ON GENERALIZING δ-OPEN FUNCTIONS

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Abstract

In this paper we introduce two classes of functions called weakly δ-open and weakly δ-closed functions. We obtain their characterizations, their basic properties and their relationships with other types of functions between topological spaces.

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1 Introduction and Preliminaries

In 1968, Veličko [20] introduced δ-open sets, which are stronger than open sets, in order to investigate the characterization of H-closed spaces and showed that τ_δ(=the collection of all δ-open sets) is a topology on X such that $\tau_\delta \subset \tau$ and so τ_δ equal with the semi-regularization topology τ_s. In 1985, M. Mršević et al [13] introduce and studied the class of δ-open functions, also in 1985, D.A.Rose [17] and D.A.Rose with D.S.Janković [18] have defined the notions of weakly open and weakly closed functions respectively. In this paper we introduce and discuss the notion of weakly δ-openness (resp. weakly δ-closedness) as a new generalization of δ-openness (resp. δ-closedness) and we obtained several characterizations and properties of these functions. We also study these functions comparing with other types of already known functions. Here it is seen that δ-openness implies weakly δ-openness but not conversely. But under a certain condition the converse is also true.

Throughout this paper, (X, τ) and (Y, σ) (or simply, X and Y) denote topological spaces on which no separation axioms are assumed unless explicitly stated. If S is any subset of a space X, then $Cl(S)$ and $Int(S)$ denote the closure and the interior of S respectively.

Recall that a set S is called regular open (resp. regular closed) if $S = Int(Cl(S))$ (resp. $S = Cl(Int(S))$. A point $x \in X$ is called a θ-cluster [20] (resp. δ-cluster [20]) of S if $S \cap Cl(U) \neq \emptyset$ (resp. $S \cap U \neq \emptyset$) for each open (resp. regular open) set U containing x. The set of all θ-cluster points (resp. δ-cluster points) of S is called the θ-closure (resp. δ-closure) of S and is denoted by $Cl_\theta(S)$ (resp. $\delta Cl(S)$). Hence, a subset S is called θ-closed (resp. δ-closed) if $Cl_\theta(S) = S$ (resp. $\delta Cl(S) = S$). The complement of a θ-closed set (resp. δ-closed set) is called θ-open (resp. δ-open). The family of all δ-open (resp. δ-closed) sets of a space X is denoted by $\delta O(X, \tau)$ (resp. $\delta C(X, \tau)$).

The δ-interior (resp. θ-interior) of S denoted by $\delta Int(S)$ (resp. $Int_\theta(S)$) is defined as follows:

$\delta Int(S) = \{x \in X :$ for some open subset U of $X, x \in U \subset Int(Cl(U)) \subset S\}$

$Int_\theta(S) = \{x \in X :$ for some open subset U of $X, x \in U \subset Cl(U) \subset S\}$.

A subset $S \subset X$ is called preopen [11] (resp. α-open [14] , β-open [1] (or semi-preopen [2])), if $S \subset Int(Cl(S))$(resp.$S \subset Int(Cl(Int(S)))$, $S \subset Cl(Int(Cl(S)))$.

Lemma 1.1.([10], [20]). Let S be a subset of X, then:

(1) S is a δ-open (resp. θ-open) set if and only if $S = \delta Int(S)$ (resp. $S = Int_\theta(S)$).

(2) $X - \delta Int(S) = \delta Cl(X - S)$ and $\delta Int(X - S) = X - \delta Cl(S)$. (resp. $X - Int_\theta(S) = Cl_\theta(X - S)$ and $Int_\theta(X - S) = X - Cl_\theta(S)$.)

(3) $\delta Cl(S)$ (resp. $\delta Int(S)$) is a closed set (resp. open set) but not necessarily is a δ-closed set (resp. δ-open set).

(4) $Cl_\theta(S)$ (resp. $Int_\theta(S)$) is a closed set (resp. open set) but not necessarily is a θ-closed set (resp. θ-open set).

Lemma. 1.2. [20].

(1) $Cl(S) \subset \delta Cl(S) \subset Cl_\theta(S)$ (resp. $Int_\theta(S) \subset \delta Int(S) \subset Int(S)$) for any subset S of X.

(2) For an open (resp. closed) subset S of X, $Cl(S) = \delta Cl(S) = Cl_\theta(S)$ (resp. $Int_\theta(S) = \delta Int(S) = Int(S)$).

Lemma. 1.3. If X is a regular space, then:

(1) $Cl(S) = \delta Cl(S) = Cl_\theta(S)$ for any subset S of X.

(2) Every closed subset of X is θ-closed (and δ-closed) and hence for any subset S, $Cl_\theta(S)$ (resp. $\delta Cl(S)$) is θ-closed (resp. δ-closed).

A space X is called extremally disconnected (E.D) [21] if the closure of each open set in X is open. A space X is called δ-connected if X can not be expressed as the union of two nonempty disjoint δ-open sets.

A function $f : (X, \tau) \to (Y, \sigma)$ is called:

(i) weakly open ([17], [18]) if $f(U) \subset Int(f(Cl(U)))$ for each open subset U of X.

(ii) weakly closed [18] if $Cl(f(Int(F))) \subset f(F)$ for each closed subset F of X.

(iii) strongly continuous [8] if for every subset A of X, $f(Cl(A)) \subset f(A)$.

(iv) almost open in the sense of Singal and Singal, written as (a.o.S) [19] if the image of each regular open set U of X is an open set of Y, equivalently $f(U) \subset Int(f(Int(Cl(U))))$ for each open subset U of X.

(v) δ-open (resp. δ-closed [7] , β-open [1], α-open [14]) if for each open set U (resp. closed set F, open set U, open set U) of X, $f(U)$ is δ-open (resp. $f(F)$ is δ-closed, $f(U)$ is β-open, $f(U)$ is α-open) set in Y.

(vi) contra δ-open (resp. contra-closed [4]) if $f(U)$ is δ-closed (resp. open) in Y for each open (resp. closed) subset U of X.

2 Weakly δ-open Functions

We define in this section the concept of weak δ-openness as a notion between δ-openness and weakly openness.

Definition 2.1. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be weakly δ-open if $f(U) \subset \delta Int(f(Cl(U)))$ for each open set U of X.

Clearly, every δ-open function is weakly δ-open and every weakly δ-open function is weakly open, but the converses are not generally true. For,

Example 2.2.

(i) A weakly δ-open function need not be δ-open.
Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$ and $\sigma = \{\emptyset, Y, \{a\}, \{a, b\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Then f is weakly δ-open since $\delta Int(f(Cl(\{a\}))) = \delta Int(f(Cl(\{a, b\}))) = \delta Int(f(Cl(\{a, c\}))) = Y$ but f is not δ-open since $f(\{a\}) \neq \delta Int(f(\{a\}))$.

(ii) A weakly open function need not be weakly δ-open.
Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$ and $\sigma = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}\}$. Let $f : (X, \tau) \to (X, \sigma)$ be the identity function. Then f is weakly open since $Int(f(Cl(\{a\}))) = \{a, b\}$, $Int(f(Cl(\{c\}))) = \{b, c\}$, $Int(f(Cl(\{a, c\}))) = X$ but f is not weakly δ-open since $f(\{a\}) \not\subset \delta Int(f(Cl(\{a\})))$.

(iii) Let $f : (X, \tau) \to (Y, \sigma)$ be the function as (i). Then it is shown that f is weakly δ-open which is not open.

Theorem 2.3. Let X be a regular space. Then $f : (X, \tau) \to (Y, \sigma)$ is weakly δ-open if and only if f is δ-open.

Proof. The sufficiency is clear.
Necessity. Let W be a nonempty open subset of X. For each x in W, let U_x be an open set such that $x \in U_x \subset Cl(U_x) \subset W$. Hence we obtain that $W = \cup\{U_x : x \in W\} = \cup\{Cl(U_x) : x \in W\}$ and, $f(W) =$

$\cup\{f(U_x) : x \in W\} \subset \cup\{\delta Int(f(Cl(U_x))) : x \in W\} \subset \delta Int(f(\cup\{Cl(U_x) : x \in W\}) = \delta Int(f(W))$. Thus f is δ-open.

Theorem 2.4. For a function $f : (X,\tau) \to (Y,\sigma)$, the following conditions are equivalent :

(i) f is weakly δ-open,

(ii) $f(Int_\theta(A)) \subset \delta Int(f(A))$ for every subset A of X,

(iii) $Int_\theta(f^{-1}(B)) \subset f^{-1}(\delta Int(B))$ for every subset B of Y,

(iv) $f^{-1}(\delta Cl(B)) \subset Cl_\theta(f^{-1}(B))$ for every subset B of Y,

(v) For each closed subset F of X, $f(Int(F)) \subset \delta Int(f(F))$,

(vi) For each open subset U of X, $f(Int(Cl(U))) \subset \delta Int(f(Cl(U)))$,

(vii) For every regular open subset U of X, $f(U) \subset \delta Int(f(Cl(U)))$,

(viii) For every α-open subset U of X, $f(U) \subset \delta Int(f(Cl(U)))$.

Proof.

(i) $\to$ *(ii)* : Let A be any subset of X and $x \in Int_\theta(A)$. Then, there exists an open set U such that $x \in U \subset Cl(U) \subset A$. Hence $f(x) \in f(U) \subset f(Cl(U)) \subset f(A)$. Since f is weakly δ-open, $f(U) \subset \delta Int(f(Cl(U))) \subset \delta Int(f(A))$. It implies that $f(x) \in \delta Int(f(A))$. Therefore $x \in f^{-1}(\delta Int(f(A)))$. Thus $Int_\theta(A) \subset f^{-1}(\delta Int(f(A)))$, and so, $f(Int_\theta(A)) \subset \delta Int(f(A))$.

(ii) $\to$ *(i)* : Let U be an open set in X. As $U \subset Int_\theta(Cl(U))$ implies, $f(U) \subset f(Int_\theta(Cl(U))) \subset \delta Int(f(Cl(U)))$. Hence f is weakly δ-open.

(ii) $\to$ *(iii)* : Let B be any subset of Y. Then by (ii), $f(Int_\theta(f^{-1}(B))) \subset \delta Int(B)$. Therefore $Int_\theta(f^{-1}(B)) \subset f^{-1}(\delta Int(B))$.

$(iii) \to (ii)$: This is obvious.

$(iii) \to (iv)$: Let B be any subset of Y. Using (iii), we have $X - Cl_\theta(f^{-1}(B)) = Int_\theta(X - f^{-1}(B)) = Int_\theta(f^{-1}(Y - B)) \subset f^{-1}(\delta Int(Y - B)) = f^{-1}(Y - \delta Cl(B)) = X - (f^{-1}(\delta Cl(B))$. Therefore, we obtain $f^{-1}(\delta Cl(B)) \subset Cl_\theta(f^{-1}(B))$.

$(iv) \to (iii)$: Similarly we obtain, $X - f^{-1}(\delta Int(B)) \subset X - Int_\theta(f^{-1}(B))$, for every subset B of Y, i.e., $Int_\theta(f^{-1}(B)) \subset f^{-1}(\delta Int(B))$.

$(i) \to (v) \to (vi) \to (vii) \to (viii) \to (i)$: This is obvious.

Theorem 2.5. Let $f : (X, \tau) \to (Y, \sigma)$ be a bijective function. Then the following statements are equivalent.

(i) f is weakly δ-open,

(ii) $\delta Cl(f(U)) \subset f(Cl(U))$ for each U open in X,

(iii) $\delta Cl(f(Int(F)) \subset f(F)$ for each F closed in X.

Proof.

$(i) \to (iii)$: Let F be a closed set in X. Then we have $f(X - F) = Y - f(F) \subset \delta Int(f(Cl(X - F)))$ and so $Y - f(F) \subset Y - \delta Cl(f(Int(F)))$. Hence $\delta Cl(f(Int(F))) \subset f(F)$.

$(iii) \to (ii)$: Let U be an open set in X. Since $Cl(U)$ is a closed set and $U \subset Int(Cl(U))$ by (iii) we have $\delta Cl(f(U)) \subset \delta Cl(f(Int(Cl(U))) \subset f(Cl(U))$.

$(ii) \to (iii)$: Similar to $(iii) \to (ii)$.

$(iii) \to (i)$: Clear.

Theorem 2.6. If X is a regular space, then for a function $f : (X, \tau) \to (Y, \sigma)$, the following conditions are equivalent:

(i) f is weakly δ-open,

(ii) For each θ-open set A in X, $f(A)$ is δ-open in Y,

(iii) For any set B of Y and any θ-closed set A in X containing $f^{-1}(B)$, there exists a δ-closed set F in Y containing β such that $f^{-1}(F) \subset A$.

Proof.

$(i) \to (ii)$: Let A be a θ-open set in X. Then $Y - f(A)$ is a set in Y such that by (i) and Theorem 2.4(iv), $f^{-1}(\delta Cl(Y - f(A))) \subset Cl_\theta(f^{-1}(Y - f(A)))$. Therefore, $X - f^{-1}(\delta Int(f(A))) \subset Cl_\theta(X - A) = X - A$. Then, we have $A \subset f^{-1}(\delta Int(f(A)))$ which implies $f(A) \subset \delta Int(f(A))$. Hence $f(A)$ is a δ-open subset of Y.

$(ii) \to (iii)$: Let B be any set in Y and A be a θ-closed set in X such that $f^{-1}(B) \subset A$. *Since* $X - A$ *is* θ-open in X, by (ii), $f(X - A)$ is δ-open in Y. Let $F = Y - f(X - A)$. Then F is δ-closed and $\beta \subset F$. Now, $f^{-1}(F) = f^{-1}(Y - f(X - A)) = X - f^{-1}(f(A)) \subset A$.

$(iii) \to (i)$: Let B be any set in Y. Let $A = Cl_\theta(f^{-1}(B))$. Then by Lemma 1.3 A is θ-closed set in X and $f^{-1}(B) \subset A$. Then there exists a δ-closed set F in Y containing B such that $f^{-1}(F) \subset A$. Since F is δ-closed $f^{-1}(\delta Cl(B)) \subset f^{-1}(F) \subset Cl_\theta(f^{-1}(B))$. Therefore by Theorem 2.4, f is a weakly δ-open function.

Theorem 2.7. If $f : (X, \tau) to (Y, \sigma)$ is weakly δ-open and strongly continuous, then f is δ-open.

Proof. Let U be an open subset of X. Since f is weakly δ-open $f(U) \subset \delta Int(f(Cl(U)))$. However, because f is strongly continuous, $f(U) \subset \delta Int(f(U))$ and therefore $f(U)$ is δ-open.

Example 2.8. A δ-open function need not be strongly continuous.
Let $X = \{a, b, c\}$, and let τ be the indiscrete topology for X. Then the identity function $f : (X, \tau) \to (X, \tau)$ is a δ-open (hence weakly δ-open) function which is not strongly continuous.

Example 2.9. A strongly continuous function need not be δ-open. Let τ be the discrete topology for the topological space X, and σ the indiscrete topology on X. Let $f : (X, \tau) \to (X, \sigma)$ be the identity function. Then f is strongly continuous, but f is not weakly δ-open (hence, f is not δ-open).

Example 2.8 and 2.9 show that:

(i) weakly δ-openness and strongly continuity are notions independent, and

(ii) δ-openness and strongly continuity are notions independent.

Theorem 2.10. If $f : (X, \tau) \to (Y, \sigma)$ is closed and a.o.S, then f is a weakly δ-open function.

Proof. Let U be an open set in X. Since f is a.o.S and $Int(Cl(U))$ is regular open, $f(Int(Cl(U)))$ is open in Y and hence $f(U) \subset f(Int(Cl(U))) \subset Int(f(Cl(U)))$. Since f is closed $f(U) \subset \delta Int(f(Cl(U)))$ (Lemma 1.2). This shows that f is weakly δ-open.

The converse of Theorem 2.10 is not true in general.

Example 2.11. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$ and $\sigma = \{\emptyset, X, \{a\}, \{b, c\}\}$. Let $f : (X, \tau) \to (X, \sigma)$ be the identity function. Then f is weakly δ-open but it is not a.o.S since c is regular open set in (X, τ) , $f(\{c\}) = \{c\}$ is not open set in (X, σ). It is easily to verify also that f is not a closed function.

Lemma 2.12. If $f : (X, \tau) \to (Y, \sigma)$ is a continuous function, then for any subset U of X , $f(Cl(U)) \subset Cl(f(U))$ [21].

Theorem 2.13. If $f : (X, \tau) \to (Y, \sigma)$ is a weakly δ-open and continuous function, then f is β-open.

Proof. Let U be a open set in X. Then by weak δ-openness of f, $f(U) \subset \delta Int(f(Cl(U)))$. Since f is continuous $f(Cl(U)) \subset Cl(f(U))$. Hence we obtain that, $f(U) \subset \delta Int(f(Cl(U))) \subset \delta Int(Cl(f(U))) \subset Cl(Int(Cl(f(U))))$. Therefore, $f(U) \subset Cl(Int(Cl(f(U)))$ which shows that $f(U)$ is a β-open set in Y. Thus, f is a β-open function.

Since every strongly continuous function is continuous, we have the following corollary.

Corollary 2.14. If $f : (X, \tau) \to (Y, \sigma)$ is an injective weakly δ-open and strongly continuous function, then f is β-open.

Theorem 2.15. If $f : (X, \tau) \to (Y, \sigma)$ is a bijective weakly δ-open of a space X onto a δ-connected space Y, then X is connected.

Proof. Let us assume that X is not connected. Then there exist non-empty open sets U_1 and U_2 such that $U_1 \cap U_2 = \phi$ and $U_1 \cup U_2 = X$. Hence we have $f(U_1) \cap f(U_2) = \phi$ and $f(U_1) \cup f(U_2) = Y$. Since f is bijective weakly δ-open, we have $f(U_i) \subset \delta Int(f(Cl(U_i)))$ for i = 1, 2 and since U_i is open and also closed, we have $f(Cl(U_i)) = f(U_i)$ for $i = 1, 2$. Hence $f(U_i)$ is δ-open in Y for i = 1, 2. Thus, Y has been decomposed into two non-empty disjoint δ-open sets . This is contrary to the hypothesis that Y is a δ-connected space. Thus X is connected.

Definition 2.16. A space X is said to be hyperconnected [16] if every nonempty open subset of X is dense in X.

Theorem 2.17. If X is a hyperconnected space, then $f : (X, \tau) \to (Y, \sigma)$ is weakly δ-open if and only if $f(X)$ is δ-open in Y.

Proof. The sufficiency is clear. For the necessity observe that for any open subset U of X, $f(U) \subset f(X) = \delta Int(f(X)) = \delta Int(f(Cl(U)))$.

3 Weakly δ-closed Functions

Definition 3.1. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be weakly δ-closed if $\delta Cl(f(Int(F))) \subset f(F)$ for each closed set F in X.

Clearly, every δ-closed function is weakly δ-closed since $\delta Cl(f(Int(A))) \subset \delta Cl(f(A)) = f(A)$ for every closed subset A of X. But not conversely.

Example 3.2. Let $f : (X, \tau) \to (Y, \sigma)$ be the function from Example 2.2 (i) (resp. Example 2.2 (ii)). Then it is shown that f is weakly δ-closed which is not δ-closed (resp. weakly closed function need not be weakly δ-closed).

Theorem 3.3. For a function $f : (X, \tau) \to (Y, \sigma)$, the following conditions are equivalent:

(i) f is weakly δ-closed,

(ii) $\delta Cl(f(U)) \subset f(Cl(U))$ for every open subset U of X,

(iii) $\delta Cl(f(U)) \subset f(Cl(U))$ for each open subset U of X,

(iv) $\delta Cl(f(Int(F))) \subset f(F)$ for each preclosed subset F of X,

(v) $\delta Cl(f(Int(F))) \subset f(F)$ for every α-closed subset F of X,

(vi) $\delta Cl(f(Int(Cl(U)))) \subset f(Cl(U))$ for each subset U of X,

(vii) $\delta Cl(f(Int(\delta Cl(U)))) \subset f(\delta Cl(U))$ for each subset U of X,

(viii) $\delta Cl(f(U)) \subset f(Cl(U))$ for each preopen subset U of X.

Proof.

$(i) \to (ii)$: Let U be any open subset of X. Then
$$\delta Cl(f(U)) = \delta Cl(f(Int(U))) \subset \delta Cl(f(Int(Cl(U)))) \subset f(Cl(U)).$$

$(ii) \to (i)$: Let F be any closed subset of X. Then,
$\delta Cl(f(Int(F))) \subset f(Cl(Int(F))) \subset f(Cl(F)) = f(F)$.
It is clear that: $(i) \to (iii) \to (iv) \to (v) \to (i)$, $(i) \to (vi) \to (viii) \to (i)$,and $(i) \to (vii)$.

$(vii) \to (viii)$: Note that $\delta Cl(U) = Cl(U)$ for each preopen subset U of X.

Theorem 3.4. If Y is a regular space, then for a function $f : (X, \tau) \to (Y, \sigma)$, the following conditions are equivalent :

(i) f is weakly δ-closed,

(ii) $\delta Cl(f(U)) \subset f(Cl(U))$ for each regular open subset U of X,

(iii) For each subset F in Y and each open set U in X with $f^{-1}(F) \subset U$, there exists a δ-open set A in Y with $F \subset A$ and $f^{-1}(F) \subset Cl(U)$,

(iv) For each point y in Y and each open set U in X with $f^{-1}(y) \subset U$, there exists a δ-open set A in Y containing y and $f^{-1}(A) \subset Cl(U)$.

Proof. It is clear that: $(i) \to (ii)$ and $(iii) \to (iv)$.

$(ii) \to (iiii)$: Let F be a subset of Y and let U be open in X with $f^{-1}(F) \subset U$. Then $f^{-1}(F) \cap Cl(X - Cl(U)) = \phi$ and consequently, $F \cap f(Cl(X - Cl(U))) = \phi$. Since $X - Cl(U)$ is regular open, $F \cap \delta Cl(f(X - Cl(U))) = \phi$ by (ii). Let $A = Y - \delta Cl(f(X - Cl(U)))$. Then A is δ-open with $F \subset A$ and $f^{-1}(A) \subset X - f^{-1}(\delta Cl(f(X - Cl(U)))) \subset X - f^{-1}f(X - Cl(U)) \subset Cl(U)$.

$(iv) \to (i)$: Let F be closed in X and let $y \in Y - f(F)$. Since $f^{-1}(y) \subset X - F$, there exists a δ-open A in Y with $y \in A$ and $f^{-1}(A) \subset Cl(X - F) = X - Int(F)$ by (iv). Therefore $A \cap f(Int(F)) = \phi$, so that $y \in Y - \delta Cl(f(Int(F)))$. Thus $\delta Cl(f(Int(F))) \subset f(F)$.

Remark 3.5. By Theorem 2.5, if $f : (X, \tau) \to (Y, \sigma)$ is a bijective function then f is weakly δ-open if and only if f is weakly δ-closed.

Next we investigate conditions under which weakly δ-closed functions are δ-closed.

Theorem 3.6. Let (Y, σ) be a regular space. If $f : (X, \tau) \to (Y, \sigma)$ is weakly δ-closed and if for each closed subset F of X and each fiber $f^{-1}(y) \subset X - F$ there exists a open U of X such that $f^{-1}(y) \subset U \subset Cl(U) \subset X - F$. Then f is δ-closed.

Proof. Let F be any closed subset of X and let $y \in Y - f(F)$. Then $f^{-1}(y) \cap F = \phi$ and hence $f^{-1}(y) \subset X - F$. By hypothesis, there exists an open U of X such that $f^{-1}(y) \subset U \subset Cl(U) \subset X - F$. Since f is weakly δ-closed by Theorem 3.4, there exists a δ-open V in Y with $y \in V$ and $f^{-1}(V) \subset Cl(U)$. Therefore , we obtain $f^{-1}(V) \cap F = \phi$ and hence $V \cap f(F) = \phi$, this shows that $y \notin \delta Cl(f(F))$. Therefore, $f(F)$ is a δ-closed subset of Y and f is δ-closed.

Recall that, a function $f : (X, \tau) \to (Y, \sigma)$ is said to be preclosed if $f(F)$ is preclosed in Y for each closed subset F of X.

Theorem 3.7.

(i) If $f : (X, \tau) \to (Y, \sigma)$ is preclosed and contra-closed, then f is weakly δ-closed.

(ii) If $f : (X, \tau) \to (Y, \sigma)$ is contra δ-open, then f is weakly δ-closed.

Proof.

(i) Let F be a closed subset of X. Since f is preclosed, $\delta Cl(Int(f(F))) = Cl(Int(f(F))) \subset f(F)$ and since f is contra-closed $f(F)$ is open. Therefore $\delta Cl(f(Int(F))) \subset \delta Cl(f(F)) = \delta Cl(Int(f(F))) \subset f(F)$.

(ii) Let F be a closed subset of X. Then, $\delta Cl(f(Int(F))) = f(Int(F)) \subset f(F)$.

Example 3.8. Example 2.2 (i) shows that weakly δ-closedness does not imply contra δ-openness. Shows also that weakly δ-closedness does not imply preclosedness.

Example 3.9. Contra-closedness and weakly δ-closedness are independent notions. Example 2.2 (i) shows that weakly δ-closedness does not imply contra-closedness while the reverse is shown in the Example 2.2 (ii).

Theorem 3.10. If Y is a regular space and if $f : (X, \tau) \to (Y, \sigma)$ is one-one and weakly δ-closed, then for every subset F of Y and every open set U in X with $f^{-1}(F) \subset U$, there exists a δ-closed set B in Y such that $F \subset B$ and $f^{-1}(B) \subset Cl(U)$.

Proof. Let F be a subset of Y and let U be an open subset of X with $f^{-1}(F) \subset U$. Put $B = \delta Cl(f(Int(Cl(U))))$, then by Lemma 1.3 B is a δ-closed subset of Y such that $F \subset B$ since $F \subset f(U) \subset f(Int(Cl(U))) \subset \delta Cl(f(Int(Cl(U)))) = B$. And since f is weakly δ-closed, $f^{-1}(B) \subset Cl(U)$.

Taking the set F in Theorem 3.10 to be $\{y\}$ for $y \in Y$ we obtain the following result,

Corollary 3.11. If Y is a regular space and if $f : (X, \tau) \to (Y, \sigma)$ is one-one and weakly δ-closed, then for every point y in Y and every open set U in X with $f^{-1}(y) \subset U$, there exists a δ-closed set B in Y containing y such that $f^{-1}(B) \subset Cl(U)$.

Recall that, a set F in X is θ-compact if for each cover Ω of F by open U in X, there is a finite family $U_1, ..., U_n$ in Ω such that $F \subset Int(\bigcup Cl(U_i) : i = 1, 2, ..., n)$ [18].

Theorem 3.12. Let (Y, σ) be a regular space. If $f : (X, \tau) \to (Y, \sigma)$

is weakly δ-closed with all fibers *theta*-closed, then $f(F)$ is δ-closed for each θ-compact F in X.

Proof. Let F be θ-compact and let $y \in Y - f(F)$. Then $f^{-1}(y) \cap F = \phi$ and for each $x \in F$ there is an open $U_x \subset X$ with $x \in U_x$ such that $Cl(U_x) \cap f^{-1}(y) = \phi$. Clearly, $\Omega = \{U_x : x \in F\}$ is an open cover of F and since F is θ-compact, there is a finite family $\{U_{x_1}, ..., U_{x_n}\} \subset \Omega$ such that $F \subset Int(A)$, where $A = \bigcup\{Cl(U_{x_i}) : i = 1, ..., n\}$. Since f is weakly δ-closed by Theorem 3.4 there exists a δ-open $B \subset Y$ with $f^{-1}(y) \subset f^{-1}(B) \subset Cl(X - A) = X - Int(A) \subset X - F$. Therefore $y \in B$ and $B \cap f(F) = \phi$. Thus $y \in Y - \delta Cl(f(F))$. This shows that $f(F)$ is δ-closed.

Two nonempty subsets A and B in X are strongly separated [18], if there exist open sets U and V in X with $A \subset U$ and $B \subset V$ and $Cl(U) \cap Cl(V) = \phi$. If A and B are singleton sets we may speak of points being strongly separated . We will use the fact that in a normal space, disjoint closed sets are strongly separated.

Recall that, a topological space (X, τ) is said to be θ-T_2 [5], if for $x, y \in X$ such that $x \neq y$ there exist disjoint θ-open sets U and V such that $x \in U$ and $y \in V$.

Theorem 3.13. Let (Y, σ) be a regular space. If $f : (X, \tau) \to (Y, \sigma)$ is a weakly δ-closed surjection and all pairs of disjoint fibers are strongly separated, then Y is θ-T_2 (hence T_2).

Proof. Let y and z be two points in Y. Let U and V be open sets in X such that $f^{-1}(y) \in U$ and $f^{-1}(z) \in V$ respectively with $Cl(U) \cap Cl(V) = \phi$. By weak δ-closedness (Theorem 3.4) there are *delta*-open sets F and B in Y such that $y \in F$ and $zinB$, $f^{-1}(F) \subset Cl(U)$ and $f^{-1}(B) \subset Cl(V)$. Therefore $F \cap B = \phi$, because $Cl(U) \cap Cl(V) = \phi$ and f surjective. Then Y is θ-T_2.

Corollary 3.14. If Y is a regular space and if $f : (X, \tau) \to (Y, \sigma)$ is weakly δ-closed surjection with all fibers closed and X is normal, then Y is θ-T_2 (hence T_2).

Corollary 3.15. If Y is a regular space and if $f : (X, \tau) \to (Y, \sigma)$ is a continuous weakly δ-closed surjection function with X compact T_2 space and Y a T_1 space, then Y is a compact θ-T_2 space.

Proof. Since f is continuous, X compact and Y is a T_1 space, Y is compact and all fibers are closed, and since X is normal by Corollary 3.14 Y is a θ-T_2 space.

Definition 3.16. A topological space X is said to be quasi H-closed [6] (resp. $N - \delta$-closed), if every open (resp. δ-closed) cover of X has a finite subfamily whose closures cover X. A subset $A \subset X$ is quasi H-closed relative to X (resp. $N - \delta$-closed relative to X) if every cover $U_i : i \in I$ of A by open (resp. δ-closed) sets of X there exists a finite subfamily I_0 of I such that $A \subset \bigcup\{Cl(U_i) : i \in I_0\}$.

Lemma 3.17. A function $f : (X, \tau) \to (Y, \sigma)$ is open if and only if for each $B \subset Y$, $f^{-1}(Cl(B)) \subset Cl(f^{-1}(B))$ [9].

Theorem 3.18. Let $\delta O(X, \tau)$ closed under intersections and let $f : (X, \tau) \to (Y, \sigma)$ be an open weakly δ-closed function which is one-one from an extremally disconnected space X into a regular space Y such that $f^{-1}(y)$ is quasi H-closed relative to X for each y in Y. If G is $N - \delta$-closed relative to Y then $f^{-1}(G)$ is quasi H-closed.

Proof. Let $\{V_\beta : \beta \in I\}$, I being the index set be an open cover of $f^{-1}(G)$. Then for each $y \in G \cap f(X)$, $f^{-1}(y) \subset \bigcup\{Cl(V_\beta) : \beta \in I(y)\} = H_y$ for some finite subfamily $I(y)$ of I. Since X is extremally disconnected each $Cl(V_\beta)$ is open and hence H_y is open in X. So by

Corollary 3.11, there exists a δ-closed set U_y containing y such that $f^{-1}(U_y) \subset Cl(H_y)$. Then, $\{U_y : y \in G \cap f(X)\} \cup Y - f(X)$ is a δ-closed cover of G, $G \subset \bigcup(\{Cl(U_y) : y \in K\} \cup \{Cl(Y - f(X))\})$ for some finite subset K of $G \cap f(X)$ since G is an $N - \delta$-closed. Hence and by Lemma 3.17, $f^{-1}(G) \subset \cup\{f^{-1}(Cl(U_y)) : y \in K\} \cup \{f^{-1}(Cl(Y - f(X)))\} \subset \bigcup\{Cl(f^{-1}(U_y)) : y \in K\} \cup \{Cl(f^{-1}(Y - f(X)))\} \subset \bigcup\{Cl(f^{-1}(U_y)) : y \in K\}$, so $f^{-1}(G) \subset \bigcup\{Cl(V_\beta) : \beta \in I(y),\ y \in K\}$. Therefore $f^{-1}(G)$ is quasi H-closed.

Corollary 3.19. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be as in Theorem 3.18. If Y is $N - \delta$-closed, then X is quasi H-closed.

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References

[1] M.E Abd-El-Monsef, S.N. El-Deeb and R.A. Mahmoud. β-open and β-continuous mappings. Bull. Fac. Sci. Assiut Univ., **12** (1983), 70-90.

[2] D. Andrijevic. Semipreopen sets. Mat. Vesnik, **38** (1986), 24-32.

[3] S.P. Arya and R.Gupta. Strongly continuous mappings. Kyungpook Math.J., **14** (1974), 131-143.

[4] C.W. Baker. Contra-open functions and contra-closed functions. Math.Today **15** (1997), 19-24.

[5] S. Bandyopadhyayi. Some Problems Concerning Covering Properties and Function Spaces. Ph.D. Thesis, University of Calcutta, (1996).

[6] D.E. Cameron. Some maximal topologies which are Q.H.C. Proc. Amer. Math. Soc., **75** (1979), 149-156.

[7] L.L. Herrington. Some properties preserved by the almost- continuous function. Boll. Un. Mat. Ital., **10** (1974), 556-568.

[8] N. Levine. Strong continuity in topological spaces. Amer. Math. Monthly, **67** (1960), 269.

[9] P.E. Long and D.A. Carnahan. Comparing almost continuous functions. Proc. Amer. Math. Soc., **38** (1973), 413-418.

[10] P.E. Long and L.L. Herrington. The τ_θ-topology and faintly continuous functions. Kyungpook Math. J., **22** (1982), 7-14.

[11] A.S. Mashhour, M.E. Abd El-Monsef and S.N. El-Deeb. Precontinuous and weak precontinuous mappings. Proc. Math. Phys. Soc. Egypt., **53** (1982), 47-53.

[12] A.S. Mashhour, I.A. Hasanein and S.N. EL-Deeb. α-continuous and α-open mappings. Acta Math. Hung., **41** (1983), 213-218.

[13] M. Mršević, I.L. Reilly and M.K. Vamanamurthy. On semi-regularization properties. J. Austral, Math. Soc. (Series A), **38** (1985), 40-54.

[14] O. Njåstad. On some classes of nearly open sets. Pacific J. Math., **15** (1965), 961-970.

[15] T. Noiri. A generalization of closed mappings. Atti. Accad. Naz. Lince Rend. Cl. Sci. Fis. Mat. Natur., **8** (1973), 210-214.

[16] T. Noiri. Properties of hyperconnected spaces. Acta Math. Hung., **66** (1995), 147-154.

[17] D.A. Rose. On weak openness and almost openness. Internat. J. Math. & Math. Sci., **7** (1984), 35-40.

[18] D.A. Rose and D.S. Janković. Weakly closed functions and Hausdorff spaces. Math. Nachr., **130** (1987), 105-110.

[19] M.K. Singal and A.R. Singal. Almost continuous mappings. Yokohama Math. J., **16** (1968), 63-73.

[20] N.V. Veličko. H-closed topological spaces. Amer. Math. Soc. Transl., **78** (1968), 103-118.

[21] S. Willard. General topology. Addition Wesley Publishing Company (1970).

Resumen

En este trabajo introducimos dos nuevas classes de funciones, llamadas funciones débilmente δ-abiertas y funciones débilmente δ-cerradas. Obtenemos sus caracterizaciones, sus propiedades fundamentales y sus relaciones con otros tipos de funciones entre espacios topológicos.

Palabras clave: Conjuntos δ- abiertos, conjuntos δ-cerrados, funciones débilmente δ-cerradas, espacios extremadamente disconexos, espacios cuasi H-cerrados.

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