ON THE CONSTANT OF HOMOTHETY FOR COVERING A CONVEX SET WITH ITS SMALLER COPIES

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Abstract

Let H_d denote the smallest integer n such that for every convex body K in $\mathbb{R}^d$ there is a $0 < \lambda < 1$ such that K is covered by n translates of λK. In [2] the following problem was posed: Is there a $0 < \lambda_d < 1$ depending on d only with the property that every convex body K in $\mathbb{R}^d$ is covered by H_d translates of $\lambda_d K$? We prove the affirmative answer to the question and hence show that the Gohberg-Markus-Boltyanski-Hadwiger Conjecture (according to which $H_d \leq 2^d$) holds if, and only if, a formally stronger version of it holds.

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1 Definitions and Results

A *convex body* in $\mathbb{R}^d$ is a compact convex set K with non-empty interior. Its volume is denoted by $\mathrm{vol}(K)$.

Definition 1.1. *For $d \geq 1$ let H_d denote the smallest integer n such that for every convex body K in $\mathbb{R}^d$ there is a $0 < \lambda < 1$ such that K is covered by n translates of λK. Furthermore, let $\overline{H}_d$ denote the smallest integer m such that there is a $0 < \lambda_d < 1$ with the property that every convex body K in $\mathbb{R}^d$ is covered by m translates of $\lambda_d K$.*

Clearly, $H_d \leq \overline{H}_d$. The following question was raised in [2] (Problem 6 in Section 3.2): Is it true that $H_d = \overline{H}_d$?

We answer the question in the affirmative using a simple topological argument.

Theorem 1.2. $H_d = \overline{H}_d$.

The famous conjecture of Gohberg, Markus, Boltyanski and Hadwiger states that $H_d \leq 2^d$ (and only the cube requires 2^d smaller positive homothetic copies to be covered). For more information on the conjecture, refer to [1], [7] and [11]. In view of Theorem 1.2, the conjecture is true if, and only if, the following, formally stronger conjecture holds:

Conjecture 1.3. *(Strong Gohberg-Markus-Boltyanski-Hadwiger Conjecture). For every $d \geq 1$ there is a $0 < \lambda_d < 1$ such that every convex body K in $\mathbb{R}^d$ is covered by 2^d translates of $\lambda_d K$.*

In Section 2 we prove the Theorem. We note that the proof provides no upper bound on λ_d in terms of d. In Section 3 we show an upper bound on the number of translates of λK required to cover K, improving a result of Januszewski and Lassak [5].

2 Proof of Theorem 1.2

We define the following function on the set of convex bodies:

$$\lambda(K) := \inf\{\lambda > 0 : K \text{ is covered by } H_d \text{ translates of } \lambda K\}.$$

By [8], H_d is finite for every d, so $\lambda(.)$ is well defined.

Remark 2.1. *Clearly, $\lambda(.)$ is affine invariant; that is, if T is an invertible affine transformation of $\mathbb{R}^d$ then $\lambda(K) = \lambda(TK)$. Moreover, $0 < \lambda(K) < 1$.*

We recall the definition of the (multiplicative) Banach-Mazur distance of two convex bodies L and K in $\mathbb{R}^d$:

$$\begin{aligned} d_{BM}(L, K) \quad = \quad & \inf\{\lambda > 0 : L - a \subseteq T(K - b) \subseteq \lambda(L - a) \\ & \text{for some } a, b \in \mathbb{R}, T \in GL(\mathbb{R}^d)\} \end{aligned} \tag{2.1}$$

The following proposition states that $\lambda(.)$ is upper semi-continuous. Similar statements have been proved before, cf. Lemma 2, in [3].

Proposition 2.2. *For every convex body K and $\varepsilon > 0$ there is a $\delta > 0$ with the property that for any convex body L, if $d_{BM}(L, K) < 1 + \delta$ then $\lambda(L) < \lambda(K) + \varepsilon$.*

Proof. Let $\lambda := \lambda(K) + \frac{\varepsilon}{2}$. Then there is a set $\Lambda \subset \mathbb{R}^d$ with card $\Lambda \leq H_d$ such that $K \subseteq \Lambda + \lambda K$. Now, let $\delta > 0$ be such that

$$1 + \delta < \frac{\lambda + \frac{\varepsilon}{2}}{\lambda} \tag{2.2}$$

Assume that $d_{BM}(L, K) < 1 + \delta$; that is,

$$L - a \subseteq \overline{K} \subseteq (1 + \delta)(L - a), \tag{2.3}$$

where $\overline{K}$ is an affine image (under an invertible affine transformation) of K. Clearly, we may assume that $\overline{K} = K$.

It follows that $L - a \subseteq \Lambda + (1+\delta)\lambda(L-a)$, and hence, $\lambda(L) \leq (1+\delta)\lambda < \lambda(k) + \varepsilon$. □

Let $\mathcal{K}_a^d$ denote the set of affine equivalence classes of convex bodies in $\mathbb{R}^d$ equipped with the topology induced by the metric d_{BM}. In [6] it is shown that $\mathcal{K}_a^d$ is a compact space. (Note that Macbeath uses a different metric on $\mathcal{K}_a^d$ however, that metric induces the same topology as d_{BM}, cf. [4].)

It follows from Remark 2.1 and Proposition 2.2 that $\lambda(.)$ is an upper semicontinuous function on a compact space. Hence, it attains its maximum, which (by Remark 2.1) is less than one. This proves Theorem 1.2.

3 Quantitative Results

Januszewski and Lassak [5] proved that for every $k + l > d^d$, any convex body $K \subset \mathbb{R}^d$ is covered by k translates of λK and l translates of $-\lambda K$, where $\lambda = 1 - \frac{1}{(d+1)d^d}$. The following argument shows that one may obtain a better estimate on the number of translates of λK required to cover K, using results of Rogers [8], Rogers and Shephard [9], and Rogers and Zhong [10].

Let K, L be convex bodies in $\mathbb{R}^d$. Let $N(K, L)$ denote the *covering number* of K and L; that is, the smallest number of translates of L required to cover K. In [10] it is shown that

$$N(K,L) \leq \frac{\mathrm{vol}(K-L)}{\mathrm{vol}(L)}\Theta(L),$$

where $\Theta(L)$ is the covering density of L. By [8], $\Theta(L) \leq d\log d +$

$\log\log d + 5d$ for every convex body L in $\mathbb{R}^d$. It follows that for any $0 < \lambda < 1$ we have

$$\begin{aligned} N(K, \lambda K) &\leq \lambda^{-d} \frac{\operatorname{vol}(K-K)}{\operatorname{vol} K}(d\log d + \log\log d + 5d) \\ &\leq \lambda^{-d} \binom{2d}{d} (d\log d + \log\log d + 5d) \qquad (3.1) \end{aligned}$$

The last inequality follows from the Rogers-Shephard Inequality [9]. Similarily,

$$\begin{aligned} N(K, -\lambda K) &\leq \lambda^{-d} \frac{\operatorname{vol}(K+K)}{\operatorname{vol} K}(d\log d + \log\log d + 5d) \\ &= \lambda^{-d} 2^d (d\log d + \log\log d + 5d) \qquad (3.2) \end{aligned}$$

By substituting $\lambda = \frac{1}{2}$ into (3.1) and (3.2), we obtain the following:

Remark 3.1. *The number of translates of $\frac{1}{2}K$ that cover K is of order not greater than $8^d\sqrt{d}\log d$; and the number of translates of $-\frac{1}{2}K$ that cover K is of order not greater than $4^d d\log d$.*

Definition 3.2. *Let $0 < \lambda < 1$, and $d \geq 1$. We denote by $\overline{H}_d(\lambda)$ the smallest integer n such that every convex body K in $\mathbb{R}^d$ is covered by n translates of λK.*

It follows from Remark 3.1 that $\overline{H}_d\left(\frac{1}{2}\right)$ is finite for every d. A natural strengthening of the question we discussed in this note is the following:

Question 3.3. *Is there a universal constant $0 < \lambda < 1$ such that for every dimension d, H_d is equal to $\overline{H}_d(\lambda)$?*

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Resumen

Llamemos H_d al menor entero positivo n con la propiedad de que para todo cuerpo convexo K en $\mathbb{R}^d$ hay una constante $0 < \lambda < 1$ tal que K se cubre por medio de n traslaciones de λK. En el libro *Research problems in discrete geometry*, de Brass, Moser y Pach, el siguiente problema fue propuesto: ¿Es posible encontrar una constante $0 < \lambda_d < 1$, que dependa solo de la dimensión d, tal que todo cuerpo convexo K en $\mathbb{R}^d$ es cubierto por H_d traslaciones de $\lambda_d K$? Demostraremos que la respuesta a esta pregunta es afirmativa, y por tanto que la conjetura de Gohberg-Markus-Boltyanski-Hadwiger (la cual postula que $H_d \leq 2^d$) se cumple si, y solo si, se satisface una versión formalmente más fuerte de la misma.

Palabras Clave: Iluminación, Conjetura de Boltyanski-Hadwiger, Conjuntos convexos, Cubrimiento de conjuntos convexos.

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