A STUDY OF THREE VARIABLE ANALOGUES OF CERTAIN FRACTIONAL INTEGRAL OPERATORS

Mumtaz Ahmad Khan[1] *and*
Bhagwat Swaroop Sharma[1]

May, 2008

Abstract

The paper deals with a three variable analogues of certain fractional integral operators introduced by M. Saigo. Besides giving three variable analogues of earlier known fractional integral operators of one variable as a special cases of newly defined operators, the paper establishes certain results in the form of theorems including integration by parts.

2000 Mathematics Subject Classification: 26A33, 26B99.

Keywords: *Fractional integral operator, Saigo operators of fractional integration, Gauss hypergeometric function, Weyl fractional integral, Fractional calculus.*

1. *Dept. Applied Math. Faculty of Engineering, Aligarh Muslim Univ. India.*

1 Introduction

The fractional calculus has been investigated by many mathematicians [16]. In their works the Riemann-Liouville operator (R-L) defined by

$$R_{0,x}^{\alpha} f = \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} (x-t)^{\alpha} f(t)\, dt \tag{1.1}$$

was the most central, while Erdelyi and Kober defined their operator (E-K) in connection with the Hankel transform [11] as

$$I_{0,x}^{\alpha,\eta} f = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_{0}^{x} (x-t)^{\alpha-1} t^{\eta} f(t)\, dt \tag{1.2}$$

Weyl and another Erdelyi-Kober fractional operators are defined as follows:

$$W_{x,\infty}^{\alpha} f = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} (t-x)^{\alpha-1} f(t)\, dt \tag{1.3}$$

and

$$K_{x,\infty}^{\eta,\alpha} f = \frac{x^{\eta}}{\Gamma(\alpha)} \int_{x}^{\infty} (t-x)^{\alpha-1} t^{-\eta-\alpha} f(t)\, dt \tag{1.4}$$

respectively.

In 1978, M. Saigo [17] defined a certain integral operator involving the Gauss hypergeometric function as follows:

Let $\alpha > \beta$ and η be real numbers. The fractional integral operator $I_x^{\alpha,\beta,\eta}$, which acts on certain functions $f(x)$ on the interval $(0, \infty)$ is defined by

$$I_x^{\alpha,\beta,\eta} f = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^{\pi} (x-t)^{\alpha-1} F\left(\alpha+\beta, -\eta;\ \alpha;\ 1-\frac{t}{x}\right) f(t)\, dt \tag{1.5}$$

where Γ is the gamma function, F denotes the Gauss hypergeometric series

$${}_2F_1\ (a,\ b;\ c;\ z) = \sum_{n=0}^{\infty} \frac{(a)_n\ (b)_n}{n!\ (c)_n}\ z^n, \quad |z| < 1 \tag{1.6}$$

and its analytic continuation into $|arg(1-z)| < \pi$, and $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$.

Such an integral was first treated by Love [12] as an integral equation. However, if one regards the integral as an operator with a slight change, it will contain as special cases both R-L and E-K owing to reduction formulas for the Gauss function by restricting the parameters. The more interesting fact is that for this operator two kinds of product rules may be made up by virtue of Erdelyi's formula [3], which were first proved by using the method of fractional integration by parts in the R-L sense. From the rules, of course, the ones for R-L and E-K are deduced. Moreover this operator is representable by products of R-L's, from which it is possible to obtain the integrability and estimations of Hardy-Littlewood type [4]. Saigo [17] also defined an integral operator on the interval (x, ∞) as an extension of operators of Weyl and another Erdelyi-Kober operators as follows:

Under the same assumptions in defining (1.5), the integral operator $J_x^{\alpha,\beta,\eta}$ is defined by

$$J_x^{\alpha,\beta,\eta} f = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1}\, t^{-\alpha-\beta}\, F\left(\alpha+\beta, -\eta;\ \alpha;\ 1-\frac{x}{t}\right) f(t)\, dt \tag{1.7}$$

Later on in 1988, Saigo and Raina [19] obtained the generalized fractional integrals and derivatives introduced by Saigo [17], [18] of the system $S_q^n(x)$, where the general system of polynomials

$$S_q^n(x) = \sum_{r=0}^{[\frac{n}{q}]} \frac{(-n)_{qr}}{r!} A_{n,r}\, x^r$$

were defined by Srivastava [20], where $q > 0$ and $n \geq 0$ are integers, and $A_{n,r}$ are arbitrary sequence of real or complex numbers.

2 Three Variable Analogues of Operators (1.5) and (1.7)

We define the three variable analogues of Saigo's operators (1.5) and (1.7) as follows:

I. Let $c > 0$, $c' > 0$, $c'' > 0$, a, b, b', b'' be real numbers. A three variable analogue of fractional integral operator $I_{0,x}^{\alpha,\beta,\eta}$ due to M. Saigo is defined as

$$ {}_1I_{0,x;0,y;0,z}^{a,b,b',b'';c,c',c''} f(x,y,z) = \frac{x^{-a}y^{-a}z^{-a}}{\Gamma(c)\Gamma(c')\Gamma(c'')} \int_0^x \int_0^y \int_0^z (x-u)^{c-1}(y-v)^{c'-1}$$

$$(z-w)^{c''-1}\, F_A^{(3)}\left[\begin{array}{l} a,b,b',b'';1-\frac{u}{x},1-\frac{v}{y},1-\frac{w}{z} \\ c,c',c'' \quad ; \end{array}\right] f(u,v,w)\, dw\, dv\, du \tag{2.1}$$

where $F_A^{(3)}$ is a Lauricella function of three variables defined by

$$F_A^{(3)}\left[\begin{array}{l} a,b,b',b'';x,y,z \\ c,c',c'' \quad ; \end{array}\right] = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(a)_{n+r+s}\ (b)_n\ (b')_r\ (b'')_s}{n!\ r!\ s!\ (c)_n\ (c')_r\ (c'')_s}\, x^n\, y^r\, z^s,$$

$|x| + |y| + |z| < 1.$

Special Cases:

(*i*) For $a = b = b' = b'' = 0,\ c = \alpha,\ c' = \beta,\ c'' = \gamma$, (2.1) reduces to

$$ {}_1I^{0,0,0,0;\alpha,\beta,\gamma}_{0,x;0,y;0,z}\ f(x,y,z) \ = \ {}_1R^{\alpha,\beta,\gamma}_{0,x;0,y;0,z}\ f(x,y,z) $$

$$ = \frac{1}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^x \int_0^y \int_0^z (x-u)^{\alpha-1}(y-v)^{\beta-1}(z-w)^{\gamma-1} \quad (2.2) $$
$$ f(u,v,w)\, dw\, dv\, du $$

Here (2.2) may be taken as a three variable analogue of Riemann-Liouville fractional integral operator $R^{\alpha}_{0,x}$.

(*ii*) For $a = c = \alpha,\ b = -\eta,\ b' = b'' = 0,\ c' = \beta,\ c'' = \gamma$, (2.1) becomes

$$ {}_1I^{\alpha,-\eta,0,0;\alpha,\beta,\gamma}_{0,x;0,y;0,z}\ f(x,y,z) = {}^{x}_{1}E^{\alpha,\beta,\gamma,\eta}_{0,x;0,y;0,z} f(x,y,z) $$

$$ = \frac{x^{-\alpha-\eta}y^{-\alpha}z^{-\alpha}}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^x \int_0^y \int_0^z \ (x-u)^{\alpha-1}(y-v)^{\beta-1}(z-w)^{\gamma-1}u^{\eta} \quad (2.3) $$
$$ f(u,v,w)\, dw\, dv\, du $$

(*iii*) For $a = c = \alpha,\ b = b'' = 0,\ b' = -\eta,\ c' = \beta,\ c'' = \gamma$, (2.1) gives

$$ {}_1I^{\alpha,0,-\eta,0;\alpha,\beta,\gamma}_{0,x;0,y;0,z}\ f(x,y,z) = {}^{y}_{1}E^{\alpha,\beta,\gamma,\eta}_{0,x;0,y;0,z}\ f(x,y,z) $$

$$ = \frac{x^{-\alpha}y^{-\alpha-\eta}z^{-\alpha}}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^x \int_0^y \int_0^z (x-u)^{\alpha-1}(y-v)^{\beta-1}\ (z-w)^{\gamma-1}v^{\eta} \quad (2.4) $$
$$ f(u,v,w)\, dw\, dv\, du $$

(*iv*) For $a = c = \alpha,\ b = b' = 0,\ b'' = -\eta,\ c' = \beta,\ c'' = \gamma$, (2.1) yields

$$ {}_1I^{\alpha,0,0,-\eta;\alpha,\beta,\gamma}_{0,x;0,y;0,z}\, f(x,y,z) = {}^{z}_{1}E^{\alpha,\beta,\gamma,\eta}_{0,x;0,y;0,z}\, f(x,y,z) $$

$$ = \frac{x^{-\alpha}y^{-\alpha}z^{-\alpha-\eta}}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^x \int_0^y \int_0^z (x-u)^{\alpha-1}(y-v)^{\beta-1}(z-w)^{\gamma-1} w^{\eta} \quad (2.5) $$

$$ f(u,v,w)\, dw\, dv\, du $$

Here (2.3), (2.4) and (2.5) may be regarded as three variable analogue of Erdelyi Kober fractional integral operator.

Under the same conditions of (2.1), a three variable analogues of $J^{\alpha,\beta,\gamma}_{x,\infty}$ is as defined below:

$$ {}_1J^{a,b,b',b'';c,c',c''}_{x,\infty;y,\infty;z,\infty}\, f(x,y,z) $$

$$ = \frac{1}{\Gamma(c)\Gamma(c')\Gamma(c'')} \int_x^\infty \int_y^\infty \int_z^\infty (u-x)^{c-1}(v-y)^{c'-1}(w-z)^{c''-1} $$

$$ F_A^{(3)} \left[\begin{array}{l} a, b, b', b''; 1-\frac{x}{u}, 1-\frac{y}{v}, 1-\frac{z}{w} \\ c, c', c'' \quad ; \end{array} \right] u^{-a} v^{-a} w^{-a} f(u,v,w)\, dw\, dv\, du \quad (2.6) $$

Special Cases:

(*i*) For $a = b = b' = b'' = 0,\ c = \alpha,\ c' = \beta,\ c'' = \gamma$, (2.6) reduces to

$$ {}_1J^{0,0,0,0;\alpha,\beta,\gamma}_{x,\infty;y,\infty;z,\infty}\, f(x,y,z) \;=\; {}_1L^{\alpha,\beta,\gamma}_{x,\infty;y,\infty;z,\infty}\, f(x,y,z) $$

$$ = \frac{1}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_x^\infty \int_y^\infty \int_z^\infty (u-x)^{\alpha-1}(v-y)^{\beta-1}(w-z)^{\gamma-1} f(u,v,w)\, dw\, dv\, du \quad (2.7) $$

We may consider (2.7) as a three variable analogue of Weyl fractional integral operator $L^{\alpha}_{x,\infty}$.

(*ii*) For $a = c = \alpha,\ b = -\eta,\ b' = b'' = 0,\ c' = \beta,\ c'' = \gamma$, (2.6) reduces to

$${}_1J^{\alpha,-\eta,0,0;\alpha,\beta,\gamma}_{x,\infty;y,\infty;z,\infty} f(x,y,z) = {}^{x}_{1}K^{\alpha,\beta,\gamma,\eta}_{x,\infty;y,\infty;z,\infty} f(x,y,z) = \frac{x^{\eta}}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}$$

$$\int_x^{\infty}\int_y^{\infty}\int_z^{\infty}(u-x)^{\alpha-1}(v-y)^{\beta-1}(w-z)^{\gamma-1}u^{-\alpha-\eta}v^{-\alpha}w^{-\alpha}$$
$$f(u,v,w)\,dw\,dv\,du \tag{2.8}$$

(*iii*) For $a = c = \alpha,\ b = b'' = 0,\ b' = -\eta,\ c' = \beta,\ c'' = \gamma$, (2.6) becomes

$${}_1J^{\alpha,0,-\eta,0;\alpha,\beta,\gamma}_{x,\infty;y,\infty;z,\infty} f(x,y,z) = {}^{y}_{1}K^{\alpha,\beta,\gamma,\eta}_{x,\infty;y,\infty;z,\infty} f(x,y,z) = \frac{y^{\eta}}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}$$

$$\int_x^{\infty}\int_y^{\infty}\int_z^{\infty}(u-x)^{\alpha-1}(v-y)^{\beta-1}(w-z)^{\gamma-1}u^{-\alpha}v^{-\alpha-\eta}w^{-\alpha}$$
$$f(u,v,w)\,dw\,dv\,du \tag{2.9}$$

(*iv*) For $a = c = \alpha,\ b = b' = 0,\ b'' = -\eta,\ c' = \beta,\ c'' = \gamma$, (2.6) gives

$${}_1J^{\alpha,0,0,-\eta;\alpha,\beta,\gamma}_{x,\infty;y,\infty;z,\infty} f(x,y,z) = {}^{z}_{1}K^{\alpha,\beta,\gamma,\eta}_{x,\infty;y,\infty;z,\infty} f(x,y,z) = \frac{z^{\eta}}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}$$

$$\int_x^{\infty}\int_y^{\infty}\int_z^{\infty}(u-x)^{\alpha-1}(v-y)^{\beta-1}(w-z)^{\gamma-1}u^{-\alpha}v^{-\alpha}w^{-\alpha-\eta}$$
$$f(u,v,w)\,dw\,dv\,du \tag{2.10}$$

We may consider (2.8), (2.9) and (2.10) as three variable analogues of Erdelyi-Kober fractional integral operator $K^{\eta,\alpha}_{x,\infty}$.

II. Let $c > 0$, a, a', a'', b, b', b'' be real numbers. Then a second three variable analogue of $I_{0,x}^{\alpha,\beta,\gamma}$ is as follows:

$$
{}_2I_{0,x;0,y;0,z}^{a,a',a'',b,b',b'';c}\, f(x,y,z) = \frac{x^{-a}y^{-a'}z^{-a''}}{\{\Gamma(c)\}^3}
$$

$$
\int_0^x \int_0^y \int_0^z (x-u)^{c-1}(y-v)^{c-1}(z-w)^{c-1}
$$

$$
F_A^{(3)}\left[\begin{matrix} a, a', a'', b, b', b''; 1-\frac{u}{x}, 1-\frac{v}{y}, 1-\frac{w}{z} \\ c \qquad\qquad ; \end{matrix}\right] f(u,v,w)\, dw\, dv\, du \tag{2.11}
$$

where

$$
F_A^{(3)}\left[\begin{matrix} a, a', a'', b, b', b''; x, y, z \\ c \qquad\qquad ; \end{matrix}\right]
$$

$$
= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(a)_n\ (a')_r\ (a'')_s\ (b)_n\ (b')_r\ (b'')_s}{n!\ r!\ s!\ (c)_{n+r+s}}\ x^n y^r z^s
$$

Special Cases:

(i) For $a = a' = a'' = 0$, $c = \alpha$, (2.11) reduces to

$$
{}_2I_{0,x;0,y;0,z}^{0,0,0;b,b',b'';\alpha}\, f(x,y,z) = {}_2R_{0,x;0,y;0,z}^{\alpha}\, f(x,y,z)
$$

$$
= \frac{1}{\{\Gamma(\alpha)\}^3} \int_0^x \int_0^y \int_0^z (x-u)^{\alpha-1}(y-v)^{\alpha-1}(z-w)^{\alpha-1} f(u,v,w)\, dw\, dv\, du \tag{2.12}
$$

Here (2.12) may be regarded as a three variable analogue of Riemann-Liouville fractional integral operator $R_{0,x}^{\alpha}$.

(ii) For $a = c = \alpha$, $a' = a'' = 0, b = -\eta$, (2.11) becomes

$$ {}_2I^{\alpha,0,0;-\eta,b',b'';\alpha}_{0,x;0,y;0,z}\ f(x,y,z) \ = \ {}^{x}_{2}E^{\alpha,\eta}_{0,x;0,y;0,z}\ f(x,y,z) $$

$$ = \frac{x^{-\alpha-\eta}}{\{\Gamma(\alpha)\}^3} \int_0^x \int_0^y \int_0^z (x-u)^{\alpha-1}(y-v)^{\alpha-1}(z-w)^{\alpha-1}u^{\eta} \tag{2.13} $$

$f(u,v,w)\,dw\,dv\,du$

(*iii*) For $a = a'' = 0,\ a' = c = \alpha,\ b' = -\eta$, (2.11) gives

$$ {}_2I^{0,\alpha,0,b,-\eta,b'';\alpha}_{0,x;0,y;0,z}\ f(x,y,z) \ = \ {}^{y}_{2}E^{\alpha,\eta}_{0,x;0,y;0,z}\ f(x,y,z) $$

$$ = \frac{y^{-\alpha-\eta}}{\{\Gamma(\alpha)\}^3} \int_0^x \int_0^y \int_0^z (x-u)^{\alpha-1}(y-v)^{\alpha-1}(z-w)^{\alpha-1}v^{\eta} \tag{2.14} $$

$f(u,v,w)\,dw\,dv\,du$

(*iv*) For $a = a' = 0,\ a'' = c = \alpha,\ b'' = -\eta$, (2.11) becomes

$$ {}_2I^{0,0,\alpha,b,b',-\eta;\alpha}_{0,x;0,y;0,z}\ f(x,y,z) \ = \ {}^{z}_{2}E^{\alpha,\eta}_{0,x;0,y;0,z}\ f(x,y,z) $$

$$ = \frac{z^{-\alpha-\eta}}{\{\Gamma(\alpha)\}^3} \int_0^x \int_0^y \int_0^z (x-u)^{\alpha-1}(y-v)^{\alpha-1}(z-w)^{\alpha-1}w^{\eta} \tag{2.15} $$

$f(u,v,w)\,dw\,dv\,du$

Here (2.13), (2.14) and (2.15) may be thought of as the second three variable analogues of Erdelyi-Kober fractional integral operator $E^{\alpha,\eta}_{0,x}$.

Under the same conditions of (2.11), a second three variable analogues of $J^{\alpha,\beta,\gamma}_{x,\infty}$ is as defined below:

$$
{}_2J^{a,a',a'',b,b',b'';c}_{x,\infty;y,\infty;z,\infty}\, f(x,y,z) = \frac{1}{\{\Gamma(c)\}^3}
$$

$$
\int_x^\infty \int_y^\infty \int_z^\infty (u-x)^{c-1}(v-y)^{c-1}(w-z)^{c-1} \tag{2.16}
$$

$$
F_A^{(3)}\left[\begin{array}{l} a,a',a'',b,b',b'';1-\frac{u}{x},1-\frac{v}{y},1-\frac{w}{z} \\ c \qquad\qquad\qquad ; \end{array}\right] u^{-a}v^{-a'}w^{-a''}
$$

$$
f(u,v,w)\,dw\,dv\,du
$$

Special Cases:

(*i*) For $a = a' = a'' = 0,\ c = \alpha$, (2.16) reduces to

$$
{}_2J^{0,0,0;b,b',b'';\alpha}_{x,\infty;y,\infty;z,\infty}\, f(x,y,z) \ = \ {}_2L^{\alpha}_{x,\infty;y,\infty;z,\infty}\, f(x,y,z)
$$

$$
= \frac{1}{\{\Gamma(\alpha)\}^3} \int_x^\infty \int_y^\infty \int_z^\infty (u-x)^{\alpha-1}\,(v-y)^{\alpha-1}\,(w-z)^{\alpha-1}\, f(u,v,w)\,dw\,dv\,du \tag{2.17}
$$

It can be considered as a three variable analogue of Weyl fractional integral operator $L^{\alpha}_{x,\infty}$.

(*ii*) For $a' = a'' = 0,\ a = c = \alpha,\ b = -\eta$, (2.16) becomes

$$
{}_2J^{\alpha,0,0,-\eta;b',b'';\alpha}_{x,\infty;y,\infty;z,\infty}\, f(x,y,z) \ = \ {}^{x}_{2}K^{\alpha,\eta}_{x,\infty;y,\infty;z,\infty}\, f(x,y,z)
$$

$$
= \frac{x^\eta}{\{\Gamma(\alpha)\}^3}\int_x^\infty \int_y^\infty \int_z^\infty (u-x)^{\alpha-1}(v-y)^{\alpha-1}(w-z)^{\alpha-1}u^{-\alpha-\eta} \tag{2.18}
$$

$$
f(u,v,w)\,dw\,dv\,du
$$

(*iii*) For $a = a'' = 0,\ a' = c = \alpha,\ b' = -\eta$, (2.16) gives

$$ {}_2J^{0,\alpha,0,b,-\eta,b'';\alpha}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z) \ = \ {}^y_2K^{\alpha,\eta}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z) $$

$$ = \frac{y^\eta}{\{\Gamma(\alpha)\}^3} \int_x^\infty \int_y^\infty \int_z^\infty (u-x)^{\alpha-1}(v-y)^{\alpha-1}(w-z)^{\alpha-1} v^{-\alpha-\eta} \tag{2.19} $$

$f(u,v,w)\,dw\,dv\,du$

(*iv*) For $a = a' = 0,\ a'' = c = \alpha,\ b'' = -\eta$, (2.16) yields

$$ {}_2J^{0,0,\alpha,b,b',-\eta;\alpha}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z) \ = \ {}^z_2K^{\alpha,\eta}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z) $$

$$ = \frac{z^\eta}{\{\Gamma(\alpha)\}^3} \int_x^\infty \int_y^\infty \int_z^\infty (u-x)^{\alpha-1}(v-y)^{\alpha-1}(w-z)^{\alpha-1} w^{-\alpha-\eta} \tag{2.20} $$

$f(u,v,w)\,dw\,dv\,du$

Here (2.18), (2.19) and (2.20) may be taken of as the second three variable analogues of Erdelyi-Kober fractional integral operator $K^{\eta,\alpha}_{x,\infty}$.

III. Let $c > 0,\ c' > 0,\ c'' > 0,\ a,\ b$ be real numbers. Then a third three variable analogue of $I^{\alpha,\beta,\eta}_{0,x}$ is as defined below:

$$ {}_3I^{a,b;c,c',c''}_{0,x;0,y;0,z}\ f(x,y,z) = \frac{x^{-a}y^{-a}z^{-a}}{\Gamma(c)\Gamma(c')\Gamma(c'')} $$

$$ \int_0^x \int_0^y \int_0^z (x-u)^{c-1}\ (y-v)^{c'-1}\ (z-w)^{c''-1} \tag{2.21} $$

$$ F_C^{(3)} \left[\begin{matrix} a, b & ; 1-\frac{u}{x}, 1-\frac{v}{y}, 1-\frac{w}{z} \\ c, c', c''; & \end{matrix} \right] f(u,v,w)\,dw\,dv\,du $$

where

$$F_C^{(3)}\left[\begin{array}{l} a,b \quad ;x,y,z \\ c,c',c''; \end{array}\right]=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(a)_{n+r+s}\,(b)_{n+r+s}}{n!\;r!\;s!\;(c)_n(c')_r(c'')_s}\,x^n y^r z^s$$

Under the same conditions of (2.21), a third three variable analogue of $J_{n,\infty}^{\alpha,\beta,\eta}$ is as given below:

$$_3J_{x,\infty;y,\infty;z,\infty}^{a,b;c,c',c''}f(x,y,z)=\frac{1}{\Gamma(c)\Gamma(c')\Gamma(c'')}$$

$$\int_x^{\infty}\int_y^{\infty}\int_z^{\infty}(u-x)^{c-1}(v-y)^{c'-1}(w-z)^{c''-1} \tag{2.22}$$

$$F_C^{(3)}\left[\begin{array}{l} a,b \quad ;1-\frac{x}{u},1-\frac{y}{v},1-\frac{z}{w} \\ c,c',c''; \end{array}\right]f(u,v,w)\,dw\,dv\,du$$

IV. Let $c > 0$, a, b, b', b'' be real numbers. A fourth three variable analogue of fractional integral operator $I_{0,x}^{\alpha,\beta,\eta}$ due to M. Saigo is defined as:

$$_4I_{0,x;0,y;0,z}^{a,b,b',b'';c}\;f(x,y,z)=\frac{x^{-a}y^{-a}z^{-a}}{\{\Gamma(c)\}^3}$$

$$\int_0^{x}\int_0^{y}\int_0^{z}(x-u)^{c-1}(y-v)^{c'-1}(z-w)^{c''-1} \tag{2.23}$$

$$F_C^{(3)}\left[\begin{array}{l} a,b,b',b'';1-\frac{u}{x},1-\frac{v}{y},1-\frac{w}{z} \\ c \qquad\quad ; \end{array}\right]f(u,v,w)\,dw\,dv\,du$$

where

$$F_C^{(3)}\left[\begin{array}{l} a,b \quad ;x,y,z \\ c,c',c''; \end{array}\right]=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(a)_{n+r+s}(b)_{n+r+s}}{n!\;r!\;s!\;(c)_n(c')_r(c'')_s}\,x^n y^r z^s$$

Special Cases:

(i) For $a = b = b' = b'' = 0,\ c = \alpha$, (2.23) reduces to

$$ {}_4I^{0,0,0,0;\alpha}_{0,x;0,y;0,z}\ f(x,y,z) \ = \ {}_2R^{\alpha}_{0,x;0,y;0,z}\ f(x,y,z) $$

$$ = \frac{1}{\{\Gamma(\alpha)\}^3} \int_0^x \int_0^y \int_0^z (x-u)^{\alpha-1}\ (y-v)^{\alpha-1} (z-w)^{\alpha-1} f(u,v,w)\, dw\, dv\, du \tag{2.24} $$

Which is (2.12) i.e. a three variable analogue of Riemann-Liouville fractional integral operator $R^{\alpha}_{0,x}$.

(ii) For $a = c = \alpha,\ b = -\eta,\ b' = b'' = 0$, (2.23) becomes

$$ {}_4I^{\alpha,-\eta,0,0;\alpha}_{0,x;0,y;0,z}\ f(x,y,z) = {}^{x}_{3}E^{\alpha,\eta}_{0,x;0,y;0,z}\ f(x,y,z) $$

$$ = \frac{x^{-\alpha-\eta} y^{-\alpha} z^{-\alpha}}{\{\Gamma(\alpha)\}^3} \int_0^x \int_0^y \int_0^z (x-u)^{\alpha-1} (y-v)^{\alpha-1} (z-w)^{\alpha-1} u^{\eta} $$

$$ f(u,v,w)\, dw\, dv\, du \tag{2.25} $$

(iii) For $a = c = \alpha,\ b = b'' = 0,\ b' = -\eta$, (2.23) gives

$$ {}_4I^{\alpha,0,-\eta,0;\alpha}_{0,x;0,y;0,z} f(x,y,z) = {}^{y}_{3}E^{\alpha,\eta}_{0,x;0,y;0,z} f(x,y,z) $$

$$ = \frac{x^{-\alpha} y^{-\alpha-\eta} z^{-\alpha}}{\{\Gamma(\alpha)\}^3} \int_0^x \int_0^y \int_0^z (x-u)^{\alpha-1} (y-v)^{\alpha-1} (z-w)^{\alpha-1} v^{\eta} $$

$$ f(u,v,w)\, dw\, dv\, du \tag{2.26} $$

(*iv*) For $a = c = \alpha,\ b = b' = 0,\ b'' = -\eta$, (2.23) yields

$$
{}_4I^{\alpha,0,0,-\eta;\alpha}_{0,x;0,y;0,z}\ f(x,y,z) = {}^{z}_{3}E^{\alpha,\eta}_{0,x;0,y;0,z}\ f(x,y,z)
$$

$$
= \frac{x^{-\alpha}y^{-\alpha}z^{-\alpha-\eta}}{\{\Gamma(\alpha)\}^3}\int_0^x\int_0^y\int_0^z (x-u)^{\alpha-1}(y-v)^{\alpha-1}(z-w)^{\alpha-1}w^{\eta}
$$

$$
f(u,v,w)dw\,dv\,du \tag{2.27}
$$

Here (2.25), (2.26) and (2.27) may be considered as third three variable analogues of Erdelyi-Kober fractional integral operator $E^{\alpha,\eta}_{0,x}$.

It may be remarked here that (2.25), (2.26) and (2.27) can also be obtained from (2.3), (2.4) and (2.5) respectively by taking $\alpha = \beta = \gamma$.

Under the same condition of (2.23), a fourth three variable analogue of another fractional integral operator $J^{\alpha,\beta,\gamma}_{x,\infty}$ due to M. Saigo is defined as follows:

$$
{}_4I^{a,b,b',b'';c}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z) = \frac{1}{\{\Gamma(c)\}^3}\int_x^\infty\int_y^\infty\int_z^\infty (u-x)^{c-1}(v-y)^{c-1}(w-z)^{c-1}
$$

$$
F^{(3)}_D\left[\begin{array}{l} a,b,b',b'';1-\frac{u}{x},1-\frac{v}{y},1-\frac{w}{z} \\ c \qquad\qquad ; \end{array}\right] u^{-a}v^{-a}w^{-a}f(u,v,w)\,dw\,dv\,du \tag{2.28}
$$

Special Cases:

(*i*) For $a = b = b' = b'' = 0,\ c = \alpha$, (2.28) reduces to

$$
{}_4I^{0,0,0,0;\alpha}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z) \ = \ {}_2L^{\alpha}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z)
$$

$$= \frac{1}{\{\Gamma(\alpha)\}^3} \int_x^\infty \int_y^\infty \int_z^\infty (u-x)^{\alpha-1}(v-y)^{\alpha-1}(w-z)^{\alpha-1} f(u,v,w)\, dw\, dv\, du \tag{2.29}$$

Which is (2.17) i.e. a second three variable analogue of Weyl fractional integral operator $L^{\alpha}_{x,\infty}$. It can be obtained from (2.7) by taking $\alpha = \beta = \gamma$.

(*ii*) For $a = c = \alpha,\ b = -\eta,\ b' = b'' = 0$, (2.28) becomes

$$_4I^{\alpha,-\eta,0,0;\alpha}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z) \ = \ {}^{x}_{3}K^{\alpha,\eta}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z)$$

$$= \frac{x^\eta}{\{\Gamma(\alpha)\}^3} \int_x^\infty \int_y^\infty \int_z^\infty (u-x)^{\alpha-1}\ (v-y)^{\alpha-1}(w-z)^{\alpha-1} u^{-\alpha-\eta} v^{-\alpha} w^{-\alpha}$$

$$f(u,v,w)\, dw\, dv\, du \tag{2.30}$$

(*iii*) For $a = c = \alpha,\ b = b'' = 0,\ b' = -\eta$, (2.28) gives

$$_4I^{\alpha,0,-\eta,0;\alpha}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z) = {}^{y}_{3}K^{\alpha,\eta}_{x,\infty;y,\infty;z,\infty} f(x,y,z)$$

$$= \frac{y^\eta}{\{\Gamma(\alpha)\}^3} \int_x^\infty \int_y^\infty \int_z^\infty (u-x)^{\alpha-1}(v-y)^{\alpha-1}(w-z)^{\alpha-1} u^{-\alpha} v^{-\alpha-\eta} w^{-\alpha}$$

$$f(u,v,w)\, dw\, dv\, du \tag{2.31}$$

(*iv*) For $a = c = \alpha,\ b = b' = 0,\ b'' = -\eta$, (2.28) yields

$$
{}_4I^{\alpha,0,0,-\eta;\alpha}_{x,\infty;y,\infty;z,\infty}\, f(x,y,z) = {}^{z}_{3}K^{\alpha,\eta}_{x,\infty;y,\infty;z,\infty}\, f(x,y,z)
$$

$$
= \frac{z^{\eta}}{\{\Gamma(\alpha)\}^3} \int_x^{\infty}\int_y^{\infty}\int_z^{\infty} (u-x)^{\alpha-1}(v-y)^{\alpha-1}(w-z)^{\alpha-1}u^{-\alpha}v^{-\alpha}w^{-\alpha-\eta}
$$

$$
f(u,v,w)\, dw\, dv\, du \tag{2.32}
$$

Here (2.30), (2.31) and (2.32) may be considered as third three variable analogues of Erdelyi-Kober fractional integral operator $K^{\alpha,\eta}_{x,\infty}$.

Further (2.30), (2.31) and (2.32) can also be obtained from (2.8), (2.9) and (2.10) respectively by taking $\alpha = \beta = \gamma$.

3 Some Results

In this section certain theorems involving the above operators will be given:

Theorem 3.1. *For functions* $f(x,y,z)$, $g(x,y,z)$, $f\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)$ *and* $g\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)$ *defined for* $0 \le x < \infty$, $0 \le y < \infty$, $0 \le z < \infty$ *and* $c > 0$, $c' > 0$, $c'' > 0$, *we have*

$$
\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} x^{a-c-1}y^{a-c'-1}z^{a-c''-1} f\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right) {}_1I^{a,b,b',b'';c,c',c''}_{0,x;0,y;0,z}\, g(x,y,z)dzdydx
$$

$$
= \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} x^{a-c-1}y^{a-c'-1}z^{a-c''-1} g\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right) {}_1I^{a,b,b',b'';c,c',c''}_{0,x;0,y;0,z}
$$

$$
f(x,y,z)dzdydx \tag{3.1}
$$

provided that each triple integral exists.

Theorem 3.2. *Under the conditions stated in theorem 3.1, we have*

$$\int_0^\infty\int_0^\infty\int_0^\infty x^{a-c-1}y^{a-c'-1}z^{a-c''-1}f\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right){}_1J^{a,b,b',b'';c,c',c''}_{0,x;0,y;0,z}$$
$$g(x,y,z)dzdydx=\int_0^\infty\int_0^\infty\int_0^\infty x^{a-c-1}y^{a-c'-1}z^{a-c''-1} \tag{3.2}$$
$$g\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right){}_1J^{a,b,b',b'';c,c',c''}_{0,x;0,y;0,z}f(x,y,z)dzdydx$$

provided that each triple integral exists.

Theorem 3.3. *For functions* $f(x,y,z)$, $g(x,y,z)$, $f\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)$ *and* $g\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)$ *defined for* $0\le x<\infty$, $0\le y<\infty$, $0\le z<\infty$ *and* $c>0$, *we have*

$$\int_0^\infty\int_0^\infty\int_0^\infty x^{a-c-1}y^{a'-c-1}z^{a''-c-1}f\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right){}_2I^{a,a',a'',b,b',b'';c}_{0,x;0,y;0,z}$$
$$g(x,y,z)dzdydx=\int_0^\infty\int_0^\infty\int_0^\infty x^{a-c-1}y^{a'-c-1}z^{a''-c-1}g\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right) \tag{3.3}$$
$${}_2I^{a,a',a'',b,b',b'';c}_{0,x;0,y;0,z}f(x,y,z)dzdydx$$

provided that each triple integral exists.

Theorem 3.4. *Under the conditions stated in theorem 3.3, we have*

$$\int_0^\infty\int_0^\infty\int_0^\infty x^{a-c-1}y^{a'-c-1}z^{a''-c-1}f\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)\,{}_2J^{a,a',a'',b,b',b'';c}_{x,\infty;y,\infty;z,\infty}$$

$$g(x,y,z)dzdydx=\int_0^\infty\int_0^\infty\int_0^\infty x^{a-c-1}y^{a'-c-1}z^{a''-c-1}g\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right) \tag{3.4}$$

$${}_2J^{a,a',a'',b,b',b'';c}_{x,\infty;y,\infty;z,\infty}f(x,y,z)dzdydx$$

provided that each triple integral exists.

Theorem 3.5. *For functions* $f(x,y,z)$, $g(x,y,z)$, $f\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)$ *and* $g\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)$ *defined for* $0\le x<\infty$, $0\le y<\infty$, $0\le z<\infty$ *and* $c>0$, $c'>0$, $c''>0$, *we have*

$$\int_0^\infty\int_0^\infty\int_0^\infty x^{a-c-1}y^{a-c'-1}z^{a-c''-1}f\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)\,{}_3I^{a,b;c,c',c''}_{0,x;0,y;0,z}g(x,y,z)dzdydx$$

$$=\int_0^\infty\int_0^\infty\int_0^\infty x^{a-c-1}y^{a-c'-1}z^{a-c''-1}g\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)\,{}_3I^{a,b;c,c',c''}_{0,x;0,y;0,z}f(x,y,z)dzdydx \tag{3.5}$$

provided that each triple integral exists.

Theorem 3.6. *Under the conditions stated in theorem 3.5, we have*

$$\int_0^\infty\int_0^\infty\int_0^\infty x^{a-c-1}y^{a-c'-1}z^{a-c''-1}f\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)\,{}_3J^{a,b;c,c',c''}_{0,x;0,y;0,z}\ g(x,y,z)dzdydx$$

$$=\int_0^\infty\int_0^\infty\int_0^\infty x^{a-c-1}y^{a-c'-1}z^{a-c''-1}g\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)\,{}_3J^{a,b;c,c',c''}_{0,x;0,y;0,z}f(x,y,z)dzdydx \tag{3.6}$$

provided that each triple integral exists.

Theorem 3.7. *For functions* $f(x, y, z)$, $g(x, y, z)$, $f\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$ *and* $g\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$ *defined for* $0 \leq x < \infty$, $0 \leq y < \infty$, $0 \leq z < \infty$ *and* $c > 0$, *we have*

$$\int_0^\infty\int_0^\infty\int_0^\infty (xyz)^{a-c-1} f\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right) {}_4I_{0,x;0,y;0,z}^{a,b,b',b'';c} g(x,y,z)dzdydx$$
$$= \int_0^\infty\int_0^\infty\int_0^\infty (xyz)^{a-c-1} g\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right) {}_4I_{0,x;0,y;0,z}^{a,b,b',b'';c} f(x,y,z)dzdydx \quad (3.7)$$

provided that each triple integral exists.

Theorem 3.8. *Under the conditions stated in theorem 3.7, we have*

$$\int_0^\infty\int_0^\infty\int_0^\infty (xyz)^{a-c-1} f\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right) {}_4J_{0,x;0,y;0,z}^{a,b,b',b'';c} g(x,y,z)dzdydx$$
$$= \int_0^\infty\int_0^\infty\int_0^\infty (xyz)^{a-c-1} g\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right) {}_4J_{0,x;0,y;0,z}^{a,b,b',b'';c} f(x,y,z)dzdydx \quad (3.8)$$

provided that each triple integral exists.

From these theorems certain interesting corollaries follow readily for the operators (2.2), (2.3), (2.4), (2.5), (2.7), (2.8), (2.9), (2.10), (2.12), (2.13), (2.14), (2.15), (2.17), (2.18), (2.19), (2.20), (2.24), (2.25), (2.26), (2.27), (2.29), (2.30), (2.31) and (2.32).

We now give proof of theorem 3.7. Proofs of other theorems follow on similar lines.

Proof of Theorem 3.7 : we have

$$\int_0^\infty\int_0^\infty\int_0^\infty (xyz)^{a-c-1} f\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right) {}_4I^{a,b,b',b'';c}_{0,x;0,y;0,z}\; g(x,y,z)dzdydx$$

$$= \int_{x=0}^\infty\int_{y=0}^\infty\int_{z=0}^\infty (xyz)^{a-c-1} f\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right) \frac{(xyz)^{-a}}{\{\Gamma(c)\}^3}$$

$$\int_{u=0}^{u=x}\int_{v=0}^{v=y}\int_{w=0}^{w=z} (x-u)^{c-1}(y-v)^{c-1}(z-w)^{c-1}$$

$$F_D^{(3)}\left[\begin{array}{l} a,b,b',b'';1-\frac{u}{x},1-\frac{v}{y},1-\frac{w}{z} \\ c \qquad\qquad ; \end{array}\right] g(u,v,w)\; dw\; dv\; du\; dz\; dy\; dx$$

$$= \int_{u=0}^\infty\int_{v=0}^\infty\int_{w=0}^\infty\int_{x=u}^\infty\int_{y=v}^\infty\int_{z=w}^\infty \frac{(xyz)^{-c-1}}{\{\Gamma(c)\}^3} f\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)$$

$$(x-u)^{c-1}(y-v)^{c-1}(z-w)^{c-1}$$

$$F_D^{(3)}\left[\begin{array}{l} a,b,b',b'';1-\frac{u}{x},1-\frac{v}{y},1-\frac{w}{z} \\ c \qquad\qquad ; \end{array}\right] g(u,v,w)\; dz\; dy\; dx\; dw\; dv\; du$$

$$= \int_{l=0}^\infty\int_{p=0}^\infty\int_{q=0}^\infty\int_{r=l}^\infty\int_{s=p}^\infty\int_{t=q}^\infty \frac{(lpq)^{-c-1}}{\{\Gamma(c)\}^3} f\left(\frac{1}{l},\frac{1}{p},\frac{1}{q}\right)$$

$$(r-l)^{c-1}(s-p)^{c-1}(t-q)^{c-1}$$

$$F_D^{(3)}\left[\begin{array}{l} a,b,b',b'';1-\frac{l}{r},1-\frac{p}{s},1-\frac{q}{t} \\ c \qquad\qquad ; \end{array}\right] g(r,s,t)\; dq\; dp\; dl\; dt\; ds\; dr$$

(Using the property of definite integrals that $\int_a^b f(x)dx = \int_a^b f(t)dt$). We now make the substitution $r = \frac{1}{x}$, $s = \frac{1}{y}$, $t = \frac{1}{z}$, $l = \frac{1}{u}$, $p = \frac{1}{v}$, $q = \frac{1}{w}$. Then the above becomes

$$= \int_{x=0}^{\infty}\int_{y=0}^{\infty}\int_{z=0}^{\infty} (xyz)^{a-c-1} g\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right) \frac{(xyz)^{-a}}{\{\Gamma(c)\}^3}$$

$$\int_{u=0}^{x}\int_{v=0}^{y}\int_{w=0}^{z} (x-u)^{c-1}(y-v)^{c-1}(z-w)^{c-1}$$

$$F_D^{(3)}\left[\begin{array}{l} a, b, b', b''; 1-\frac{u}{x}, 1-\frac{v}{y}, 1-\frac{w}{z} \\ c \qquad ; \end{array}\right] f(u,v,w)\, dw\, dv\, du\, dz\, dy\, dx$$

$$= \int_{x=0}^{\infty}\int_{y=0}^{\infty}\int_{z=0}^{\infty} (xyz)^{a-c-1} g\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)\ {}_4I_{0,x;0,y;0,z}^{a,b,b',b'';c}\ f(x,y,z) dz dy dx$$

This completes the proof of Theorem 3.7. Similarly, we can prove theorems 3.1 to 3.6 and 3.8.

A general triple hypergeometric series $F^{(3)}[x,y,z]$ [cf. Srivastava [21], p.428] is defined as

$$F^{(3)}[x,y,z] \equiv F^{(3)}\left[\begin{array}{l} (a) :: (b); (b'); (b'') \ : (d); (d'); (d''); \\ (e) :: (g); (g'); (g'') : (h); (h'); (h''); \end{array} \ x, y, z\right]$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \wedge(m,n,p) \frac{x^m y^n z^p}{m! n! p!}$$

where, for convenience

$$\wedge(m,n,p) = \frac{\prod_{j=1}^{A}(a_j)_{m+n+p}\prod_{j=1}^{B}(b_j)_{m+n}\prod_{j=1}^{B'}(b'_j)_{n+p}\prod_{j=1}^{B''}(b''_j)_{m+p}\prod_{j=1}^{D}(d_j)_m\prod_{j=1}^{D'}(d'_j)_n\prod_{j=1}^{D''}(d''_j)_p}{\prod_{j=1}^{E}(e_j)_{m+n+p}\prod_{j=1}^{G}(g_j)_{m+n}\prod_{j=1}^{G'}(g'_j)_{n+p}\prod_{j=1}^{G''}(g''_j)_{m+p}\prod_{j=1}^{H}(h_j)_m\prod_{j=1}^{H'}(h'_j)_n\prod_{j=1}^{H''}(h''_j)_p}$$

Further, we can prove the following theorems

Theorem 3.9. *If*

$$F(x,y,z) = \int_0^\infty \int_0^\infty \int_0^\infty (rx)^{\lambda-1}(sy)^{\mu-1}\ (tz)^{\nu-1}$$

$$F^{(3)}\left[\begin{array}{l}(a)::(b);(b');(b'')\ :(d);(d');(d'');\\ (e)::(g);(g');(g''):(h);(h');(h'');\end{array} rx,sy,tz\right] g(r,s,t)dt\,ds\,dr$$

and $\psi(x,y,z) = {}_1I^{a,b,b',b'';c,c',c''}_{0,x;0,y;0,z}\ f(x,y,z)$ *for* $c > 0,\ c' > 0,\ c'' > 0$ *and* $a,\ b,\ b',b''$ *real numbers, then we have*

$$\psi(x,y,z) = \int_0^\infty \int_0^\infty \int_0^\infty (r)^{\lambda-1}(s)^{\mu-1}(t)^{\nu-1}g(r,s,t)$$

$${}_1I^{a,b,b',b'';c,c',c''}_{0,x;0,y;0,z}\ (x)^{\lambda-1}(y)^{\mu-1}\ (z)^{\nu-1} \tag{3.9}$$

$$F^{(3)}\left[\begin{array}{l}(a)::(b);(b');(b'')\ :(d);(d');(d'');\\ (e)::(g);(g');(g''):(h);(h');(h'');\end{array} rx,sy,tz\right] dtdsdr$$

provided that the triple integrals involved exist.

Theorem 3.10. *If*

$$f(x,y,z) = \int_0^\infty \int_0^\infty \int_0^\infty (rx)^{\lambda-1}(sy)^{\mu-1}\ (tz)^{\nu-1}$$

$$F^{(3)}\left[\begin{array}{l}(a)::(b);(b');(b'')\ :(d);(d');(d'');\\(e)::(g);(g');(g''):(h);(h');(h'');\end{array} rx,sy,tz\right] g(r,s,t)dt\,ds\,dr$$

and $\psi(x,y,z) = {}_1J^{a,b,b',b'';c,c',c''}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z)$ *for* $c > 0,\ c' > 0,\ c'' > 0$ *and* $a,\ b,\ b',b''$ *real numbers, then we have*

$$\psi(x,y,z) = \int_0^\infty\int_0^\infty\int_0^\infty (r)^{\lambda-1}(s)^{\mu-1}\ (t)^{\nu-1}\ g(r,s,t)$$

$${}_1J^{a,b,b',b'';c,c',c''}_{x,\infty;y,\infty;z,\infty}\ (x)^{\lambda-1}(y)^{\mu-1}\ (z)^{\nu-1} \tag{3.10}$$

$$F^{(3)}\left[\begin{array}{l}(a)::(b);(b');(b'')\ :(d);(d');(d'');\\(e)::(g);(g');(g''):(h);(h');(h'');\end{array} rx,sy,tz\right] dtdsdr$$

provided that the triple integrals involved exist.

Theorem 3.11. *If*

$$f(x,y,z) = \int_0^\infty\int_0^\infty\int_0^\infty (rx)^{\lambda-1}(sy)^{\mu-1}\ (tz)^{\nu-1}$$

$$F^{(3)}\left[\begin{array}{l}(a)::(b);(b');(b'')\ :(d);(d');(d'');\\(e)::(g);(g');(g''):(h);(h');(h'');\end{array} rx,sy,tz\right] g(r,s,t)dt\ ds\ dr$$

and $\psi(x,y,z) = {}_2I^{a,a',a'',b,b',b'';c}_{0,x;0,y;0,z}\ f(x,y,z)$ *for* $c > 0$, *and* $a,\ a',\ a'',\ b,\ b',\ b''$ *real numbers, then we have*

$$\psi(x,y,z) = \int_0^\infty\int_0^\infty\int_0^\infty (r)^{\lambda-1}(s)^{\mu-1}(t)^{\nu-1}\ g(r,s,t)$$

$${}_2I^{a,a',a'',b,b',b'';c}_{0,x;0,y;0,z}\ (x)^{\lambda-1}(y)^{\mu-1}\ (z)^{\nu-1} \tag{3.11}$$

$$F^{(3)}\left[\begin{array}{l}(a)::(b);(b');(b'')\ :(d);(d');(d'');\\(e)::(g);(g');(g''):(h);(h');(h'');\end{array} rx,sy,tz\right] dtdsdr$$

provided that the triple integrals involved exist.

Theorem 3.12. *If*

$$f(x,y,z) = \int_0^\infty \int_0^\infty \int_0^\infty (rx)^{\lambda-1}(sy)^{\mu-1}\ (tz)^{\nu-1}$$

$$F^{(3)}\left[\begin{array}{l} (a) :: (b); (b'); (b'') \ : (d); (d'); (d''); \\ (e) :: (g); (g'); (g'') : (h); (h'); (h''); \end{array} rx, sy, tz\right] g(r,s,t)dt\ ds\ dr$$

and $\psi(x,y,z) = {}_2J^{a,a',a'',b,b',b'';c}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z)$ *for* $c > 0$, *and* a, a', a'', b, b', b'' *real numbers, then we have*

$$\psi(x,y,z) = \int_0^\infty \int_0^\infty \int_0^\infty (r)^{\lambda-1}(s)^{\mu-1}(t)^{\nu-1} g(r,s,t)$$

$${}_2J^{a,a',a'',b,b',b'';c}_{x,\infty;y,\infty;z,\infty}\ (x)^{\lambda-1}(y)^{\mu-1}\ (z)^{\nu-1} \tag{3.12}$$

$$F^{(3)}\left[\begin{array}{l} (a) :: (b); (b'); (b'') \ : (d); (d'); (d''); \\ (e) :: (g); (g'); (g'') : (h); (h'); (h''); \end{array} rx, sy, tz\right] dtdsdr$$

provided that the triple integrals involved exist.

Theorem 3.13. *If*

$$f(x,y,z) = \int_0^\infty \int_0^\infty \int_0^\infty (rx)^{\lambda-1}(sy)^{\mu-1}\ (tz)^{\nu-1}$$

$$F^{(3)}\left[\begin{array}{l} (a) :: (b); (b'); (b'') \ : (d); (d'); (d''); \\ (e) :: (g); (g'); (g'') : (h); (h'); (h''); \end{array} rx, sy, tz\right] g(r,s,t)dt\ ds\ dr$$

and $\psi(x,y,z) = {}_3I^{a,b;c,c',c''}_{0,x;0,y;0,z}\ f(x,y,z)$ *for* $c > 0,\ c' > 0,\ c'' > 0$ *and* $a,\ b$ *real numbers, then we have*

$$\psi(x,y,z) = \int_0^\infty\int_0^\infty\int_0^\infty (r)^{\lambda-1}(s)^{\mu-1}(t)^{\nu-1}g(r,s,t)$$

$${}_3I^{a,b;c,c',c''}_{0,x;0,y;0,z}\ (x)^{\lambda-1}(y)^{\mu-1}\ (z)^{\nu-1} \tag{3.13}$$

$$F^{(3)}\left[\begin{array}{l}(a) :: (b); (b'); (b'')\ : (d); (d'); (d'');\\ (e) :: (g); (g'); (g'') : (h); (h'); (h'');\end{array} rx, sy, tz\right] dtdsdr$$

provided that the triple integrals involved exist.

Theorem 3.14. *If*

$$f(x,y,z) = \int_0^\infty\int_0^\infty\int_0^\infty (rx)^{\lambda-1}(sy)^{\mu-1}\ (tz)^{\nu-1}$$

$$F^{(3)}\left[\begin{array}{l}(a) :: (b); (b'); (b'')\ : (d); (d'); (d'');\\ (e) :: (g); (g'); (g'') : (h); (h'); (h'');\end{array} rx, sy, tz\right]\ g(r,s,t)dt\ ds\ dr$$

and $\psi(x,y,z) = {}_3J^{a,b;c,c',c''}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z)$ *for* $c > 0,\ c' > 0,\ c'' > 0$ *and* $a,\ b$ *real numbers, then we have*

$$\psi(x,y,z) = \int_0^\infty\int_0^\infty\int_0^\infty (r)^{\lambda-1}(s)^{\mu-1}\ (t)^{\nu-1}g(r,s,t)$$

$${}_3J^{a,b;c,c',c''}_{x,\infty;y,\infty;z,\infty}\ (x)^{\lambda-1}(y)^{\mu-1}\ (z)^{\nu-1} \tag{3.14}$$

$$F^{(3)}\left[\begin{array}{l}(a) :: (b); (b'); (b'')\ : (d); (d'); (d'');\\ (e) :: (g); (g'); (g'') : (h); (h'); (h'');\end{array} rx, sy, tz\right] dtdsdr$$

provided that the triple integrals involved exist.

Theorem 3.15. *If*

$$f(x,y,z)=\int_0^\infty\int_0^\infty\int_0^\infty (rx)^{\lambda-1}(sy)^{\mu-1}\ (tz)^{\nu-1}$$

$$F^{(3)}\left[\begin{array}{l}(a)::(b);(b');(b'')\ :(d);(d');(d'');\\ (e)::(g);(g');(g''):(h);(h');(h'');\end{array} rx,sy,tz\right]\ g(r,s,t)dt\ ds\ dr$$

and $\psi(x,y,z)={}_4I^{a,b,b',b'';c}_{0,x;0,y;0,z}\ f(x,y,z)$ *for* $c\ >\ 0$*, then we have*

$$\psi(x,y,z)=\int_0^\infty\int_0^\infty\int_0^\infty (r)^{\lambda-1}(s)^{\mu-1}(t)^{\nu-1}g(r,s,t)$$

$${}_4I^{a,b,b',b'';c}_{0,x;0,y;0,z}\ (x)^{\lambda-1}(y)^{\mu-1}\ (z)^{\nu-1} \tag{3.15}$$

$$F^{(3)}\left[\begin{array}{l}(a)::(b);(b');(b'')\ :(d);(d');(d'');\\ (e)::(g);(g');(g''):(h);(h');(h'');\end{array} rx,sy,tz\right]dtdsdr$$

provided that the triple integrals involved exist.

Proof of Theorem 3.15 : We have

$$\psi(x,y,z)={}_4I^{a,b,b',b'';c}_{0,x;0,y;0,z}\ f(x,y,z)=\tfrac{(xyz)^{-a}}{\{\Gamma(c)\}^3}$$

$$\int_0^x\int_0^y\int_0^z (x-u)^{c-1}\ (y-v)^{c-1}(z-w)^{c-1}$$

$$F_D^{(3)}\left[\begin{array}{ll}a,b,b',b'';1-\frac{u}{x},1-\frac{v}{y},1-\frac{w}{z}\\ c\qquad\qquad\ ;\end{array}\right]\ f(u,v,w)\ dw\ dv\ du$$

$$=\tfrac{(xyz)^{-a}}{\{\Gamma(c)\}^3}\int_0^x\int_0^y\int_0^z (x-u)^{c-1}(y-v)^{c-1}(z-w)^{c-1}$$

$$F_D^{(3)}\left[\begin{array}{l} a,b,b',b'';1-\frac{u}{x},1-\frac{v}{y},1-\frac{w}{z} \\ c \qquad\qquad ; \end{array}\right] f(u,v,w)\ dw\ dv\ du$$

$$= \int_0^\infty\int_0^\infty\int_0^\infty (ru)^{\lambda-1}(sv)^{\mu-1}\ (tw)^{\nu-1}$$

$$F^{(3)}\left[\begin{array}{l} (a)::(b);(b');(b'')\ :(d);(d');(d''); \\ (e)::(g);(g');(g''):(h);(h');(h''); \end{array} ru,sv,tw\right]$$

$$g(r,s,t)\,dt\,ds\,dr\,dw\,dv\,du$$

$$= \frac{(xyz)^{-a}}{\{\Gamma(c)\}^3}\int_0^x\int_0^y\int_0^z (r)^{\lambda-1}(s)^{\mu-1}\ (t)^{\nu-1}g(r,s,t)$$

$$\int_0^x\int_0^y\int_0^z (u)^{\lambda-1}(v)^{\mu-1}(w)^{\nu-1}$$

$$(x-u)^{c-1}\ (y-v)^{c-1}\ (z-w)^{c-1}$$

$$F_D^{(3)}\left[\begin{array}{l} a,b,b',b'';1-\frac{u}{x},1-\frac{v}{y},1-\frac{w}{z} \\ c \qquad\qquad ; \end{array}\right]$$

$$F^{(3)}\left[\begin{array}{l} (a)::(b);(b');(b'')\ :(d);(d');(d''); \\ (e)::(g);(g');(g''):(h);(h');(h''); \end{array} ru,sv,tw\right] dw\ dv\ du\ dt\ ds\ dr$$

$$= \int_0^x\int_0^y\int_0^z (r)^{\lambda-1}(s)^{\mu-1}\ (t)^{\nu-1}\ g(r,s,t)\ {}_4I_{0,x;0,y;0,z}^{a,b,b',b'';c}\ x^{\lambda-1}y^{\mu-1}\ z^{\nu-1}$$

$$F^{(3)}\left[\begin{array}{l}(a)::(b);(b');(b''):(d);(d');(d'');\\(e)::(g);(g');(g''):(h);(h');(h'');\end{array} rx,sy,tz\right] dt\,ds\,dr$$

This completes the proof of 3.15. Similarly, we can prove theorem 3.16 stated below:

Theorem 3.16. *If*

$$f(x,y,z)=\int_0^\infty\int_0^\infty\int_0^\infty (rx)^{\lambda-1}(sy)^{\mu-1}\ (tz)^{\nu-1}$$

$$F^{(3)}\left[\begin{array}{l}(a)::(b);(b');(b''):(d);(d');(d'');\\(e)::(g);(g');(g''):(h);(h');(h'');\end{array} rx,sy,tz\right] g(r,s,t)dt\,ds\,dr$$

and $\psi(x,y,z)={}_4J^{a,b;c,c',c''}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z)$ *for* $c>0$, *then we have*

$$\psi(x,y,z)=\int_0^\infty\int_0^\infty\int_0^\infty (r)^{\lambda-1}(s)^{\mu-1}(t)^{\nu-1}g(r,s,t)$$

$${}_4J^{a,b;c,c',c''}_{x,\infty;y,\infty;z,\infty}\ (x)^{\lambda-1}(y)^{\mu-1}\ (z)^{\nu-1} \tag{3.16}$$

$$F^{(3)}\left[\begin{array}{l}(a)::(b);(b');(b''):(d);(d');(d'');\\(e)::(g);(g');(g''):(h);(h');(h'');\end{array} rx,sy,tz\right] dtdsdr$$

provided that the triple integrals involved exist.

The proofs of theorems 3.9 to 3.14 follow on similar lines.

4 Results Analogues to Integration by Parts

Certain results analogues to integration by parts for the operators (2.1), (2.6), (2.11), (2.16), (2.21), (2.22), (2.23) and (2.28) are given in this section in the form of the following theorems:

Theorem 4.1. *For functions of three variables $f(x,y,z)$ and $g(x,y,z)$ defined in the positive octant of the three dimensional space and $c > 0$, $c' > 0$, $c'' > 0$, we have*

$$\int_0^\infty \int_0^\infty \int_0^\infty f(x,y,z)\ {}_1I_{0,x;0,y;0,z}^{a,b,b',b'';c,c',c''}\ g(x,y,z)\ dz\ dy\ dx$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty g(x,y,z)\ {}_1J_{x,\infty;y,\infty;z,\infty}^{a,b,b',b'';c,c',c''}\ f(x,y,z)\ dz\ dy\ dx \qquad (4.1)$$

provided that each triple integral exists.

Theorem 4.2. *For functions of three variables $f(x,y,z)$ and $g(x,y,z)$ defined in the positive octant of the three dimensional space and $c > 0$, we have*

$$\int_0^\infty \int_0^\infty \int_0^\infty f(x,y,z)\ {}_2I_{0,x;0,y;0,z}^{a,a',a'',b,b',b'';c}\ g(x,y,z)\ dz\ dy\ dx$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty g(x,y,z)\ {}_2J_{x,\infty;y,\infty;z,\infty}^{a,a',a'',b,b',b'';c}\ f(x,y,z)\ dz\ dy\ dx \qquad (4.2)$$

provided that each triple integral exists.

Theorem 4.3. *For functions of three variables $f(x,y,z)$ and $g(x,y,z)$ defined in the positive octant of the three dimensional space and $c > 0\ c' > 0$, $c'' > 0$, we have*

$$\int_0^\infty \int_0^\infty \int_0^\infty f(x,y,z)\ {}_3I_{0,x;0,y;0,z}^{a,b;c,c',c''}\ g(x,y,z)\ dz\ dy\ dx$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty g(x,y,z)\ {}_3J_{x,\infty;y,\infty;z,\infty}^{a,b;c,c',c''}\ f(x,y,z)\ dz\ dy\ dx \qquad (4.3)$$

provided that each triple integral exists.

Theorem 4.4. *For functions of three variables $f(x,y,z)$ and $g(x,y,z)$ defined in the positive octant of the three dimensional space and $c > 0$, we have*

$$\int_0^\infty \int_0^\infty \int_0^\infty f(x,y,z)\ {}_4I^{a,b,b',b'';c}_{0,x;0,y;0,z}\ g(x,y,z)\ dz\ dy\ dx$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty g(x,y,z)\ {}_4J^{a,b,b',b'';c}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z)\ dz\ dy\ dx \qquad (4.4)$$

provided that each triple integral exists.

Proof of Theorem 4.4 : We have

$$\int_0^\infty \int_0^\infty \int_0^\infty f(x,y,z)\ {}_4I^{a,b,b',b'';c}_{0,x;0,y;0,z}\ g(x,y,z)\ dz\ dy\ dx$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty f(x,y,z) \frac{(xyz)^{-a}}{\{\Gamma(c)\}^3} \int_{u=0}^{u=x} \int_{v=0}^{v=y} \int_{w=0}^{w=z} (x-u)^{c-1}(y-v)^{c-1}(z-w)^{c-1}$$

$$F_D^{(3)} \left[\begin{array}{l} a, b, b', b''; 1-\frac{u}{x}, 1-\frac{v}{y}, 1-\frac{w}{z} \\ c \qquad\qquad ; \end{array} \right] f(u,v,w)\ dw\ dv\ du$$

$$= \int_{u=0}^\infty \int_{v=0}^\infty \int_{w=0}^\infty \int_{x=u}^\infty \int_{y=v}^\infty \int_{z=w}^\infty \frac{(xyz)^{-a}}{\{\Gamma(c)\}^3} (x-u)^{c-1}(y-v)^{c-1}(z-w)^{c-1} f(x,y,z)$$

$$F_D^{(3)}\left[\begin{array}{l} a,b,b',b'';1-\frac{u}{x},1-\frac{v}{y},1-\frac{w}{z} \\ c \qquad\qquad ; \end{array}\right] g(u,v,w)\ dz\ dy\ dx\ dw\ dv\ du$$

$$= \int_{u=0}^{\infty}\int_{v=0}^{\infty}\int_{w=0}^{\infty} g(u,v,w)\frac{1}{\{\Gamma(c)\}^3}$$

$$\int_{x=u}^{\infty}\int_{y=v}^{\infty}\int_{z=w}^{\infty} (x-u)^{c-1}(y-v)^{c-1}(z-w)^{c-1}$$

$$F_D^{(3)}\left[\begin{array}{l} a,b,b',b'';1-\frac{u}{x},1-\frac{v}{y},1-\frac{w}{z} \\ c \qquad\qquad ; \end{array}\right]$$

$$x^{-a}\ y^{-a}\ z^{-a}\ f(x,y,z)\ dz\ dy\ dx\ dw\ dv\ du$$

$$= \int_{u=0}^{\infty}\int_{v=0}^{\infty}\int_{w=0}^{\infty} g(u,v,w)\ {}_4J^{a,b,b',b'';c}_{x,\infty;y,\infty;z,\infty}\ f(u,v,w)\ dw\ dv\ du$$

$$= \int_{0}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} g(x,y,z)\ {}_4J^{a,b,b',b'';c}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z)\ dz\ dy\ dx$$

This completes the proof of Theorem 4.4. Similarly, we can prove theorems 4.1, 4.2 and 4.3.

From the theorems of this section certain interesting corollaries readily follows for the operators (2.2), (2.3), (2.4), (2.5), (2.7), (2.8), (2.9), (2.10), (2.12), (2.13), (2.14), (2.15), (2.17), (2.18), (2.19), (2.20), (2.24), (2.25), (2.26), (2.27), (2.29), (2.30), (2.31) and (2.32).

References

[1] Buschman, R.G.: *Fractional integration*, Math. Japan., Vol. 9 (1964), pp. 99-106.

[2] Buschman, R.G.: *Integrals of hypergeometric functions*, Math. Z., Vol. 89 (1965), pp. 74-76.

[3] Erdelyi, A. : *Transformation of hypergeometric integrals by means of fractional integration by parts*, Quart. J. Math., Vol. 10 (1939), pp. 176-189.

[4] Hardy, G.H. and Littlewood, J.E.: *Some properties of Fractional integrals*, I. Math. Z., Vol. 27 (1928), pp. 565-606.

[5] Khan, M.A.: *Certain Fractional q-integrals and q-Derivatives*, Nanta Mathematica, Vol. VII, No. 1 (1974), pp. 52-60.

[6] Khan, M.A.: *On Generalized Fractional q-Integrals*, Mathematica Balkanica, Vol. 6, Fasc. 2 (1992), pp. 155-163.

[7] Khan, M.A.: *Transformation of Generalized Fractional q-Integrals*, I. Mathematica Balkanica, Vol. 6, Fasc. 2 (1992), pp. 305-312.

[8] Khan, M.A.: *On Certain Transformations of Generalized Fractional q-Integrals*, II. Mathematica Balkanica, Vol. 7, Fasc. 1 (1993), pp. 27-33.

[9] Khan, M.A. and Abukhammash, G.S.: *A Study of Two Variable Analogues of Certain Fractional Integral Operators*, Pro Mathematica, 17(33)(2003),pp. 31-48.

[10] Khan, M.A. and Abukhammash, G.S.: *On Certain Fractional Integral Operators of Two Varibles And Integral Transforms*, Pro Mathematica, 17(34)(2003),pp. 169-193.

[11] Kober, H.: *On Fractional Integrals and Derivatives*, Quart. J. Math., Vol. 11 (1940), pp. 193-211.

[12] Love, E.R.: *Some integral equations involving hypergeometric functions*, Proc. Edinburgh Math. Soc., Vol. 15 (1967), pp. 169-198.

[13] Raina, R.K.: *On the Weyl fractional differentiation*, Pur. Appl. Math. Sci., Vol. 10 (1979), pp. 37-41.

[14] Raina, R.K.: *A note on the fractional derivatives of a general system of polynomials*, Indian J. Pure. Appl. Math., Vol. 16 (1985), pp. 770-774.

[15] Raina, R.K.: *The Weyl fractional operator of a system of polynomials*, Rend. Sem. Mat. Univ. Padova, Vol. 76 (1986), pp. 171-176.

[16] Ross, B.(ed.): *Fractional Calculus and its applications*, Lecture Note in Mathematica, Vol. 457, Springer-verlag (Berlin), (1975).

[17] Saigo, M.: *A remark on integral operators involving the Gauss hypergeometric functions*, Math. Rep. Kyushu Univ., Vol. 11, (1978), pp. 135-143.

[18] Saigo, M.: *A certain boundary value problem for the Euler-Darboux equation*, Math. Japan., Vol. 25, (1979), pp. 377-385.

[19] Saigo, M. and Raina, R.K.: *Fractional Calculus operators associated with a general class of polynomials*, Fukuoka Univ. Sci. Rep., Vol. 18, (1988), pp. 15-22.

[20] Srivastava, H.M.: *The Weyl fractional integral of a general class of polynomials*, Bull. Un. Mat. Ital. B (6), Vol. 2 (1983), pp. 219-228.

[21] Srivastava, H.M. and Manocha, H.L.: *A treatise on generating functions*, Ellis Horwood, Chichester, (1984).

Resumen

El artículo trata los análogos en tres variables de ciertos operadores de integración fraccionarios presentados por M. Saigo. Además de dar los análogos en tres variables de operadores de integración fraccionarios en una variable anteriormente conocidos como casos especiales de los operadores que se acaban de definir, en el artículo se establecen ciertos resultados en forma de teoremas incluyendo la integración por partes.

Palabras Clave: Operador integral fraccionario, Operadores de integración fraccionaria de Saigo, Función hipergeométrica de Gauss, Integral fraccionaria de Weyl, Cálculo fraccionario.

Mumtaz Ahmad Khan
Department of Applied Mathematics,
Faculty of Engineering and Technology,
Aligarh Muslim University. Aligarh - 202002, U.P., India.
mumtaz_ahmad_khan_2008@yahoo.com