AN EXAMPLE OF TWO NON-UNITARILY EQUIVALENT COMPACT OPERATORS WITH THE SAME TRACES AND KERNEL

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Abstract

We give examples of two non nuclear non normal Hilbert-Schmidt operators that are not unitarily equivalent but have the same traces and kernel.

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1 Introduction

We begin reviewing briefly some properties of compact operators on Hilbert spaces that will be needed later on [3]. If H is a Hilbert space and $B(H)$ is the set of bounded operators on H, an element $A \in B(H)$ is said to be compact if it transforms weakly convergent sequences in H into strongly convergent sequences. If $K(H)$ is the set of compact operators on H then $K(H) \neq B(H)$ if and only if $\dim H = \infty$, since in this case the identity operator I is not compact. Also when $\dim H = \infty, K(H)$ is the only norm closed two sided proper ideal in $B(H)$ (Calkin's theorem).

If $A \in B(H)$ the resolvent set $\rho(A)$ of A is the set

$$\{\lambda \in \mathbb{C} : (\lambda I - A)^{-1} \in B(H)\}$$

and the spectrum $\sigma(A)$ of A is the nonempty compact set $\mathbb{C}\backslash\rho(A)$. The function $\lambda \mapsto (\lambda I - A)^{-1}$ is an analytic function from $\rho(A)$ into $B(H)$ and it is called the resolvent of A. When A is compact $\sigma(A)$ is at most countable with 0 as its only possible limit point, moreover $0 \in \sigma(A)$ if $A \in K(H)$ and $\dim H = \infty$.

If A is a compact operator then every $\mu \in \sigma(A)\backslash\{0\}$ is an eigenvalue, that is $\ker(\mu I - A) \neq \{0\}$ and $\dim\ker(\mu I - A) < \infty$ is called the geometric multiplicity of the eigenvalue μ; for such μ the number

$$\dim \bigcup_{k=1}^{\infty} \ker(\mu I - A)^k$$

is also finite and it is called its algebraic multiplicity. Evidently the geometric multiplicity is not greater than the algebraic multiplicity, but for normal compact operators they are equal. If $\mu \in \sigma(A)\backslash\{0\}$, A is compact and $\epsilon > 0$ is such that $\Omega = \{\lambda \in \mathbb{C} : 0 < |\lambda - \mu| < \epsilon\}$ does not meet $\sigma(A)$, then in Ω is valid the following Laurent series expansion for the resolvent.

$$(\lambda I - A)^{-1} = \sum_{n=1}^{q} (\lambda - \mu)^{-n}(A - \mu I)^{n-1}P + W(\lambda), \quad q < \infty,$$

where W is analytic in $\Omega \cup \{\mu\}$ and P is the projection onto the finite dimensional subspace $\bigcup_{k=1}^{\infty} \ker(\mu I - A)^k$. Moreover it holds that

$$P^2 = P, AP = PA, (A - \mu I)^n P \neq 0$$

if $0 \leq n \leq q-1, (A - \mu I)^q P = 0$ and if μ has geometric multiplicity one then $q =$ algebraic multiplicty of $\mu = \dim \bigcup_{k=1}^{\infty} \ker(\mu I - A)^k$. For compact normal operators $q = 1$.

If A is a compact operator, let $\{s_n^2(A) : n \in \mathbb{N}\}$ be the sequence of eigenvalues of A^*A, each one of them repeated a number of times equal to its algebraic multiplicity and ordered in such a way that

$$s_n(A) \geq s_{n+1}(A) \geq 0 \; \forall n \in \mathbb{N}.$$

If $0 < p < \infty$ we say that $A \in K_p(H)$ when

$$\sum_{n=1}^{\infty} s_n(A)^p < \infty; K_p(H)$$

is a two sided ideal in $B(H)$. Until now the more useful of these ideals have been $K_1(H)$, the ideal of nuclear operators, and $K_2(H)$, the ideal of Hilbert-Schmidt operators. If $A, B \in K_2(H)$ then $AB \in K_1(H)$. Let $\{\mu_n(A)\}_{n\geq 1}$ be the sequence of eigenvalues of the compact operator A, ordered in such a way that $|\mu_n(A)| \geq |\mu_{n+1}(A)| \forall n \in \mathbb{N}$ and each one of them being counted according to its algebraic multiplicity, then $\forall A \in K_p(H)$ hold the inequalities of Weyl,

$$\sum_{j=1}^{n} |\mu_j(A)|^p \leq \sum_{j=1}^{n} s_j(A)^p \qquad \forall n \geq 1.$$

Therefore if $A \in K_1(H)$ the series $\sigma_1(A) = \sum_{j=1}^{\infty} \mu_j(A)$ converges absolutely and it is called the trace of A. The trace of A^r is denoted by $\sigma_r(A)$. There are two results of Lidskii that enable us in some cases to find the trace without prior knowledge of the spectrum: if $\{\varphi_n\}_{n\geq 1}$ is a complete orthonormal set in H then $\sigma_r(A) = \sum_{n=1}^{\infty}\langle A^r\varphi_n, \varphi_n\rangle$ and when A is an integral operator $\sigma_r(A)$ is given as an integral of the kernel of the operator A^r. If $A \in K_1(H)$ then $\mu \in \sigma(A)\setminus\{0\}$ if and only if $D(\mu^{-1}) = \det_1(I - \mu^{-1}A) = 0$ where D is the entire function, known as the Fredholm determinant of $I - \mu^{-1}A$, given by the following formulae of Plemelj,

$$D(\lambda) = \sum_{n=0}^{\infty} d_n\lambda^n, \qquad d_0 = 1,$$

$$d_n = \frac{(-1)^n}{n!}\begin{vmatrix} \sigma_1(A) & n-1 & 0 & \cdots & 0 & 0 \\ \sigma_2(A) & \sigma_1(A) & n-2 & \cdots & 0 & 0 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \sigma_{n-1}(A) & \sigma_{n-2}(A) & \sigma_{n-3}(A) & \cdots & \sigma_1(A) & 1 \\ \sigma_n(A) & \sigma_{n-1}(A) & \sigma_{n-2}(A) & \cdots & \sigma_2(A) & \sigma_1(A) \end{vmatrix} \quad n \geq 1.$$

If k is the algebraic multiplicity of μ then $D^{(l)}(\mu^{-1}) = 0$ for $0 \leq l \leq k-1$ and $D^{(k)}(\mu^{-1}) \neq 0$. The Hadamard factorization of the entire function $\det_1(I - \lambda A)$ is

$$\prod_{1=1}^{\infty}[1 - \lambda\mu_i(A)].$$

Hilbert proved that if $A \in K_2(H)$ the above formulae for the Fredholm determinant hold if we take $\sigma_1(A) = 0$. The function defined in this way is called the modified or renormalized Fredholm determinat for Hilbert-Schmidt operators and it is denoted by $\det_2(I - \lambda A)$. The Hadmard factorization of the entire function $\det_2(I - \lambda A)$ is

$$\prod_{i=1}^{\infty}[1 - \lambda\mu_i(A)]e^{\lambda\mu_i(A)}.$$

In a previous work [1] we reformulated the Riemann Hypothesis as a problem of Functional Analysis by means of the following theorem.

Theorem. *Let* $[A_\rho f](\theta) = \int_0^1 \rho(\theta/x)f(x)dx$*, where*

$$\rho(x) = x - [x], x \in \mathbb{R}, [x] \in \mathbb{Z}, [x] \leq x \leq [x] + 1,$$

be considered as an operator on $L^2(0,1)$*. Then the Riemann Hypothesis holds if and only if* $\ker A_\rho = \{0\}$.

Among other things, we also proved that:

i) A_ρ is Hilbert-Schmidt, but neither nuclear nor normal.

ii) $\lambda \neq 0$ is an eigenvalue of A_ρ if and only if $T(\lambda^{-1}) = 0$ where

$$T(\mu) = 1 - \mu + \sum_{r=1}^{\infty}(-1)^{r+1}\frac{\prod_{l=1}^{r}\zeta(l+1)}{(r+1)!(r+1)}\mu^{r+1}$$

is an entire function of order one and type one. Moreover each nonzero eigenvalue $\lambda = \mu^{-1}$ has geometric multiplicity one and associated eigenfunction $\psi_\mu(x) = \mu x T'(\mu x)$.

iii) If $\{\lambda_n\}_{n\geq 1}$ is the sequence of nonzero eigenvalues of A_ρ where the ordering is such that $|\lambda_n| \geq |\lambda_{n+1}| \forall n \in \mathbb{N}$ and each one of them being repeated according to its algebraic multiciplity then the largest eigenvalue λ_1 is positive and has algebraic multiplicity one, $|\lambda_n| \leq \frac{e}{n} \forall n \in \mathbb{N}, \sum_{n=1}^{\infty} |\lambda_n| = \infty$ and $\lambda_n \notin \mathbb{R}$ for an infinite number of n's.

iv) If $D^*(\mu) = \det_2(I - \mu A_\rho)$ is the modified Fredholm determinant of $I - \mu A_\rho$ then $D^*(\mu) = e^\mu T(\mu) \forall \mu \in \mathbb{C}$.

By this same method it is shown in [2] that if $Q_f\xi =< \xi, f > h$ then

$$\begin{aligned} \det_2(I - \mu A_\rho - \mu Q_f) &= e^{(i+<h,f>)\mu} \times \\ &\times (T(\mu) + < \psi_\mu, f >), \qquad \mu \in \mathbb{C} \end{aligned} \tag{1}$$

Note also that

$$A_\rho \psi_\mu = \mu^{-1}\psi_\mu + T(\mu)h, \qquad \mu \in \mathbb{C}\backslash\{0\}$$

If in equation (1) we take $f_\alpha(x) = -\alpha^{-1}\rho(\alpha/x)$, $\alpha \in]0,1]$, bearing in mind that

$$\begin{aligned} < h, f_\alpha > &= \frac{\zeta(2)}{2}\alpha - 1 \quad \text{and} \\ < \psi_\mu, f_\alpha > &= -T(\mu) - \alpha T'(\mu\alpha) \end{aligned}$$

we get that the operators $\alpha[A_\rho + Q_{f_1}]$ and $A_\rho + Q_{f_\alpha}$ where $\alpha \in]0,1[$ have the same nonzero eigenvalues with the same algebraic and geometric multiplicities. It can be shown that these operators have the same kernel and that they are not unitarily equivalent. The kernel of these operators is trivial if and only if the Riemann Hypothesis is true. In particular, it holds that

$$\sigma_n(\alpha[A_\rho + Q_{f_1}]) = \sigma_n(A_\rho + Q_{f_\alpha}) \qquad \forall n \geq 2.$$

These operators are Hilbert-Schmidt but not nuclear nor normal. The non unitary equivalence is shown by proving that

$$\sigma_1(\alpha^2(A_\rho + Q_{f_1})^*(A_\rho + Q_{f_1})) \neq \sigma_1((A_\rho + Q_{f_\alpha})^*(A_\rho + Q_{f_\alpha}))$$

References

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Resumen

Se dan ejemplos de 2 operadores Hilbert-Schmidt no nucleares, no normales que tienen las mismas trazas y núcleo y no son unitariamente equivalentes.

Palabras clave: Determinantes de Fredholm, trazas, equivalencia unitaria.

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