GENERALIZED CLOSED SETS VIA IDEALS AND OPERATORS

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Abstract

Given a topological space (X, τ), *three operators* α, β, γ *associated to a topology* τ *and* I *an ideal on* X. *The concepts of:* α*-closed set,* α*-semi closed set,* (α, β)*-semi closed set an* (I, γ) *g-closed set are generalized. Also new separation axioms are introduced and characterized and new spaces are obtained in such way that the spaces* $\alpha - T_{\frac{1}{2}}$, α*-semi-*$T_{\frac{1}{2}}$, (α, β)*-semi-*$T_{\frac{1}{2}}$ *and* $\gamma - T_I$ *respectively are generalized.*

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1 Introduction

The study of the concepts of the notions of generalized closed sets goes back to the classic paper of N. Levine [9], where using the basic definition of closed set, he introduces the notion of generalized closed set (abbreviated g-closed) and using this concept in order to define the $T_{\frac{1}{2}}$ spaces. Later on Dunham [5] $T_{\frac{1}{2}}$ spaces are characterized proving that a topological space is $T_{\frac{1}{2}}$ if and only if the unitary sets are open or closed. Khalimsky et al. [7] shown that the digital line is typical example of $T_{\frac{1}{2}}$ space. Using the same idea, many authors have defined and studied many types of generalized closed sets in order to introduce new separations axioms and new spaces.

Recently J Donchev et al. [4], using the theory of topological ideals introduced by R. L. Newcomb [13], the local function defined by D. Jankovic et al. [6] and the operator theory introduced by Kasahara [8], provide the definition of (I, γ) generalized closed sets and they introduced a class of spaces denominated $\gamma - T_I$ spaces, that are a generalization of the $T_{\frac{1}{2}}$ spaces given by Levine [9].

In this paper a new variant of a local function given by Jancovik et al. [6],is introduced, in order to define new concepts of g-closed sets that generalize the notions of (I, γ) g-closed sets [4], sg-closed set [1], g$\alpha-$closed [11], $\alpha-$sg-closed set [16], $(\alpha, \beta)-$sg-closed set [17], etc. It can be used in order to introduce new spaces and new separation axioms that generalize the well known results given in [2], [4], [9], [15], [16] and [17].

2 Preliminary

Let X be a nonempty set, we say that $\alpha : \mathcal{P}(X) \to \mathcal{P}(X)$ is an expansive operator on a family Γ of subsets of X if $U \subset \alpha(U)$ for all $U \in \Gamma$. If (X, τ) is a topological space and α is expansive on the topology τ, then, we say that α is an associated operator on the topology τ [2].

We denote by (X, τ, α) the topological space (X, τ) with the operator α associated to the topology, also, if $\alpha(A) \subseteq \alpha(B)$ whenever $A \subseteq B$, we say that the operator α is monotone.

Let (X, τ, α) and A be a subset of X, we say that A is α-open [8] if for each $x \in A$ there exists an open set U containing x such that $\alpha(U) \subseteq A$. The complement of an α-open set is called α-closed set. It is easy to prove that every α-closed set is closed and the intersection of an arbitrary family of α-closed sets is an α-closed set, in this way, we can define the α-closure of a subset A of X, denoted by $\alpha - cl(A)$, as the intersection of all α-closed sets that contain A. In this case, we can see that $x \in \alpha - cl(A)$ if and only if for all α-open set U containing x, $U \cap A \neq \emptyset$. We say that the set A is an α-generalized closed, denoted by α-g-closed, if $\alpha - cl(A) \subset U$ whenever $A \subset U$ and U is α-open. Every α-closed set is α-g-closed. We say that X is an $\alpha - T_{\frac{1}{2}}$ space if all α-g-closed set is α-closed.

We say that A subset of X is an α-semi-open ([2]) if there exists an open set $U \in \tau$ such that $U \subseteq A \subseteq \alpha(U)$. The complement of an α-semi-open set is called α-semi-closed set. All closed set is an α-semi-closed set, in general, the intersection of an arbitrary family of α-semi closed sets is not an α-semi closed set; but, if we consider that α is a monotone operator, then the intersection of an arbitrary family of α-semi-closed sets is an α-semi-closed set, in this case, we can define the α-semi-closure of A, denoted by $\alpha - scl(A)$, as the intersection of all α-semi-closed sets containing A; it verifies that $x \in \alpha - scl(A)$ if and only if all α-semi-open set U containing x, $U \cap A \neq \emptyset$. We say that A is an α-semi-generalized closed set, denoted by α-sg-closed, if $\alpha - scl(A) \subseteq U$ whenever $A \subseteq U$ and U is an α-semi-open set. If α is a monotone operator, all α-semi-closed set is an α-sg-closed set. We say that X is an $\alpha - semiT_{\frac{1}{2}}$ space if all α-sg-closed set is an α-semi-closed set.

If β is another associated operator with τ, then we say that A subset of X is an (α, β)-semi-open set [15] if for each $x \in A$ there exists a β-semi-open set V such that $x \in V$ and $\alpha(V) \subseteq A$. The complement of an (α, β)-semi-open set is called (α, β)-semi-closed set. We can see the following [15]:

1. If A is an open set, then A is an (id, β)-semi-open.
2. If $\alpha = \beta = id$, A is an (α, β)-semi-open set if and only if A is an open set.
3. If $\beta = id$ and α is an arbitrary operator then A is an (α, β)-semi-open set if and only if A is an α-open set [8].
4. If $\alpha = id$, β is a monotone operator then the collection of all (α, β)-semi-open sets agree with the collection of all β-semi-open sets.

The intersection of an arbitrary family of (α, β)-semi-closed sets is an (α, β)-semi-closed set, and we define the (α, β)-semi-closure of A, denoted by $(\alpha, \beta) - scl(A)$, as the intersection of all (α, β)-semi-closed sets containing A; we can see that $x \in (\alpha, \beta) - scl(A)$ if and only if for all (α, β)-semi-open set U containing x, $U \cap A \neq \emptyset$. We say that A is an (α, β)-semi-generalized closed set, abbreviated by (α, β)-sg-closed, if $(\alpha, \beta) - scl(A) \subset U$ whenever $A \subset U$ and U is an (α, β)-semi-open set. All (α, β)-semi-closed set is an (α, β)-sg-closed set. We say that X is an $(\alpha, \beta) - semiT_{\frac{1}{2}}$ space if all (α, β)-sg-closed set is an (α, β)-semi-closed set.

3 Generalized Local Function

In this section, we generalize the concept of local function given in [6]. Also, we study some of its properties.

Definition 3.1 *A non-empty collection I of subsets of a set X is said to be an ideal on X if it satisfies the following two conditions.*

1. *If $A_1 \in I$ and $A_2 \in I$, then $A_1 \cup A_2 \in I$.*

2. *If $A_1 \in I$ and $A_2 \subset A_1$, then $A_2 \in I$.*

Definition 3.2 *Let X be a set, $\mathfrak{F}$ be a collection of subsets of X and I be an ideal on X. The generalized local function with respect to I, is a map that assign each subset A of X, the set $A^*(I, \mathfrak{F})$, defined as follows:*

$$A^*(I, \mathfrak{F}) = \{x \in X : A \cap U_x \notin I \text{ for all } U_x \in \mathfrak{F} \text{ such that } x \in U_x\}.$$

In the above definition $A^*(I, \mathfrak{F})$ can be empty and in general can not contain A

Remark 3.1 *Observe that*

1. *When $\mathfrak{F} = \tau$ a topology on X, the concept of generalized local function agree with the concept of local function given in [6].*

2. *When $\mathfrak{F} = \tau$ a topology on X and the ideal is $\{\emptyset\}$, then $A^*(I, \mathfrak{F}) = cl(A)$.*

We describe some properties that satisfies $A^*(I, \mathfrak{F})$.

Theorem 3.1 *Let X be a set, $\mathfrak{F}$ be a collection of subsets of X, A and B subsets of X, then*

1. *If $I = \mathcal{P}(X)$, then $A^*(I, \mathfrak{F}) = \emptyset$.*

2. *If $A \subset B$, then $A^*(I, \mathfrak{F}) \subset B^*(I, \mathfrak{F})$.*

3. *$\emptyset^*(I, \mathfrak{F}) = \emptyset$.*

4. *$A^*(I, \mathfrak{F}) \cup B^*(I, \mathfrak{F}) \subset (A \cup B)^*(I, \mathfrak{F})$.*

5. *$(A^*(I, \mathfrak{F}))^*(I, \mathfrak{F}) \subset A^*(I, \mathfrak{F})$.*

6. *If J is an ideal on X such that $I \subset J$, then $A^*(J, \mathfrak{F}) \subset A^*(I, \mathfrak{F})$.*

Definition 3.3 *Let X be a set, $\mathfrak{F}$ be a family of subsets X, I be an ideal on X and A a subset of X. We define the $\mathfrak{F}$ closure of A, denoted by $\mathfrak{F} - cl^*(A)$ as: $\mathfrak{F} - cl^*(A) = A \cup A^*(I, \mathfrak{F})$.*

Remark 3.2 *If τ is a topology on X and $\mathfrak{F} = \tau$, then $\mathfrak{F} - cl^*$ is a Kuratowski operator and therefore it induce a topology on X denoted by $\tau^*(I)$.*

It is easy to prove that $\mathfrak{F} - cl^*$ satisfies the following properties:

Theorem 3.2 *Let X be a set, $\mathfrak{F}$ be a family of subsets of X, I be an ideal on X, A and B subsets of X, then:*

1. $A \subset \mathfrak{F} - cl^*(A)$.
2. *If* $A \subset B$, *then* $\mathfrak{F} - cl^*(A) \subset \mathfrak{F} - cl^*(B)$.
3. $\mathfrak{F} - cl^*(\emptyset) = \emptyset$.
4. $\mathfrak{F} - cl^*(\mathfrak{F} - cl^*(A)) \subset \mathfrak{F} - cl^*(A)$.
5. $\mathfrak{F} - cl^*(A) \cup \mathfrak{F} - cl^*(B) \subset \mathfrak{F} - cl^*(A \cup B)$.

4 $(I, \gamma)_\alpha$-g-closed Sets and $(\alpha, \gamma) - T_I$ Spaces

In this section, we show that using the concept of generalized local function, we can obtain immediately the notion of α-g-closed set given in [9].

Let (X, τ, α), I an ideal on X and τ_α the collection of all α-open sets in X, then the generalized local function taking $\mathfrak{F} = \tau_\alpha$ satisfies the following properties:

Theorem 4.1 *Let (X, τ, α), I be an ideal on X and A be a subset of X, then*

1. $A^*(I, \tau_\alpha) \subset \alpha - cl(A)$.
2. *If* $I = \{\emptyset\}$ *and* α *is any operator, then* $A^*(I, \tau_\alpha) = \alpha - cl(A)$.

3. *If $\alpha = id$, I any ideal, then $A^*(I, \tau_\alpha) = A^*(I, \tau)$.*

4. $A^*(I, \tau_\alpha) = \alpha - cl\,(A^*(I, \tau_\alpha)) \subset \alpha - cl(A)$.

Proof.

1. Let $x \in A^*(I, \tau_\alpha)$, then for all α-open set U_x such that $x \in U_x$ we have $U_x \cap A \notin I$, then $U_x \cap A \neq \emptyset$ therefore $x \in \alpha - cl(A)$.

2. Let $x \in \alpha - cl(A)$, then for all α-open set U_x such that $x \in U_x$ we have $U_x \cap A \neq \emptyset$, that is, $U_x \cap A \notin \{\emptyset\} = I$.

3. If α is the identity operator, the collection of all α-open sets agree with the collection of all open sets.

4. Let $x \in A^*(I, \tau_\alpha)$, then for all α-open set U_x such that $x \in U_x$ we have $U_x \cap A^*(I, \tau_\alpha) \neq \emptyset$, that is $x \in \alpha - cl\,(A^*(I, \tau_\alpha))$. If $x \in \alpha - cl\,(A^*(I, \tau_\alpha))$, then for all α-open set U_x such that $x \in U_x$, $U_x \cap A^*(I, \tau_\alpha) \neq \emptyset$, that is, $U_x \cap A \notin I$ and therefore $x \in A^*(I, \tau_\alpha)$.

Finally, the last part of 4, follows from 1. □

Remark 4.1 *If in Definition 3.3, we take $\mathfrak{F} = \tau_\alpha$, and denote by $\tau_\alpha - cl^*(A)$ for $\alpha - cl^*(A)$. We observe that,*

1. *If $\alpha = id$, then $\alpha - cl^*(A) = cl^*(A)$.*

2. *If $I = \{\emptyset\}$, then $\alpha - cl^*(A) = \alpha - cl(A)$.*

Now we introduce our generalization of the concepts of α-g-closed sets and (I, γ)-g-closed sets given in [9] and [4] respectively.

Consider now, the triple (X, τ, I) where I is an ideal defined on X, as the topological space.

Definition 4.1 *Let (X, τ, I), and consider two operators α, γ associated with τ. A subset A of X is said to be $(I, \gamma)_\alpha$-generalized closed set, abbreviated $(I, \gamma)_\alpha$-g-closed, if $\alpha - cl^*(A) \subset \gamma(U)$ whenever $A \subset U$ and $U \in \tau_\alpha$.*

Remark 4.2 *When α is the identity operator, the (I,γ)-g-closed sets and the $(I,\gamma)_\alpha$-g-closed sets are the same. In the case that $I = \{\emptyset\}$, the (I, id)-g-closed sets are α-g-closed sets.*

We can resume the above in the following:

Theorem 4.2 *Let (X, τ, I), $A \subset X$, α and γ be two operators associated with τ.*

1. *If $\alpha = id$, A is an (I,γ)-g-closed set [4] if and only if A is an $(I,\gamma)_\alpha$-g-closed set.*
2. *If $I = \{\emptyset\}$ and A is an $(I, id)_\alpha$-g-closed set, then A is an α-g-closed set [16].*
3. *If $\alpha = \gamma = id$ and $I = \{\emptyset\}$, A is a g-closed set [9] if and only if A is an $(I,\gamma)_\alpha$-g-closed set.*

Proof.

1. Suppose that $\alpha = id$ and A is an $(I,\gamma)_\alpha$-g-closed set, let U an open set such that $A \subset U$, then $cl^*(A) \subset \gamma(U)$, since $A^*(I,\tau) \subset cl^*(A)$, we conclude that A is an (I,γ)-g-closed set.

   Reciprocally, suppose that A is an (I,γ)-g-closed set, let U an α-open set such that $A \subset U$. Since $\alpha = id$, then U is open and therefore $A^*(I,\tau) \subset \gamma(U)$, using Theorem 4.1, $A^*(I,\tau) = A^*(I,\tau_\alpha)$, in consequence $\alpha - cl^*(A) \subset \gamma(U)$.

2. Suppose that A is an $(I, id)_\alpha$-g-closed set of X and let U an α-open set such that $A \subset U$, then $\alpha - cl^*(A) \subset id(U)$, since $I = \{\emptyset\}$; it follows that $\alpha - cl^*(A) = \alpha - cl(A)$, in consequence A is an α-g-closed set.

3. Is an immediate consequence of parts 1., 2. □

Theorem 4.3 *Let (X, τ, I), α and γ two operators associated with τ. Then all α-g-closed set is an $(I,\gamma)_\alpha$-g-closed set.*

The following example shows the existence of a set that is $(I,\gamma)_\alpha$-g-closed but is not α-g-closed.

Example 4.1 *Consider* $\mathbb{R}$, *the set of real numbers with the finite complement topology* $\tau_f = \{U \subset \mathbb{R} : \mathbb{R} \setminus U$ *is finite or* $\mathbb{R}\}$, *and the operator* $\alpha : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ *associated with the topology defined as* $\alpha(U) = int(U)$. *The set of the rational numbers* $\mathbb{Q}$, *is not an* α*-g-closed set, because* $\mathbb{R}\setminus\{\sqrt{2}\}$ *is an* α*-open set that contains* $\mathbb{Q}$*; but* $\alpha-cl(\mathbb{Q}) = \mathbb{R} \not\subseteq \mathbb{R}\setminus\{\sqrt{2}\}$. *We prove that,* $\mathbb{Q}$ *is an* $(I,\gamma)_\alpha$*-g-closed set, if we consider the ideal* $I = \mathcal{P}(\mathbb{R})$ *and* γ *any operator associated with the topology.*

Using the fact that there exists $(I,\gamma)_\alpha$-g-closed sets that are not α-g-closed. We introduce a new class of spaces.

Definition 4.2 *Let* (X,τ,I), α *and* γ *be two operators associated with* τ. *We say that* X *is an* $(\alpha,\gamma)-T_I$ *space if all* $(I,\gamma)_\alpha$*-g-closed set is an* α*-closed set.*

If we analyze the above definition, we can see that it gives us a general context in comparison with the one described in [4]. The following theorem indicates that taking adequate operators and ideals, we can obtain as particular cases the following well known results in the literature

Theorem 4.4 *Let* (X,τ,I), α *and* γ *be two operators associated with* τ.

1. *If* $\alpha = id$ *and* X *is an* $(\alpha,\gamma)-T_I$, *then* X *is* $\gamma - T_I$ *[4].*

2. *If* X *is an* $(\alpha,\gamma)-T_I$ *then* X *is* $\alpha - T_{\frac{1}{2}}$ *[16].*

3. *If* $I = \{\emptyset\}$, $\gamma = id$ *and* X *is an* $\alpha - T_{\frac{1}{2}}$, *then* X *is* $(\alpha,\gamma)-T_I$.

4. *If* $I = \{\emptyset\}$, $\gamma = \alpha = id$. X *is* $(\alpha,\gamma)-T_I$ *if and only if* X *is* $T_{\frac{1}{2}}$ *[9].*

5 $(I, \gamma)_\alpha$-sg-closed Sets and $(\alpha, \gamma) - Semi - T_I$ Spaces

In the same way as in the above section, we use the generalized local function in order to obtain the concept of α-sg-closed set given in [16].

Let (X, τ, α) where α a monotone operator, I an ideal on X and $\alpha - SO(X)$ the collection of all α-semi-open sets in X, then the generalized local function taking $\mathfrak{F} = \alpha - SO(X)$ satisfies the following properties:

Theorem 5.1 *Let A be a subset of X an α be a monotone operator then:*

1. $A^*(I, \alpha - SO(X)) \subset \alpha - scl(A)$.
2. *If* $I = \{\emptyset\}$, *then*

$$A^*(I, \alpha - SO(X)) = \alpha - scl(A).$$

3. *If* $\alpha = id$ *and* I *is any ideal, then* $A^*(I, \alpha - SO(X)) = A^*(I, \tau)$.
4. $A^*(I, \alpha - SO(X)) = \alpha - scl\,(A^*(I, \alpha - SO(X))) \subset \alpha - scl(A)$, *that is,* $A^*(I, \alpha - SO(X))$ *is an* α*-semi-closed set.*

Proof.

1. Let $x \in A^*(I, \alpha - SO(X))$, then for all α-semi-open set U_x such that $x \in U_x$, we have $U_x \cap A \notin I$, then $U_x \cap A \neq \emptyset$. Therefore $x \in \alpha - scl(A)$.
2. Let $x \in \alpha - scl(A)$, then for all α-semi-open set U_x such that $x \in U_x$ we have $U_x \cap A \neq \emptyset$, that is, $U_x \cap A \notin \{\emptyset\} = I$.
3. If α is the identity operator, the α-semi-open sets are the same as the open sets.

4. Let $x \in A^*(I, \alpha - SO(X))$ then for all α-semi-open set U_x such that $x \in U_x$ we have $U_x \cap A^*(I, \alpha - SO(X)) \neq \emptyset$; that is $x \in \alpha - scl\,(A^*(I, \alpha - SO(X)))$.

   If $x \in \alpha - scl\,(A^*(I, \alpha - SO(X)))$, then for all α-semi-open set U_x such that $x \in U_x$ we have $U_x \cap A^*(I, \alpha - SO(X)) \neq \emptyset$, then there exists $y \in U_x \cap A^*(I, \alpha - SO(X))$, that is, $y \in A^*(I, \alpha - SO(X))$ and U_x is an α-semi-open set containing y,it follows that $U_x \cap A \notin I$, in consequence $x \in A^*(I, \alpha - SO(X))$ and therefore $\alpha - scl\,(A^*(I, \alpha - SO(X))) \subset A^*(I, \alpha - SO(X))$.

   Finally, the last part of 4, follows from 1. □

Remark 5.1 *If $\mathfrak{F} = \alpha - SO(X)$. In Definition 3.3, we denote $\mathfrak{F} - cl^*(A)$ by $\alpha - scl^*(A)$ that is, $\alpha - scl^*(A) = A \cup A^*(I, \alpha - SO(X))$. Also satisfies the following properties*

1. *$\alpha - scl^*(A) \subset \alpha - scl(A)$, for all monotone operator α.*
2. *If $\alpha = id$, then $\alpha - scl^*(A) = cl^*(A)$.*
3. *If $I = \{\emptyset\}$ and α is a monotone operator, then $\alpha - scl^*(A) = \alpha - scl(A)$.*

Definition 5.1 *Let (X, τ, I), and consider two operators α, γ associated with τ. A subset A of X is said to be $(I, \gamma)_\alpha$-semi-generalized closed set, abbreviated $(I, \gamma)_\alpha$-sg-closed, if $\alpha - scl^*(A) \subset \gamma(U)$ whenever $A \subset U$ and $U \in \alpha - SO(X)$.*

Remark 5.2 *Observe that when α is the identity operator, the (I, γ)-g-closed sets and $(I, \gamma)_\alpha$-sg-closed sets agree. If we choose $I = \{\emptyset\}$, then all $(I, id)_\alpha$-sg-closed set is a g-closed set.*

We can resume the above in the following theorem:

Theorem 5.2 *Let (X, τ, I), $A \subset X$, α and γ be two operators associated with τ an α monotone.*

1. *If $\alpha = id$, the set A is an (I,γ)-g-closed set ([4]) if and only if A is an $(I,\gamma)_\alpha$-sg-closed set.*

2. *If $I = \{\emptyset\}$ and the set A is an $(I, id)_\alpha$-sg-closed, then A is an α-sg-closed ([16]).*

3. *If $\alpha = id$, $I = \{\emptyset\}$, and A is an $(I, id)_\alpha$-sg-closed, then A is a g-closed set ([9]).*

Proof.

1. Suppose that $\alpha = id$ and A is an $(I,\gamma)_\alpha$-sg-closed set. Let U an open set such that $A \subset U$, since all open set is an α-semi-open set, then $\alpha - scl^*(A) \subset \gamma(U)$. Since $\alpha = id$, we have that $cl^*(A) \subset \gamma(U)$ and therefore $A^*(I,\tau) \subset cl^*(A) \subset \gamma(U)$, it follows that A is an (I,γ)-g-closed set.

   The converse follows in the same way.

2. Suppose that A is an $(I, id)_\alpha$-sg-closed set of X and U an α-semi-open set such that $A \subset U$, then $\alpha - scl^*(A) \subset id(U)$, since $I = \{\emptyset\}$; it follows $\alpha - scl^*(A) = \alpha - scl(A)$ in consequence, A is an α-sg-closed set.

3. It is an immediate consequence of parts 1., 2. □

Theorem 5.3 *Let (X, τ, I), α be a monotone operator associated with τ and γ an expansive operator on $\alpha - SO(X)$. Then all α-sg-closed set is an $(I,\gamma)_\alpha$-sg-closed set.*

Proof.
Let A an α-sg-closed subset of X and U an α-semi-open set such that $A \subset U$, then $\alpha - scl(A) \subset U$; therefore $\alpha - scl^*(A) \subset U$, since γ is expansive on $\alpha - SO(X)$, then $U \subset \gamma(U)$, it follows that A is an $(I,\gamma)_\alpha$-sg-closed set. □

The following example shows the existence of an $(I,\gamma)_\alpha$-sg-closed set that is not an α-sg-closed set.

Example 5.1 *Consider $\mathbb{R}$, the set of the real numbers, with the finite complement topology $\tau_f = \{U \subset \mathbb{R} : \mathbb{R} \setminus U$ is finite or $\mathbb{R}\}$, the operator $\alpha : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ associated with this topology is defined as $\alpha(U) = cl(U)$.*

The open interval $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ is not an α-sg-closed set because $\mathbb{R} \setminus \{b\}$ is an α-semi-open set containing (a, b) and the $\alpha - scl(A) \nsubseteq \mathbb{R} \setminus \{b\}$. (a, b) is an $(I, \gamma)_\alpha$-sg-closed set when we consider the ideal $I = \mathcal{P}(X)$ and γ the identity operator.

We have shown the existence of an $(I, \gamma)_\alpha$-sg-closed set that is not an α-semi-closed. Now we introduce a new class of spaces in the following definition

Definition 5.2 *Let (X, τ, I), α and γ be two operators associated with τ. We say that X is an $(\alpha, \gamma) - semi - T_I$ space if all $(I, \gamma)_\alpha$-sg-closed set is an α-semi-closed set.*

Theorem 5.4 *Let (X, τ, I), and consider two operators α, γ associated with τ and α monotone.*

1. *If $\alpha = id$ and X is an $(\alpha, \gamma) - semi - T_I$ space, then X is a $\gamma - T_I$ space ([4]).*

2. *If X is an $(\alpha, \gamma) - semi - T_I$ space and γ is expansive on $\alpha - SO(X)$, then X is an $\alpha - semi - T_{\frac{1}{2}}$ space ([16]).*

3. *If $I = \{\emptyset\}$ and X is an $\alpha - semi - T_{\frac{1}{2}}$ space, then X is an $(\alpha, id) - semi - T_I$ space.*

4. *If $I = \{\emptyset\}$ and $\gamma = \alpha = id$. X is an $(\alpha, \gamma) - semi - T_I$ space if and only if X is an $T_{\frac{1}{2}}$ space ([9]).*

6 $(I,\gamma)_{(\alpha,\beta)}$-sg-closed Sets and $(\alpha,\beta,\gamma)-semi-T_I$spaces

Consider (X,τ), I an ideal on X, α and β operators on τ and $(\alpha,\beta)-SO(X)$ the collection of all (α,β)-semi-open sets in X, then the generalized local function taking $\mathfrak{F}=(\alpha,\beta)-SO(X)$ satisfies the following properties:

Theorem 6.1 *Let A be a subset of X, then:*

1. $A^*(I,(\alpha,\beta)-SO(X))\subset(\alpha,\beta)-scl(A)$.
2. *If $I=\{\emptyset\}$, α and β any two operators, then* $A^*(I,(\alpha,\beta)-SO(X))=(\alpha,\beta)-scl(A)$.
3. *If $\alpha=id$, β a monotone operator and I any ideal, then* $A^*(I,(\alpha,\beta)-SO(X))=A^*(I,\beta-SO(X))$.
4. *If $\beta=id$, α any operator and I any ideal, then* $A^*(I,(\alpha,\beta)-SO(X))=A^*(I,\tau_\alpha)$.
5. *If $\alpha=\beta=id$, and I any ideal, then* $A^*(I,(\alpha,\beta)-SO(X))=A^*(I,\tau)$.
6. $A^*(I,(\alpha,\beta)-SO(X))=(\alpha,\beta)-scl\,(A^*(I,(\alpha,\beta)-SO(X)))\subset(\alpha,\beta)-scl(A)$, *that is,* $A^*(I,(\alpha,\beta)-SO(X))$ *is an (α,β)-semi-closed set.*

Proof.

1. Let $x\in A^*(I,(\alpha,\beta)-SO(X))$, then for all (α,β)-semi-open set U_x such that $x\in U_x$ we have $U_x\cap A\notin I$; but $\emptyset\in I$ for any ideal I, then $U_x\cap A\neq\emptyset$; it follows that $x\in(\alpha,\beta)-scl(A)$.
2. Let $x\in(\alpha,\beta)-scl(A)$, then for all (α,β)-semi-open set U_x such that $x\in U_x$, $U_x\cap A\neq\emptyset$, that is, $U_x\cap A\notin\{\emptyset\}=I$, it follows that $x\in A^*(I,(\alpha,\beta)-SO(X))$.

   Parts 3, 4 and 5 follow directly from the definition 2.5.

6. Let $x \in A^*(I,(\alpha,\beta)-SO(X))$ then for all (α,β)-semi-open set U_x such that $U_x \cap A^*(I,(\alpha,\beta)-SO(X)) \neq \emptyset$; it follows that $x \in (\alpha,\beta)-scl\,(A^*(I,(\alpha,\beta)-SO(X)))$.

   Suppose that $x \in (\alpha,\beta)-scl\,(A^*(I,(\alpha,\beta)-SO(X)))$, then for all (α,β)-semi-open set U_x such that $x \in U_x$ we have $U_x \cap A^*(I,(\alpha,\beta)-SO(X)) \neq \emptyset$, then there exists $y \in U_x \cap A^*(I,(\alpha,\beta)-SO(X))$, that is, $y \in A^*(I,(\alpha,\beta)-SO(X))$ and U_x is an (α,β)-semi-open set containing y, then $U_x \cap A \notin I$, therefore $x \in A^*(I,(\alpha,\beta)-SO(X))$ then $(\alpha,\beta)-scl\,(A^*(I,(\alpha,\beta)-SO(X))) \subset A^*(I,(\alpha,\beta)-SO(X))$.

   Finally, the last part of 6, follows from 1. □

We denote by $(\alpha,\beta)-scl^*(A) = A \cup A^*(I,(\alpha,\beta)-SO(X))$, when $\mathfrak{F} = (\alpha,\beta)-SO(X)$ in the Definition 3.3. It is clear that, $(\alpha,\beta)-scl^*(A) \subset (\alpha,\beta)-scl(A)$, satisfies the following properties:

1. If $\alpha = id$ and β is a monotone operator, then $(\alpha,\beta)-scl^*(A) = \beta - cl^*(A)$.

2. If α is any operator and $\beta = id$, then $(\alpha,\beta)-scl^*(A) = \alpha - cl^*(A)$.

3. If $\alpha = \beta = id$, then $(\alpha,\beta)-scl^*(A) = cl^*(A)$.

4. If $I = \{\emptyset\}$, then $(\alpha,\beta)-scl^*(A) = (\alpha,\beta)-scl(A)$.

We now introduce the concept of $(I,\gamma)_{(\alpha,\beta)}$-sg-closed set as a generalization of the concepts of (I,γ)-g-closed set ([4]) and (α,β)-sg-closed set ([17]).

Definition 6.1 *Let (X,τ,I), and α, β, γ operators associated with τ. A subset A of X is called $(I,\gamma)_{(\alpha,\beta)}$-semi-generalized-closed set, abbreviated $(I,\gamma)_{(\alpha,\beta)}$-sg-closed, if $(\alpha,\beta)-scl^*(A) \subset \gamma(U)$ whenever $A \subset U$ and $U \in (\alpha,\beta)-SO(X)$.*

Remark 6.1 *1. Observe that when $\alpha = \beta = id$, the (I,γ)-g-closed sets and the $(I,\gamma)_{(\alpha,\beta)}$-sg-closed sets agree.*

2. *If α is any operator and β is a monotone, the $(I,\gamma)_\alpha$-g-closed sets and the $(I,\gamma)_{(\alpha,\beta)}$-sg-closed sets agree.*

3. *If $\alpha = id$ and β is a monotone, the $(I,\gamma)_\beta$-g-closed sets and the $(I,\gamma)_{(\alpha,\beta)}$-sg-closed sets agree.*

4. *When $I = \{\emptyset\}$, the $(I,\gamma)_{\alpha,\beta)}$-sg-closed sets and the (α,β)-sg-closed sets agree.*

We can resume the above in the following theorem.

Theorem 6.2 *Let (X,τ,I), α, β and γ operators associated with τ and $A \subset X$, then*

1. *If $\alpha = id$ and β is a monotone operator, A is an $(I,\gamma)_{(\alpha,\beta)}$-sg-closed set if and only if A is an $(I,\gamma)_\beta$-sg-closed set.*

2. *If α is any operator and $\beta = id$, A is an $(I,\gamma)_{(\alpha,\beta)}$-sg-closed set if and only if A is an $(I,\gamma)_\alpha$-g-closed set.*

3. *If $\alpha = \beta = id$, A is an $(I,\gamma)_{(\alpha,\beta)}$-sg-closed set if and only if A is an (I,γ)-g-closed set ([4]).*

4. *If $I = \{\emptyset\}$, α, β any operators, $\gamma = id$ and A is an $(I,\gamma)_{(\alpha,\beta)}$-sg-closed set, then A is an (α,β)-sg-closed set ([17]).*

Theorem 6.3 *Let (X,τ,I), α, β be operators on τ and γ an expansive operator on $(\alpha,\beta) - SO(X)$. All (α,β)-sg-closed set is an $(I,\gamma)_{(\alpha,\beta)}$-sg-closed set.*

Proof.
Let A an (α,β)-sg-closed set of X and U an (α,β)-semi-open set such that $A \subset U$, then $(\alpha,\beta) - scl(A) \subset U$; that is $(\alpha,\beta) - scl^*(A) \subset U$, since γ is expansive on $(\alpha,\beta) - SO(X)$, then $U \subset \gamma(U)$, therefore, $(\alpha,\beta) - scl^*(A) \subset \gamma(U)$; it follows that A is an $(I,\gamma)_{(\alpha,\beta)}$-sg-closed set. □

The following example shows the existence of an $(I,\gamma)_{(\alpha,\beta)}$-sg-closed set that is not an (α,β)-sg-closed set.

Example 6.1 *Consider $\mathbb{R}$, the set of the real numbers, with the finite complement topology $\tau_f = \{U \subset \mathbb{R} : \mathbb{R} \setminus U$ is finite or $\mathbb{R}\}$, the operators $\alpha, \beta : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ associated with this topology are defined as follow: $\alpha(U) = int(U)$, $\beta(U) = cl(U)$.*

The set $\mathbb{Q}$ of the rational numbers is not an (α, β)-sg-closed set because $\mathbb{R} \setminus \{\sqrt{2}\}$ is an (α, β)-semi-open set containing $\mathbb{Q}$; but $(\alpha, \beta) - scl(\mathbb{Q}) = \mathbb{R} \not\subseteq \mathbb{R} \setminus \{\sqrt{2}\}$. $\mathbb{Q}$ is an $(I, \gamma)_{(\alpha,\beta)}$-sg-closed set if we consider the ideal $I = \mathcal{P}(X)$ and γ the identity operator.

Using the fact that there exist $(I, \gamma)_{(\alpha,\beta)}$-sg-closed sets that are not (α, β)-semi-closed. We introduce a new class of spaces in the following definition.

Definition 6.2 *Let (X, τ, I), α, β and γ be operators associated with τ. X is called an $(\alpha, \beta, \gamma) - semi - T_I$ space if all $(I, \gamma)_{(\alpha,\beta)}$-sg-closed set is an (α, β)-semi-closed set.*

Theorem 6.4 *Let (X, τ, I), α, β, γ be operators associated with τ and A a subset of X, then*

1. *If $\alpha = id$ and β is monotone, X is an $(\alpha, \beta, \gamma) - semi - T_I$ space if and only if X is an $(\beta, \gamma) - semi - T_I$ space.*

2. *If α is any operator and $\beta = id$, X is an $(\alpha, \beta, \gamma) - semi - T_I$ space if and only if X is an $(\alpha, \gamma) - semi - T_I$ space.*

3. *If $\alpha = \beta = id$ and X is $(\alpha, \beta, \gamma) - semi - T_I$ space, then X is an $\gamma - T_I$ space ([4]).*

4. *If $I = \{\emptyset\}$, $\gamma = id$ and X is an $(\alpha, \beta) - semi - T_{\frac{1}{2}}$ space, then X is an $(\alpha, \beta, \gamma) - semi - T_I$ space.*

5. *If α, β are any operators, γ expansive on $(\alpha, \beta) - SO(X)$ and X is an $(\alpha, \beta, \gamma) - semi - T_I$ space, then X is an $(\alpha, \beta) - semi - T_{\frac{1}{2}}$ space ([17]).*

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Resumen

Son dados un espacio topológico (X,τ), tres operadores α,β,γ asociados a una topología τ, es un ideal I en X. Los conceptos de conjunto α-cerrado, conjunto α-semicerrado, conjunto (α,β)-semicerrado y conjunto (I,γ) g-cerrado son generalizados. También nuevos axiomas de separación son introducidos y caracterizados, y nuevos espacios son obtenidos de tal manera que los espacios $\alpha - T_{\frac{1}{2}}$, α- semi $T_{\frac{1}{2}}$, (α,β)- semi $T_{\frac{1}{2}}$ y $\gamma - T_1$, respectivamente, son generalizados.

Palabras clave: (α, β)-semicerrado, (I, γ) g-cerrado, (α, β)-semi-$T_{\frac{1}{2}}$, (α, β, γ)- $semi - T_I$.

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