4), we obtain the following trinomial operator representations:

$$(D_{x^{-1}} + D_{y^{-1}} + D_{z^{-1}})^n x^\alpha y^\beta z^\gamma = n!(-1)^n x^\alpha y^\beta z^\gamma g_n^{(\alpha,\beta,\gamma)}(x,y,z) \quad (4.5)$$

$$(D_x D_y + D_x D_z + D_y D_z)^n x^{\alpha+n} y^{\beta+n} e^{-z} = (n!)^2 x^\alpha y^\beta e^{-z} L_n^{(\alpha,\beta)}(x,y) \quad (4.6)$$

$$(D_x D_y D_z + D_x D_y D_w + D_x D_z D_w + D_y D_z D_w)^n \left\{x^{n+\alpha} y^{n+\beta} z^{\gamma+n} e^{-w}\right\} = (n!)^3 x^\alpha y^\beta z^\gamma e^{-w} L_n^{(\alpha,\beta,\gamma)}(x,y,z) \quad (4.7)$$

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Resumen

Se desarrolla una nueva técnica para representar ciertos polinomios mediante operadores.

Palabras Clave: Operador binomial, operador trinomial, representaciones mediante operadores

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