AN EASY AND REMARKABLE INEQUALITY DERIVED FROM (ACTUALLY EQUIVALENT TO) FERMAT'S LAST THEOREM

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Abstract

A remarkable inequality among integer numbers is given. Easily deduced from Fermat's Last Theorem, it would be nevertheless very difficult to establish through other means.

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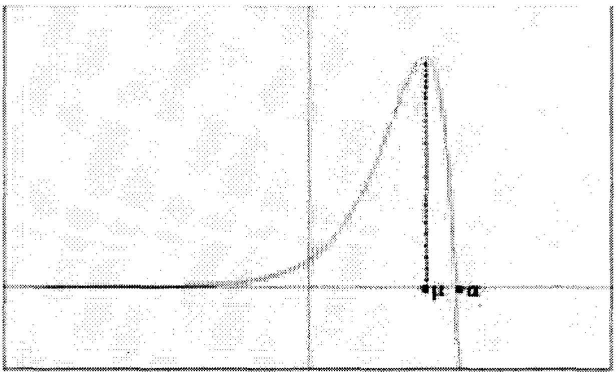
Proposition. *Let $a, b, c,$ be non negative coprime integers such that $a < b < c$; $a^2 + b^2 > c^2$. Then there exists a unique integer $\lambda \geq 2$ such that*

$$a^{2\lambda+1} + b^{2\lambda+1} + c^{2\lambda+1} + (a+b)(ab)^\lambda < (a+c)(ac)^\lambda + (b+c)(bc)^\lambda$$

Proof. Consider the function $f(x) = a^x + b^x - c^x$. It is well defined, continuous and derivable on $\mathbb{R}$, takes its only maximum at some μ, increases over $x < \mu$, decreases over $x > \mu$ and $f(\mathbb{R}) =]-\infty, f(\mu)]$. The condition $a^2 + b^2 > c^2$ implies $f(2) > 0$ hence there exists a unique real $\alpha > 2$ such that $f(\alpha) = 0$. By Fermat's Last Theorem (Wiles-Taylor and Company) α is not an integer; moreover $f(x)$ is positive if $x < \alpha$ and negative if $x > \alpha$. The inequation concerned is simply $f(x)f(x+1) < 0$ whose unique integer solution is $x = \lambda = [\alpha]$, where [,] denotes the integer part (floor function), because it must be positive for any other integer value.



$\mathbf{f(x) = a^x + b^x - c^x; a < b < c;\ a^2 + b^2 > c^2}$

NOTE. It is not difficult to show, by linear independence over $\mathbb{Q}$, that α must be irrational. Is it algebraic or transcendental? This question is not easy at all, it is an open problem. Note that if $a = b$ i.e. $2a^\alpha = c^\alpha$, then α is transcendental by a powerful theorem of Alan Baker.

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Theorem (A. Baker, 1966) If $\alpha \notin \{0,1\}$ is algebraic and β is irrational algebraic then the number α^β is transcendental.

(1) *Baker's theorem is much more general than this indeed. (Gelfond-Schneider's is the given form).*

(2) *By "the number α^β", we mean any fixed determination ($\alpha^\beta = e^{\beta \ln \alpha}$) of the considered multi-valued function.*

(3) *The given inequality is quite easy ... because of FLT. Without FLT it would become extremely hard.*

References

[1] A. Baker, (1979). *Transcendental Number Theory.* Cambridge University Press, London.

Resumen

Una desigualdad notable entre números enteros es dada. Fácilmente deducida del Último Teorema de Fermat, sería, sin embargo, muy difícil de establecer por otros medios.

Palabras clave: Desigualdades diofánticas, Números algebraicos y Trascendentes.

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