b-OPEN SETS AND A NEW CLASS OF FUNCTIONS

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Abstract

The concept of (b, s)-continuous functions in topological spaces is introduced and studied. Some of their characteristic properties are considered. Also we investigate the relationships between these classes of functions and other classes of functions.

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1 Introduction and Preliminaries

Andrijević [4] introduced a new class of generalized open sets called b-open sets into the field of topology. This class is a subset of the class of semi-preopen sets [5], i.e. a subset of a topological space which is contained in the closure of the interior of its closure. Also the class of b-open sets is a superset of the class of semi-open sets [26], i.e. a set which is contained in the closure of its interior, and the class of locally dense sets [12] or preopen sets [28], i.e. a set which is contained in the interior of its closure. Andrijević studied several fundamental and interesting properties of b-open sets.

Throughout the present paper, the space (X, τ) always means a topological space on which no separation axioms are assumed unless explicitly stated. A subset A is said to be semiopen [26] (resp. β-open [1], preopen [28], α-open [30]) if $A \subset Cl(Int(A))$ (resp. $A \subset Cl(Int(Cl)A)))$, $A \subset Int(Cl(A)))$, $A \subset Int(Cl(Int(A))))$. The complement of a semiopen (resp. β-open, preopen, α-open) set is said to be semiclosed (resp. β-closed, preclosed, α-closed). We denote the collection of all semiopen (resp. β-open, α-open) sets by $SO(X)$ (resp. $\beta O(X)$, $\alpha O(X)$). We set $SO(X,x) = \{U : x \in U \in SO(X)\}$, $\beta O(X,x) = \{U : x \in U \in \beta O(X)\}$ and $\alpha O(X,x) = \{U : x \in U \in \alpha O(X)\}$.

Let $A \subseteq X$, then A is said to be *b-open* [4] if $A \subseteq Cl(Int(A)) \cup Int(Cl(A))$, where $Cl(A)$ and $Int(A)$ denotes the closure and the interior of A in (X, τ), respectively. The complement $X \setminus A$ of a b-open set A is called *b-closed* and the *b-closure* of a set A, denoted by $bCl(A)$, is the intersection of all b-closed sets containing A. The *b-interior* of a set A denoted by $bInt(A)$, is the union of all b-open sets contained in A.

The family of all b-open (resp. b-closed) sets in (X, τ) will be denoted by $BO(X, \tau)$ (resp. $BC(X, \tau)$).

Proposition 1.1 *(Andrijević [4])* (a) The union of any family of b-open sets is b-open.
(b) The intersection of an open and a b-open set is a b-open set.

Lemma 1.2 The *b-closure* of a subset A of X, denoted by $bCl(A)$, is the set of all $x \in X$ such that $O \cap A \neq \emptyset$ for every $O \in BO(X,x)$, *where* $BO(X,x) = \{U \mid x \in U \in BO(X,\tau)\}$.

A point $x \in X$ is said to be a θ-semi cluster point of a subset A of X if $Cl(U) \cap A \neq \emptyset$ for every $U \in SO(X,x)$. The set of all θ-semi cluster points of A is called the θ-semiclosure of A and is denoted by $sCl_\theta(A)$. A subset A is called θ-semiclosed [23] if $A = sCl_\theta(A)$. The complement of a θ-semiclosed set is called θ-semiopen. A subset A is said to be regular open (resp. regular closed) if $A = In(Cl(A))$ (resp. $A = Cl(Int(A))$.

Definition 1 *A function $f : X \to Y$ is said to be:*

(i) perfectly continuous [31] if $f^{-1}(V)$ is clopen in X for every open set V of Y.

(ii) contra-continuous [13] if $f^{-1}(V)$ is closed in X for every open set V of Y.

(iii) regular set-connected [14] if $f^{-1}(V)$ is clopen in X for every regular open set V of Y.

(iv) s-continuous [8] if for each point $x \in X$ and each semiopen set V of Y with $f(x) \in V$, there exists an open set U of X containing x such that $f(U) \subset V$.

(v) almost s-continuous [32] if for each point $x \in X$ and each semiopen set V of Y with $f(x) \in V$, there exists an open set U of X containing x such that $f(U) \subset sC(V)$.

(vi) (θ, s)-continuous [23] if for each point $x \in X$ and each semiopen set V of Y with $f(x) \in V$, there exists an open set U of X containing x such that $f(U) \subset Cl(V)$.

(vii) *β-quasi-irresolute [20] if for each point $x \in X$ and each semiopen set V of Y with $f(x) \in V$, there exists an β-open set U of X containing x such that $f(U) \subset Cl(V)$.*

(viii) *(p, s)-continuous [19] if for each point $x \in X$ and each semiopen set V of Y with $f(x) \in V$, there exists a preopen set U of X containing x such that $f(U) \subset Cl(V)$.*

(ix) *α-quasi-irresolute [21] if for each point $x \in X$ and each semiopen set V of Y with $f(x) \in V$, there exists an α-open set U of X containing x such that $f(U) \subset Cl(V)$.*

(x) *weakly θ-irresolute [21] if for each point $x \in X$ and each semiopen set V of Y with $f(x) \in V$, there exists an semiopen set U of X containing x such that $f(U) \subset Cl(V)$.*

2 (b, s)-continuous Functions

Definition 2 *A function $f : X \to Y$ is said to be (b, s)-continuous functions if for each point $x \in X$ and each semiopen set V of Y with $f(x) \in V$, there exists a b-open set U of X containing x such that $f(U) \subset Cl(V)$.*

Definition 3 *Let A be a subset of a space (X, τ). The set $\bigcap\{U \in RO(X) : A \subset U\}$ is called the r-kernel of A [17] and is denoted by $rker(A)$.*

Lemma 2.1 *(Ekici [17]) The following properties hold for the subsets A, B of a space X :*

(1) $x \in rker(A)$ if and only if $A \cap F \neq \emptyset$ for any $F \in RC(X, x)$.

(2) $A \subset rker(A)$ and $A = rker(A)$ if A is regular open in X.

(3) If $A \subset B$, then $rker(A) \subset rker(B)$.

Theorem 2.2 *For a function $f : X \to Y$, the following are equivalent:*

(1) f is (b, s)-continuous;

(2) $f^{-1}(V) \subset bInt(f^{-1}(Cl(V)))$ for every semiopen set V of Y;

(3) $f^{-1}(F)$ is b-open in X for every regular closed set F of Y;

(4) $f^{-1}(V)$ is b-closed in X for every regular open set V of Y;

(5) $f^{-1}(A)$ is b-open in X for every θ-semiopen set A of Y;

(6) $f^{-1}(B)$ is b-closed in X for every θ-semiclosed set B of Y;

(7) $f(bCl(A)) \subset sCl_\theta(f(A))$ for every subset A of X;

(8) $bCl(f^{-1}(B)) \subset f^{-1}(sCl_\theta(B))$ for every subset B of Y.

(9) $f(bCl(A)) \subset rker(f(A))$ for every subset A of X;

(10) $bCl(f^{-1}(B)) \subset f^{-1}(rker(B))$ for every subset B of Y;

(11) For each point x in X and each regular closed set V in Y containing $f(x)$, there is a b-open set U in X containing x such that $f(U) \subset V$.

Proof. (1) $\Rightarrow$ (2): Let $V \in SO(Y)$ and $x \in f^{-1}(V)$. Since f is (b, s)-continuous, there exists an b-open set U of X such that $f(U) \subset Cl(V)$. It follows that $x \in U \subset f^{-1}(Cl(V))$. Hence $x \in bInt(f^{-1}(Cl(V)))$. Therefore $f^{-1}(V) \subset bInt(f^{-1}(Cl(V)))$.

(2) $\Rightarrow$ (3): Let F be any regular closed set of Y. Since $F \in SO(Y)$, then by (2), it follows that $f^{-1}(F) \subset bInt(f^{-1}(Cl(F)))$. This shows that $f^{-1}(F)$ is b-open in X.

(3) $\Leftrightarrow$ (4): This is obvious.

(4) $\Rightarrow$ (5): This follows from the fact that any θ-semiopen set is a union of regulares closed sets.

(5) $\Leftrightarrow$ (6): This is obvious.

(5) $\Rightarrow$ (7): Let A be any subset of X and $y \notin sCl_\theta(f(A))$. Then there

exist $V \in SO(Y, y)$ such that $f(A) \cap Cl(V) = \emptyset$. Since $Cl(V)$ is θ-semiopen, $f^{-1}(Cl(V))$ is b-open in X and $A \cap f^{-1}(Cl(V)) = \emptyset$. Therefore $bCl(A) \cap f^{-1}(Cl(V)) = \emptyset$ and $f(bCl(A)) \cap Cl(V) = \emptyset$. Consequently, we obtain $y \notin f(bCl(A))$ and hence $f(bCl(A)) \subset sCl_\theta(f(A))$.

(7) $\Rightarrow$ (8): Let B be any subset of Y. Then by (7) $f(bCl(f^{-1}(B))) \subset sCl_\theta(f(f^{-1}(B))) \subset sCl_\theta(B)$. Therefore, we obtain $bCl(f^{-1}(B)) \subset f^{-1}(sCl_\theta(B))$.

(8) $\Rightarrow$ (6): Let B be any θ-semiclosed set of Y. By (8) we have $bCl(f^{-1}(B)) \subset f^{-1}(sCl_\theta(B)) = f^{-1}(B)$ and hence $f^{-1}(B)$ is b-closed in X.

(5) $\Rightarrow$ (1): Let $x \in X$ and $V \in SO(Y, f(x))$. Then $Cl(V)$ is θ-semiopen in Y. Set $U = f^{-1}(Cl(V))$, then $U \in BO(X, x)$ and $f(U) \subset Cl(V)$. Therefore f is (b, s)-continuous.

(11)$\Rightarrow$(9): Let A be any subset of X. Suppose that $y \notin rker(f(A))$. Then, by Lemma 2.1 there exists $V \in RC(Y, y)$ such that $f(A) \cap V = \emptyset$. For any $x \in f^{-1}(V)$, by (11) there exists $U_x \in BO(X, x)$ such that $f(U_x) \subset V$. Hence $f(A \cap U_x) \subset f(A) \cap f(U_x) \subset f(A) \cap V = \emptyset$ and $A \cap U_x = \emptyset$. This shows that $x \notin bCl(A)$ for any $x \in f^{-1}(V)$. Therefore, $f^{-1}(V) \cap bCl(A) = \emptyset$ and hence $V \cap f(bCl(A)) = \emptyset$. Thus, $y \notin f(bCl(A))$. Consequently, we obtain $f(bCl(A)) \subset rker(f(A))$.

(9)$\Leftrightarrow$ (10): Let B be any subset of Y. By (9) and Lemma 3.1, we have $f(bCl(f^{-1}(B))) \subset rker(ff^{-1}(B)) \subset rker(B)$ and $bCl(f^{-1}(B)) \subset f^{-1}(rker(B))$.

Conversely, suppose that (10) holds. Let $B = f(A)$, where A is a subset of X. Then $bCl(A) \subset bCl(f^{-1}(B)) \subset f^{-1}(rker(f(A)))$. Therefore $f(bCl(A)) \subset rker(f(A))$.

(10)$\Rightarrow$ (4): Let V be any regular open set of Y. Then, by (10) and Lemma 2.1(2) we have $bCl(f^{-1}(V)) \subset f^{-1}(rker(V)) = f^{-1}(V)$ and $bCl(f^{-1}(V)) = f^{-1}(V)$. This shows that $f^{-1}(V)$ is b-closed in X. Therefore f is (b, s)-continuous.

(3)$\Rightarrow$(11): Let $x \in X$ and V be a regular closed set of Y containing $f(x)$.

Then $x \in f^{-1}(V)$. Since by hypothesis $f^{-1}(V)$ is b-open, there exists $U \in BO(X,x)$ such that $x \in U \subset f^{-1}(V)$. Hence $x \in U$ and $f(U) \subset V$.

Theorem 2.3 *For a function $f : X \to Y$, the following are equivalent:*
(1) f is (b,s)-continuous;
(2) $f^{-1}(V)$ is b-closed in X for every regular open set V of Y;
(3) $f^{-1}(Int(Cl(F)))$ is b-closed in X for every open set F of Y (4) $f^{-1}(Cl(Int(G)))$ is b-open in X for every closed set G of Y.

Proof. (1) $\Leftrightarrow$ (2): Theorem 2.2.
(3) $\Leftrightarrow$ (4): Let G be a closed set in Y. Then $Y \backslash G$ is open. We have that $f^{-1}(Int(Cl(Y\backslash G))) = f^{-1}(Y \backslash Cl(Int(G))) = X \backslash f^{-1}(Cl(Int(G)))$ is a b-closed set in X. Hence $f^{-1}(Cl(Int(G)))$ is b-open.
The converse can be obtained similarly.
(2) $\Leftrightarrow$ (3): Let F be a open set in Y. We have $Int(Cl(F)) \in RO(Y)$. By (2) $f^{-1}(Int(Cl(F)))$ is b-closed in X.
The converse can be obtained similarly.

Remark 2.4 *From the above definitions, we have the Diagram following:*

$$
\begin{tikzcd}
 & & perfectly-cont. \arrow[r, Rightarrow] \arrow[d, Rightarrow] & contra-cont. \arrow[d, Rightarrow] \\
s-cont. \arrow[r, Rightarrow] & almost-s-cont. \arrow[r, Rightarrow] & regular-set-connect. \arrow[r, Rightarrow] & (\theta,s)-cont. \arrow[d, Rightarrow] \\
 & & & (b,s)-cont. \arrow[d, Rightarrow] \\
 & & & \beta-quasi-irre.
\end{tikzcd}
$$

By the following examples, remarks and ([31], [13], [14], [8], [23], [20]), the inverse claims in the implication above are not usually true.

Example 2.5 *In [[33], Examples 6.1–6.6.], Noiri and Jafari showed that:*
(1) Contra continuity (resp. regular set-connected, (θ, s)-continuity) does not necessarily imply perfectly continuity.
(2) (θ, s)-continuity does not necessarily imply regular set-connected.
(3) (θ, s)-continuity does not necessarily imply contra-continuity.
(4) continuity and (θ, s)-continuity are independent concepts.

Example 2.6 *(1) In [[14], Example 3.8], Dontchev et al. showed that a regular set-connected function need not be almost s-continuous.*
(2) In [[32], Example 5.3], Noiri et al. showed that a almost s-continuous function need not be s-continuous.

Example 2.7 *Let $X = \{a, b, c, \}$, $\tau = \{\emptyset, X, \{b\}, \{a, c\}\}$ and*

$$\sigma = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}.$$

Then the identity function $f : (X, \tau) \to (X, \sigma)$ is (b, s)-continuous, but it is not continuous.

Example 2.8 *The identity function in the real line with the usual topology is continuous. The inverse image of $(0, 1)$ is not b-closed and the function is not (b, s)-continuous.*

Remark 2.9 *By Examples 2.7 and 2.8, continuity and (b, s)-continuity are independent of each other.*

We recall that a space X is called extremally disconnected [7] if the closure of each open set of X is open in X, equivalently if every semiopen set is α-open. The space X is called submaximal [7] if every dense subset of X is open in X, equivalently if every preopen set is open.

Lemma 2.10 *If (X, τ) be a submaximal extremally disconnected space, then $\tau = \tau^{\alpha} = SO(X, \tau) = PO(X, \tau) = BO(X, \tau) = \beta O(X, \tau)$*

Theorem 2.11 *If (X, τ) is submaximal extremally disconnected, then the following are equivalently for a function $f : (X, \tau) \to (Y, \sigma)$:*
(1) f is (b, s)-continuous;
(2) f is (θ, s)-continuous;
(3) f is (p, s)-continuous;
(4) f is α-quasi-irresolute;
(5) f is weakly θ-irresolute;
(6) f is β-quasi-irresolute.

Proof. It follows from Lemma 2.10.

3 Separation Axioms Related to b-open Sets

Recall, that a space X is said to be:

(i) Weakly Hausdorff [37] if each element of X is an intersection of regular closed sets.

(ii) s-Urysohn [6] if for each pair of distinct points x and y in X, there exist $U \in SO(X, x)$ and $U \in SO(X, y)$ such that $Cl(U) \cap Cl(V) = \emptyset$.

(iii) b-T_2 [10] if for each pair of distinct points x and y in X, there exist $U \in BO(X, x)$ and $V \in BO(X, y)$ such that $U \cap V = \emptyset$.

Theorem 3.1 *If $f : X \to Y$ is a (b, s)-continuous injection and Y is s-Urysohn, then X is b-T_2.*

Proof. Let x and y be distinct points of X. Then $f(x) \neq f(y)$. Since Y is s-Urysohn, there exist $V \in SO(Y, f(x))$ and $W \in SO(Y, f(y))$ such that $Cl(V) \cap Cl(W) = \emptyset$. Since f is (b, s)-continuous, there exist $U \in BO(X, x)$ and $G \in BO(X, y)$ such that $f(U) \subset Cl(V)$ and $f(G) \subset Cl(W)$. It follows that $U \cap G = \emptyset$. This shows that X is b-T_2.

Theorem 3.2 *If $f, g : X \to Y$ are (b, s)-continuous functions, X is submaximal extremally disconnected and Y is s-Urysohn, then $E = \{x \in X : f(x) = g(x)\}$ is closed in X.*

Proof. Suppose that $x \notin E$. Then $f(x) \neq g(x)$. Since Y is s-Urysohn, there exist $V \in SO(Y, f(x))$ and $W \in SO(Y, g(x))$ such that $Cl(V) \cap Cl(W) = \emptyset$. Since f and g are (b,s)-continuous, there exist $U \in BO(X,x)$ and $G \in BO(X,x)$ such that $f(U) \subset Cl(V)$ and $f(G) \subset Cl(W)$. Set $D = U \cap G$. By Lemma 2.8 $D \in O(X)$ since X is submaximal extremally disconnected. Therefore $D \cap E = \emptyset$ and it follows that E is closed in X.

Lemma 3.3 *[29]. Let A be a subset of X and B a subset of Y. If $A \in BO(X)$ and $Y \in BO(Y)$, then $A \times B \in BO(X \times Y)$.*

Theorem 3.4 *If $f : X \to Y$ be a (b,s)-continuous function and Y is s-Urysohn, then $E = \{(x,y) : f(x) = f(y)\}$ is b-closed in $X \times X$.*

Proof. Suppose that $(x,y) \notin E$. Then $f(x) \neq f(y)$. Since Y is s-Urysohn, there exist $V \in SO(Y, f(x))$ and $W \in SO(Y, f(y))$ such that $Cl(V) \cap Cl(W) = \emptyset$. Since f is (b,s)-continuous, there exist $U \in BO(X,x)$ and $G \in BO(X,y)$ such that $f(U) \subset Cl(V)$ and $f(G) \subset Cl(W)$. Set $D = U \times G$. By Lemma 3.3 $(x,y) \in D \in BO(X \times X)$ and $D \cap E = \emptyset$. This means that $bCl(E) \subset E$ and therefore E is b-closed in $X \times X$.

Theorem 3.5 *Let $f : X \to Y$ be a function and $g : X \to X \times Y$ the graph function, given by $g(x) = (x, f(x))$ for every $x \in X$. Then f is (b,s)-continuous if g is (b,s)-continuous.*

Proof. Let F be a regular closed set of Y. Then we have $X \times F = X \times (Cl(Int(X)) \times Cl(Int(F)) = Cl(Int(X \times F)$. Therefore $X \times F$ is regular closed in $X \times Y$. By Theorem 2.1, $f^{-1}(F) = g^{-1}(X \times F)$ is b-open in X and hence f is (b,s)-continuous.

The following lemma is due to El-Atik [15].

Lemma 3.6 *Let A and X_o be subsets of (X,τ). If $A \in BO(X)$ and $X_o \in \alpha O(X)$, then $A \cap X_o \in BO(X_o)$.*

Theorem 3.7 *If $f : (X, \tau) \to (Y, \sigma)$ is (b, s)-continuous and X_o is a α-open subset of X, then the restriction $f_{X_o} : X_o \to Y$ is (b, s)-continuous.*

Proof. Let V be an regular closed set of Y. Since f is (b, s)-continuous, $f^{-1}(V)$ is b-open in X. By Lemma 3.6, $(f_{X_o})^{-1}(V) = X_o \cap f^{-1}(V)$ is b-open in X_o and hence f_{X_o} is (b, s)-continuous.

Lemma 3.8 *Let A and X_o be subsets of (X, τ). If $A \in BO(X_o)$ and $X_o \in \alpha O(X)$, then $A \in BO(X)$. [29].*

Theorem 3.9 *Let $f : (X, \tau) \to (Y, \sigma)$ be a function and $\{A_i : i \in \Omega\}$ be a cover of X by α-open sets of (X, τ). Then f is (b, s)-continuous if $f_{A_i} : A_i \to Y$ is (b, s)-continuous for each $i \in \Omega$.*

Proof. Let V be any regular closed set of Y and f_{A_i} be (b, s)-continuous. Then $(f_{A_i})^{-1}(V) = f^{-1}(V) \cap A_i$ is b-open in A_i and hence, by Lemma 3.8, $(f_{A_i})^{-1}(V)$ is b-open in X for each $i \in \Omega$. Therefore $f^{-1}(V) = X \cap f^{-1}(V) = \cup\{A_i \cap f^{-1}(V) : i \in \Omega\} = \cup\{f_{A_i}^{-1}(V) : i \in \Omega\}$ is b-open in X. Hence f is (b, s)-continuous.

Theorem 3.10 *Let $\{X_i : i \in \Omega\}$ be any family of topological spaces. If $f : X \to \prod X_i$ is a (b, s)-continuous. function . Then $Pr_i \circ f : X \to X_i$ is (b, s)-continuous. for each $i \in \Omega$, where Pr_i is the projection of $\prod X_i$ onto X_i.*

Proof. Let U_i be an arbitrary regular open set in X_i. Since Pr_i is continuous open, it is an R-map and hence $Pr_i^{-1}(U_i)$ is regular open in $\prod X_i$. Since f is (b, s)-continuous. , we have by Theorem 2.2 $f^{-1}(Pr_i^{-1}(U_i)) = (Pr_i \circ f)^{-1}(U_i)$ is b-closed in X. Therefore $Pr_i \circ f$ is (b, s)-continuous. for each $i \in \Omega$.

4 Applications

Definition 4 *A function $f : X \to Y$ is said to be:*
(i) b-irresolute if for each point $x \in X$ and each b-open set V of Y

containing $f(x)$, there exists a b-open set U of X containing x such that $f(U) \subset V$.
(ii) θ-irresolute [25] if for each point $x \in X$ and each semiopen set V of Y containing $f(x)$, there exists a semiopen set U of X containing x such that $f(Cl(U)) \subset Cl(V)$.
(iii) b-open [16] if $f(V) \in BO(Y)$ for every $V \in BO(X)$.
(iv) b-closed if $f(V) \in BC(Y)$ for every $V \in BC(X)$.

Theorem 4.1 *If $f : X \to Y$ is b-irresolute and $g : Y \to Z$ is (b, s)-continuous, then $g \circ f : X \to Z$ is (b, s)-continuous.*

Proof. Let $x \in X$ any W be a semiopen set in Z containing $(g \circ f)(x)$. Since g is (b, s)-continuous, there exists $V \in BO(Y, f(x))$ such that $g(V) \subset Cl(W)$. Since f is b-irresolute, there exists $U \in BO(X, x)$ such that $f(U) \subset V$. This shows that $(g \circ f)(U) \subset Cl(W)$. Therefore $g \circ f$ is (b, s)-continuous.

Theorem 4.2 *If $f : X \to Y$ is (b, s)-continuous and $g : Y \to Z$ is θ-irresolute, then $g \circ f : X \to Z$ is (b, s)-continuous.*

Proof. Similar to Theorem 4.1.

Theorem 4.3 *If $f : X \to Y$ is a b-open surjective function and $g : Y \to Z$ is a function such that $g \circ f : X \to Z$ is (b, s)-continuous. then g is (b, s)-continuous.*

Proof. Suppose that x and y are in X and Y respectively, such that $f(x) = y$. Let W be a semiopen set in Z containing $(g \circ f)(x)$. Then there exists $U \in BO(X, x)$ such that $g(f(U)) \subset Cl(W)$. Since f is b-open, then $f(U) \in BO(Y, y)$ such that $g(f(U)) \subset Cl(W)$. This implies that g is (b, s)-continuous.

Recall that Caldas et al. [11] defined the b-frontier of A denoted by b-$fr(A)$, as b-$fr(A) = bCl(A) \backslash bInt(A)$, equivalently b-$fr(A) = bCl(A) \cap bCl(X \backslash A)$.

Theorem 4.4 *The set of points $x \in X$ which $f : (X, \tau) \to (Y, \sigma)$ is not (b, s)-continuous is identical with the union of the b-frontiers of the inverse images of $Cl(V)$ sets where V is a semiopen set of Y containing $f(x)$.*

Proof. Necessity. Suppose that f is not (b, s)-continuous at a point x of X. Then, there exists a semiopen set $V \subset Y$ containing $f(x)$ such that $f(U)$ is not a subset of $Cl(V)$ for every $U \in BO(X, x)$. Hence we have $U \cap (X \setminus f^{-1}(Cl(V))) \neq \emptyset$ for every $U \in BO(X, x)$. It follows that $x \in bCl(X \setminus f^{-1}(Cl(V)))$. We also have $x \in f^{-1}(Cl(V)) \subset bCl(f^{-1}(Cl(V)))$. This means that $x \in b\text{-}fr(f^{-1}(Cl(V)))$.
Sufficiency. Suppose that $x \in b\text{-}fr(f^{-1}(Cl(V)))$ for some $V \in SO(Y, f(x))$ Now, we assume that f is (b, s)-continuous at $x \in X$. Then there exists $U \in BO(X, x)$ such that $f(U) \subset Cl(V)$. Therefore, we have $x \in U \subset f^{-1}(Cl(V))$ and hence $x \in bInt(f^{-1}(Cl(V))) \subset X \setminus b\text{-}fr(f^{-1}(Cl(V)))$. This is a contradiction. This means that f is not (b, s)-continuous.

Definition 5 *For a function $f : X \to Y$, the graph $G(f) = \{(x, f(x)) : x \in X\}$ is called (b, s)-closed if for each $(x, y) \in (X \times Y) - G(f)$, there exist $U \in BO(X, x)$ and $V \in SO(Y, y)$ such that $U \times Cl(V) \cap G(f) = \emptyset$.*

Lemma 4.5 *A function $f : X \to Y$ has the (b, s)-closed graph $G(f)$ if for each $(x, y) \in (X \times Y) - G(f)$, there exist $U \in BO(X, x)$ and $V \in SO(Y, y)$ such that $f(U) \cap Cl(V) = \emptyset$.*

Proof. It is an immediate consequence of Definition 4 and the fact that for any subsets $U \subset X$ and $V \subset Y$, $(U \times Cl(V)) \cap G(f) = \emptyset$ if and only if $f(U) \cap Cl(V) = \emptyset$.

Theorem 4.6 *If $f : X \to Y$ is (b, s)-continuous and Y is s-Urysohn, then $G(f)$ is (b, s)-closed in $X \times Y$.*

Proof. Let $(x, y) \in (X \times Y) - G(f)$. It follows that $f(x) \neq y$. Since Y is s-Urysohn, there exist semiopen sets V and W in Y containing

$f(x)$ and y, respectively such that $Cl(U) \cap Cl(W) = \emptyset$. Since f is (b,s)-continuous, there exists $U \in BO(X,x)$ such that $f(U) \subset Cl(V)$. Therefore $f(U) \cap Cl(W) = \emptyset$ and $G(f)$ is (b,s)-closed in $X \times Y$.

Theorem 4.7 *If $f : X \to Y$ is a (b,s)-continuous injection with a (b,s)-closed graph, then X is b-T_2*

Proof. Let x and y be any distinct points of X. Then, since f is injective, we have $f(x) \neq f(y)$ and thus $(x, f(y)) \in (X \times Y) - G(f)$. Since $G(f)$ is (b,s)-closed, there exist $U \in BO(X,x)$ and $V \in SO(Y, f(y))$ such that $f(U) \cap Cl(V) = \emptyset$. Since f is (b,s)-continuous, there exists $G \in BO(Y,y)$ such that $f(G) \subset Cl(V)$. Therefore, we have $f(U) \cap f(G) = \emptyset$ and hence $U \cap G = \emptyset$. This shows that X is b-T_2.

Definition 6 *(i) A space X is called b-normal if for disjoint bclosed subsets A and B of X, there exist disjoint b-open sets U and V such that $A \subset U$ and $B \subset V$.*
(ii) Any two subsets A and B of X are called b-separated if there exist disjoint b-open sets U and V such that $A \subset U$ and $B \subset V$.

Observe that $X = \{a, b, c\}$ with Sierpinski topology is b-normal but not normal.

Theorem 4.8 *If $f : X \to Y$ is a (b,s)-continuous, b-closed function of a b-normal X onto a space Y, then any two disjoint θ-semiclosed subsets of Y can be b-separated.*

Proof. Let F_1 and F_2 be any distinct θ-semiclosed sets of Y. since f is (b,s)-continuous, $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are disjoint b-closed sets of X. By b-normality of X, there exist $U_1, U_2 \subset BO(X)$ such that $f^{-1}(F_1) \subset U_1$ and $f^{-1}(F_2) \subset U_2$ and $U_1 \cap U_2 = \emptyset$. Let $V_i = Y - f(X - U_i)$ for $i = 1, 2$. Since f is b-closed, the sets V_1, V_2 are b-open in Y and $F_i \subset V_i$ for $i = 1, 2$. Since U_1 and U_2 are disjoint and $f^{-1}(F_i) \subset U_i$ for $i = 1, 2$, we obtain $V_1 \cap V_2 = \emptyset$. This shows that F_1 and F_2 are b-separated.

Theorem 4.9 *If $f : X \to Y$ is a b-irresolute b-closed surjection and X is b-normal, then Y is b-normal.*

Proof. Let F_1 and F_2 be any distinct b-closed sets of Y. since f is b-irresolute, $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are disjoint b-closed sets of X. By b-normality of X, there exist $U_1, U_2 \subset BO(X)$ such that $f^{-1}(F_1) \subset U_1$ and $f^{-1}(F_2) \subset U_2$ and $U_1 \cap U_2 = \emptyset$. Let $V_i = Y - f(X - U_i)$ for $i = 1, 2$. Since f is b-closed, the sets V_1, V_2 are b-open in Y and $F_i \subset V_i$ for $i = 1, 2$. Since U_1 and U_2 are disjoint and $f^{-1}(F_i) \subset U_i$ for $i = 1, 2$, we obtain $V_1 \cap V_2 = \emptyset$. This shows that Y is b-normal.

Definition 7 *i) Let A be a subset of X, then we say that A is S-closed relative [34] to X if every cover $\{V_\alpha : \alpha \in \nabla\}$ of A by semiopen sets of X, there exists a finite subset ∇_o of ∇ such that $A \subset \cup\{Cl(V_\alpha) : \alpha \in \nabla_o\}$. A space X is said to be S-closed if X is S-closed relative to X equivalently if every regular closed cover of X has a finite subcover.*
ii) Let A be a subset of X, then we say that A is b-compact relative to X [15] if every cover of A by b-open sets of X has a finite subcover. A space X is said to be b-compact if X is b-compact relative to X.
iii) Let A be a subset of X, then we say that A is nearly compact [36] if every cover of A by regular open sets of X has a finite subcover.

Theorem 4.10 *If $f : X \to Y$ is a (b, s)-continuous function and A is b-compact relative to X, then $f(A)$ is S-closed relative to Y.*

Proof. Suppose that $f : X \to Y$ is (b, s)-continuous and let A be b-compact relative to X. Let $\{V_\alpha : \alpha \in \nabla\}$ be an semiopen cover of $f(A)$. For each point $x \in A$, there exists $\alpha(x) \in \nabla$ such that $f(x) \in V_{\alpha(x)}$. Since f is (b, s)-continuous, there exists $U_x \in BO(X, x)$ such that $f(U_x) \subset Cl(V_{\alpha(x)})$. The family $\{U_x : x \in A\}$ is a cover of A by b-open sets of X and hence there exists a finite set A_0 of A such that $A \subset \cup_{x\in A_0} U_x$. Therefore, we obtain $f(A) \subset \cup_{x\in A_0} Cl(V_{\alpha(x)})$. This shows that $f(A)$ is S-closed relative to Y.

Lemma 4.11 *Let A be a subset of a topological space X. Then $A \in BO(X)$ if and only if $bCl(A)$ is b-clopen in X (i.e., b-open and b-closed).*

Proof. This follows immediately from Proposition 3.5 of [29].

Definition 8 *A space X is said to be: (i) b-connected [29] if X can not expressed as the union of two disjoint non-empty b-open sets of X.*
(ii) θ-irreducible [22] if every pair of non-empty regular closed sets of X has a non-empty intersection.

It should be noted that $X = \{a, b, c\}$ with Sierpinski topology is connected but not b-connected. A space with indiscrete topology is connected but not b-connected since b-open sets establish a discrete topology. Also a space with partition topology is neither connected nor b-connected.

Theorem 4.12 *If $f : X \to Y$ is a (b, s)-continuous function surjection and X is b-connected, then Y is θ-irreducible.*

Proof. Suppose that Y is not θ-irreducible. Then, there exist non-empty disjoint regular closed set F and G of Y. Since f is (b, s)-continuous and surjective, by Theorem 2.1 $f^{-1}(F)$ and $f^{-1}(G)$ are non-empty disjoint b-open sets of X. Thus, we have $bCl(f^{-1}(F)) \cap f^{-1}(G) = \emptyset$. By Lemma 4.11, $bCl(f^{-1}(F))$is b-open and b-closed. This shows that X is not b-connected.

Recall, that a function $f : X \to Y$ is said almost α-continuous [35] if $f^{-1}(V)$ is α-open in X for every regular open set V of Y.

Theorem 4.13 *Let $f : X \to Y$ be a (b, s)-continuous almost α-continuous surjection. If X is nearly compact (resp. S-closed), then Y is nearly compact (resp. S-closed).*

Proof. If V is regular open in Y, then $f^{-1}(V)$ is b-closed α-open in X since f is (b, s)-continuous almost α-continuous. Since, every b-closed is β-closed, we have
$Int(Cl(Int(f^{-1}(V)))) \subset f^{-1}(V) \subset Int(Cl(Int(f^{-1}(V))))$ and $f^{-1}(V) = Int(Cl(Int(f^{-1}(V))))$. Thus, we obtain that $f^{-1}(V)$ is regular open in

X. Suppose that X is nearly compact. Let $\{V_\alpha : \alpha \in \nabla\}$ be any regular open cover of Y. Then $\{f^{-1}(V_\alpha) : \alpha \in \nabla\}$ be any regular open cover of X and there exists a finite subset ∇_o of ∇ such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in \nabla_o\}$. Therefore, we obtain $Y = \cup\{V_\alpha : \alpha \in \nabla_o\}$ since f is surjective. This shows that Y is nearly compact.
The proof for S-closednees is similar and therefore is omitted.

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References

[1] M. E. Abd El-Monsef, S. N. Ei-Deeb and R. A. Mahmoud, *β-open sets and β-continuous mappings.* Bull. Fac. Assiut Univ., **12** (1983), 77-90.

[2] M. E. Abd El-Monsef, A. N. Geaisa and R. A. Mahmoud, *β-regular spaces.* Proc. Math. Phys. Soc. Egypt, **60** (1985), 47-52.

[3] M. E. Abd El-Monsef, R. A. Mahmoud and E. R. Lashin, *β-closure and β-interior.* J. Fac. Ed. Ain Shams Univ., **10** (1986), 235-245.

[4] D. Andrijević, *On b-open sets.* Mat. Vesnik 48(1996), 59-64.

[5] D. Andrijević, *Semi-preopen sets.* Mat. Vesnik 38(1986), 24-32.

[6] S. P. Arya and M. P. Bhamini, *Some generalizations of pairwise Urysohn spaces.* Indian J. Pure Appl. Math., **18** (1987), 1088-1093.

[7] N. Bourbaki, *Eléments de mathématique*, Livre III: Topologie générale, Châp. IX-Paris 1948.

[8] D. E. Cameron and G. Woods, *s-continuous and s-open mappings.* Preprint.

[9] M. Caldas, S. Jafari and T. Noiri, *On Λ_b-sets and the associated topology τ^{Λ_b}.* Acta Math. Hungar. 4(2006), 337-345.

[10] M. Caldas and S. Jafari, *On some applicationsof b-open sets in topological espaces.* Kochi J. Math. **2**(2007), 11-19.

[11] M. Caldas, S. Jafari and N. Rajesh, *On properties of totally b-continuous functions.* Annalele Atiintifice Uni. Ali Cuzi Din Iasi Mat., **55** (2009), 119-130.

[12] H. H. Corson and E. Michael, *Metrizability of certain countable unions.* Illinois J. Math. **8**(1964), 351-360.

[13] J. Dontchev, *Contra-continuous functions and strongly S-closed spaces.* Internat. J. Math. Math. Sci., **19** (1996), 303-310.

[14] J. Dontchev, M. Ganster and I. Reilly, *More on almost s-continuity.* Topology Atlas, Preprint N.212.

[15] A. A. El-Atik, *A study of some types of mappings on topological spaces.* Masters Thesis, Faculty of Science, Tanta University. Egypt 1997.

[16] E. Ekici and M. Caldas, *Slightl γ-continuous functions.* Bol. Soc. Paran. Mat. **22** (2004), 63-74.

[17] E. Ekici, *Another form of contra-continuity.* Kochi J. Math., **1** (2006), 21-29.

[18] M. Ganster, T. Noiri and I. L. Reilly, *Weak and strong forms of θ-irresolute functions.* J. Inst. Math. Comput. Sci. Math. Ser. **1** (1988), 19-29.

[19] S. Jafari, *A new type of continuity.* (Submitted).

[20] S. Jafari and T. Noiri, *On β-quasi irresolute functions.* Mem. Fac. Sci. Kochi Univ. Ser A. Math. **21**(2000), 53-62.

[21] S. Jafari and T. Noiri, *On α-quasi irresolute functions.* (Submitted).

[22] D. S. Jankovic and P. E. Long, *θ-irreducible spaces.* Kyungpook Math. J. **26**(1986), 33-66.

[23] J. E. Joseph and M. H. Kwack, *On S-closed spaces.* Proc. Amer. Math. Soc. **80**(1980), 341-348.

[24] J. E. Joseph, *Characterization of nearly-compact spaces.* Boll. Un. Mat. Ital. **5**(1976), 311-321.

[25] F. H. Khedr and T. Noiri, *Onθfunctions.* Indian J. Math., **28** (1986), 211-217.

[26] N. Levine, *Semi-open sets and semi-continuity in topologyical spaces.* Amer. Math. Monthly **70**(1963), 36-41.

[27] A. S. Mashhour, I. A. Hasamein and S. N. El-Deeb, *A note on semi-continuity and precontinuity.* Indian J. Pure Appl. Math., **13** (1982), 1119-1123.

[28] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, *On pre-continuous and weak precontinuous mappings.* Proc. Math. Phys. Soc. Egypt **53**(1982), 47-53.

[29] A. A. Nasef, *b-locally Closed sets and related topics.* Chaos, Solitons and Fractals, **12** (2001), 1909-1915.

[30] O. Njåstad, *On some classes of nearly open sets.* Pacific J. Math., **15** (1965), 961-970.

[31] T. Noiri, *Super-continuity and some strong forms of continuity.* Indian J. Pure Appl. Math., **15** (1984), 241-250.

[32] T. Noiri, B. Ahmad and M. Khan, *Almost s-continuous functions.* Kyungpook Math. J., **35** (1995), 311-322.

[33] T. Noiri and S. Jafari, *Properties of (θ, s)-continuous functions.* Topology and it Applications, **123** (2002), 167-179.

[34] T. Noiri, *On S-closed subpaces.* Atti Accad. Naz. Lincei Rend. Clo. Sci. FizMat. Narur., **64** (1978), 157-162.

[35] T. Noiri, Almost *α-continuous functions.* Kyungpook Math. J., **28** (1998), 71-77.

[36] M. K. Singal and A. Mathur, *On nearly compact spaces.* Boll. Un. Mat. Ital., **4** (1969), 702-710.

[37] T. Soundararajan, *Weakly Hausdorff spaces and the cardinality, of topological spaces,in: General Topology and its Relation to Modern Analysis and Algebra.* III, Proc. Conf. Kampur, 1968 Academia, Praga, **50** (1971), 301-306.

Resumen

Se introduce y estudia el concepto de funciones (b, s)-continuas en espacios topológicos. Asimismo, se consideran algunas de sus propiedades características. También se investigan las relaciones entre estas clases de funciones (b, s)-continuas y otras clases de funciones.

Palabras clave: Espacios topológicos, conjuntos b-abiertos, conjuntos b-cerrados, funciones (b, s)-continuas, funciones b-irresolutas.

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