REMARKS ON STRONGLY PRECLOSED FUNCTIONS IN TOPOLOGICAL SPACES

M. Caldas[1] *and S. Jafari*[2]

June, 2009

Abstract

In this note, we present some of the basic properties of the classes of functions called strongly preclosed closed and pre-irresolute functions in topological spaces.

2000 Mathematics Subject Classification: 54B05, 54C08.

Keywords: *Topological spaces, preclosed set, pre-limit point, pre-irresolute, strongly preclosed function.*

[1] *Departamento de Matemática Aplicada, Univ. Federal Fluminense, Brasil.*
[2] *College of Vestsjaelland South, Denmark.*

1 Introduction

The notion of preopen [8] set plays a significant role in General Topology. Preopen sets are also called nearly open and locally dense [3] by several authors in the literature. They are not only important in the context of covering properties and decompositions of continuity but also in functional analysis in the context of open mapping theorems and closed graph theorems. One of the most important generalizations of continuity is the notion of nearly continuity [11] (= precontinuity [8] or almost continuity [5]) which involves preopen sets and is investigated by different authors under different terms (c.f. [1], [2], [4], [5], [8], [12]). In the present note, we show some of the basic properties of the notions of pre-derived and strongly preclosed functions. Also, we give some additional properties of the preclosure of a set due to N. El-Deeb et al. [4].

Throughout this paper (X, τ) (simply X) alway means topological spaces. A subset A of (X, τ) is called *preopen* [8] if $A \subset Int(Cl(A))$. The complement of a preopen set is called *preclosed*. The intersection of all preclosed sets containing A is called the *preclosure* [4] of A, denoted by $PCl(A)$. A subset A is preclosed if $A = PCl(A)$. We denote the family of preopen sets of (X, τ) by $PO(X, \tau)$.

Lemma 1.1 *(El-Deeb et al. [4]). Let (X, τ) be a topological space and A, B subsets of X. Then the following hold:*

(1) $x \in PCl(A)$ if and only if $A \cap V \neq \emptyset$ for every $V \in PO(X, \tau), x \in V$.

(2) A is preclosed in (X, τ) if and only if $A = PCl(A)$.

(3) $PCl(A) \subset PCl(B)$ if $A \subset B$.

(4) $PCl(PCl(A)) = PCl(A)$.

2 Some Basic Notions and their Properties, Strongly Preclosed Functions

Definition 1 *Let A be a subset of a space X. A point $x \in A$ is said to be pre-limit point of A [7] if for each preopen set U containing x, $U \cap (A \backslash \{x\}) \neq \emptyset$. The set of all pre-limit points of A is called the pre-derived set of A and is denoted by $d_p(A)$.*

Theorem 2.1 *For subsets A, B of a space X, the following statements hold:*

(1) $d_p(A) \subset d(A)$ where $d(A)$ is the derived set of A.

(2) If $A \subset B$, then $d_p(A) \subset d_p(B)$.

(3) $d_p(A) \cup d_p(B) \subset d_p(A \cup B)$ and $d_p(A \cap B) \subset d_p(A) \cap d_p(B)$.

(4) $d_p(d_p(A)) \backslash A \subset d_p(A)$.

(5) $d_p(A \cup d_p(A)) \subset A \cup d_p(A)$.

Proof. (1) It suffices to observe that every open set is preopen.
(3) Follows from (2).
(4) If $x \in d_p(d_p(A)) \backslash A$ and U is a preopen set containing x, then $U \cap (d_p(A) \backslash \{x\}) \neq \emptyset$. Let $y \in U \cap (d_p(A) \backslash \{x\})$. Then since $y \in d_p(A)$ and $y \in U$, $U \cap (A \backslash \{y\}) \neq \emptyset$. Let $z \in U \cap (A \backslash \{y\})$. Then $z \neq x$ for $z \in A$ and $x \notin A$. Hence $U \cap (A \backslash \{x\}) \neq \emptyset$. Therefore $x \in d_p(A)$.
(5) Let $x \in d_p(A \cup d_p(A))$. If $x \in A$, the result is obvious. So let $x \in d_p(A \cup d_p(A)) \backslash A$, then for preopen set U containing x, $U \cap (A \cup d_p(A) \backslash \{x\}) \neq \emptyset$. Thus $U \cap (A \backslash \{x\}) \neq \emptyset$ or $U \cap (d_p(A) \backslash \{x\}) \neq \emptyset$. Now it follows similarly from (4) that $U \cap (A \backslash \{x\}) \neq \emptyset$. Hence $x \in d_p(A)$. Therefore, in any case $d_p(A \cup d_p(A)) \subset A \cup d_p(A)$.

In general the converse of (1) may not be true and the equality does not hold in Theorem 2.1 (3).

Example 2.2 *Let $X = \{a, b, c, d\}$ with topology $\tau = \{\emptyset, \{b\}, \{d\}, \{b, d\}, X\}$ Therefore $PO(X, \tau) = \{\emptyset, \{b\}, \{d\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}, X\}$. Take:*
(i) $A = \{a\}$. Then $d_p(A) = \emptyset$ and $d(A) = \{c\}$. Hence $d(A) \not\subset d_p(A)$.
(ii) $A = \{a, c, d\}$ and $B = \{b, c\}$. Then $d_p(A) = \{a, c\}$, $d_p(B) = \{a, c\}$ and $d_p(A \cap B) = \emptyset$. Hence $d_p(A \cap B) \neq d_p(A) \cap d_p(B)$.

Example 2.3 *Let $X = \{a, b, c, d\}$ with topology $\tau = \{\emptyset, \{a, b\}, \{a, b, c\}, X\}$. Take $A = \{a, c, d\}$ and $B = \{b, c, d\}$. Then $d_p(A) = \emptyset$, $d_p(B) = \emptyset$ and $d_p(A \cup B) = \{c, d\}$. Hence $d_p(A \cup B) \neq d_p(A) \cup d_p(B)$.*

Theorem 2.4 *$PCl(A) = A \cup d_p(A)$.*

Proof. Since $d_p(A) \subset PCl(A)$, $A \cup d_p(A) \subset PCl(A)$. On the other hand, let $x \in PCl(A)$. In the case that $x \in A$, the proof is complete. If $x \notin A$, then each preopen set U containing x intersects A at a point distinct from x, so $x \in d_p(A)$. Thus $PCl(A) \subset A \cup d_p(A)$, which completes the proof.

Corollary 2.5 *A subset A is preclosed if and only if it contains the set of its pre-limit points.*

Definition 2 *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be*
(1) pre-irresolute [13] if $f^{-1}(V)$ is preopen in X for every preopen set V of Y, Mashhour et al. [9] termed pre-irresoluteness as M-pre-continuity.
(2) strongly preclosed if $f(F)$ is preclosed in Y for every preclosed set F of X.

Lemma 2.6 *A function $f : (X, \tau) \to (Y, \sigma)$ is a strongly preclosed function if and only if $pCl(f(A)) \subset f(pCl(A))$ for each subset A of X.*

Theorem 2.7 *A function $f : (X, \tau) \to (Y, \sigma)$ is a strongly preclosed function if and only if $d_p(f(A)) \subset f(d_{\lambda_g}(A))$ for each subset A of X.*

Proof. If $d_p(f(A)) \subset f(d_p(A))$ for each subset A of X and A is a preclosed subset of X, then $pCl(f(A)) = f(A) \cup d_p(f(A)) \subset f(A) \cup f(d_p(A)) \subset f(A)$.

Conversely, assume that $f : (X, \tau) \to (Y, \sigma)$ is a strongly preclosed function such that $A \subset X$. Also suppose $y \in d_p(f(A))$. It follows that $y \in pCl(f(A) - \{y\}) = pCl(f(A - f^{-1}(y))) \subset f(pCl(A - f^{-1}(y)))$. This means that $f^{-1}(y) \cap pCl(A - f^{-1}(y)) \neq \emptyset$. let $x \in f^{-1}(y) \cap pCl(A - f^{-1}(y))$. Hence $x \in d_p(A)$ and also $y = f(x) \in f(d_p(A))$.

A topological space is called preBolzano-Weierstrass if every every countably infinite subset of the space has a *pre*-limit point.

Corollary 2.8 *If $f : (X, \tau) \to (Y, \sigma)$ is a strongly preclosed function with Y preBolzano-Weierstrass and $f^{-1}(y)$ preBolzano-Weierstrass for each $y \in Y$, then X is preBolzano-Weierstrass.*

Proof. Suppose A is a countably infinite subset of X. If $f(A)$ is finite, then $A \cap f^{-1}(y)$ is infinite for some $y \in Y$. This means that $d_p(A) \neq \emptyset$. On the other hand, if $f(A)$ is infinite, then $d_p(f(A)) \neq \emptyset$. By Theorem 2.7, $d_p(A) \neq \emptyset$.

Theorem 2.9 *A function $f : (X, \tau) \to (Y, \sigma)$ is strongly preclosed if and only if $f(X)$ is preclosed in Y and $f(U) - f(X - U)$ is preopen in $f(X)$ whenever U is preopen in X.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be strongly preclosed . Observe that $f(X)$ is preclosed in Y and $f(U) - f(X - U) = f(X) - f(X - U)$ is preopen in $f(X)$ whenever U is preopen in X. Also suppose $f(X)$ is preclosed in Y and $f(U) - f(X - U)$ is preopen in $f(X)$ whenever U is preopen in X. Moreover, let D be preclosed in X. Therefore $f(D) = f(X) - (f(X - D) - f(D))$ is preclosed in $f(X)$. Hence $f(D)$ is preclosed in Y.

Corollary 2.10 *A function $f : (X, \tau) \to (Y, \sigma)$ is a strongly preclosed surjection function if and only if $f(U) - f(X - U)$ is preopen in Y whenever U is preopen in X.*

Corollary 2.11 *If $f : (X, \tau) \to (Y, \sigma)$ is a pre-irresolute, strongly preclosed surjection, then the topology on Y is $\{g(U) - g(X - U) \mid U \text{ is preopen in } X\}$.*

Proof. Assume that V is preopen in Y. Then by hypothesis, $f^{-1}(V)$ is preopen in X. So we have $f(f^{-1}(V)) - f(X - f^{-1}(V)) = V$. Thus, all preopen sets in Y are of the form $f(U) - f(X - U)$ whenever U is preopen in X. By Corollary 2.6, all the sets of the form $f(U) - f(X - U)$ whenever U is preopen in X, are preopen in Y.

By $PO(X, \tau)$, we denote the collection of set of preopen subsets of X.

Definition 3 *A topological space X is said to be strongly pre-normal [10] if for any pair of disjoint preclosed sets D and E of X, there exist disjoint preopen sets U and V such that $D \subset U$ and $E \subset V$.*

Theorem 2.12 *If $f : (X, \tau) \to (Y, \sigma)$ is a pre-irresolute, strongly pre-closed surjection and X is strongly pre-normal, then Y is strongly pre-normal.*

Proof. Suppose N and M are preclosed subsets of Y and $N \cap M = \emptyset$. Thus $f^{-1}(N)$ and $f^{-1}(M)$ are preclosed disjoint subsets of X. Then there exist $U, V \in PO(X)$, disjoint such that $f^{-1}(N) \subset U$ and $f^{-1}(M) \subset V$. Hence $N \subset f(U) - f(X - U)$ and $M \subset f(V) - f(X - V)$. By Corollary 2.11, $f(U) - f(X - U)$ and $f(V) - f(X - V)$ are preopen in Y and observe that they are disjoint.

References

[1] M. Caldas, *On weak preopenness and a decomposition on preopenness*, Southast Assian Bulletin of Mathematics, 2009 (to appear).

[2] M. Caldas, S. Jafari and T. Noiri, *Characterizations of pre-R_0 and pre-R_1 topological spaces*, Topology Proceedings, 25(2000), 17-30.

[3] H. Corson and E. Michael, *Metrizability of certain countable unions*, Illinois J. Math., 8(1964), 351-360.

[4] N.El-Deeb, I.A. Hasanein, A.S. Mashhour and T. Noiri, *On p-regular spaces*, Bull. Math. Soc. Sci. R.S. Roumanie, 27(1983), 311-315.

[5] T. Husain, *Almost continuous mappings*, Pr. Mat., 10(1966), 1-7.

[6] S. Jafari and T. Noiri, *On almost precontinuous functions*, Internat. J. Math. Math. Sci., 24(2000), 193-201.

[7] S. Jafari, *On some certain types of sets via preopen sets*, Tamkang J. Math. 37(4)(2006), 391-398.

[8] A.S. Mashhour, M.E. Abd El-Monsef and S.N. El-Deeb, *On pre-continuous and weak pre-continuous mappings*, Proc. Math. Phys. Soc. Egypt, 53(1982), 47-53.

[9] A.S. Mashhour, M.E. Abd EL-Monsef and I.A. Hasanein, *On pre-topological spaces*, Bull. Math. de la Soc. Sci. Math. de la R.S. Roumanie, 28(1984), 39-45.

[10] T.M. Nour, *Contributions to the theory of bitopological spaces*, Ph.D. Thesis, Delhi Uni., (1989).

[11] V. Ptak, *Completeness and open mappings theorem*, Bull. Soc. Math. France, 86(1958), 41-74.

[12] V. Popa, *Properties of H-almost continuous functions*, Bull. Math. Soc. Sci. Math. R.S. Roum., 31(1987), 163-168.

[13] I.L. Reilly and M.K Vamanmurthy, *On α-continuity in topological spaces*, Acta Math. Hung., 45(1985), 27-32.

Resumen

En esta nota, presentamos algunas de las propiedades básicas de las clases de funciones denominadas funciones fuertemente pre cerradas, cerradas y pre-irresolutas en espacios topológicos.

Palabras clave: Espacios topológicos, con punto pre cerrado, punto pre-límite, función fuertemente pre cerrada.

Miguel Caldas
Departamento de Matemática Aplicada
Universidade Federal Fluminense
Rua Mário Santos Braga, s/n
24020-140, Niterói, RJ-Brasil
gmamccs@vm.uff.br

Saeid Jafari
College of Vestsjaelland South
Herrestraede 11
4200 Slagelse-Denmark
jafari@stofanet.dk