# A STUDY OF A TWO VARIABLES LEGENDRE POLYNOMIALS

*Mumtaz Ahmad Khan*[1] *and Mukesh Pal Singh*[2]

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## *Abstract*

*The present paper deals with a study of a two variable polynomial* $P_n(x)$ *analogues to the Legendre polynomial* $P_n(x)$. *The paper contains differential recurrence relations, a partial differential equation, double generating functions, double and triple hypergeometric forms, a special property and a bilinear double generating function for the newly defined polynomials* $P_{n,k}(x, y)$.

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[1] *Dept. Applied Math. Faculty of Engineering, Aligarh Muslim Univ. India.*
[2] *Dept. of Math. Accurate Institute of Management and Technology, India.*

# 1 Introduction

The Hermite polynomials $H_n(x)$ and Legendre polynomials $P_n(x)$ are respectively defined by

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!} \tag{1.1}$$

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)t^n \tag{1.2}$$

A careful inspection of the L.H.S. of (1.1) and (1.2) reveals the fact that L.H.S. of (1.1) is $e^u$ and that of (1.2) is $(1-u)^{-\frac{1}{2}}$, where $u = 2xt - t^2$. Thus $H_n(x)$ and $P_n(x)$ are examples of polynomials generated by a function of the form $G(2xt-t^2)$. The expansions of $(1-u)^{-\frac{1}{2}}$ and $(1-u-v)^{-\frac{1}{2}}$ are given by

$$(1-u)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n u^n}{n!} \tag{1.3}$$

and

$$(1-u-v)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(\frac{1}{2})_{n+k} u^n v^k}{n!k!} \tag{1.4}$$

The expansion (1.4) motivates a two variable analogue of Legendre polynomials by taking $u = 2xt - s^2$ and $v = 2yt - t^2$ in (1.4). Thus we first attempt to define two variable analogues of polynomials by means of generating function of the form $G(u,v)$ where $u = 2xs - s^2$ and $v = 2yt - t^2$ before embarking on a particular example of it namely the two variable analogue of Legendre polynomial.

# 2 Double Generating Functions of the Form $G(2xt - s^2,\ 2yt - t^2)$

Consider the double generating relation

$$G(2xt - s^2,\ 2yt - t^2) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} g_{n,k}(x,y)s^n t^k \tag{2.1}$$

in which $G(u,v)$ has a formal double power series expansion.

Where a formal double power series is one for which the radius of convergence is not necessarily greater than zero.

Thus $G$ determines the coefficient set $\{g_{n,k}(x,y)\}$ even if the double series is divergent for $s \neq 0,\ t \neq 0$. Let

$$\left.\begin{aligned} &F = G(u,v), \qquad where \\ &u = 2xs - s^2 \quad and \\ &v = 2yt - t^2 \end{aligned}\right\} \tag{2.2}$$

Now, from partial differentiation, we have

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} \tag{2.3}$$

$$\frac{\partial F}{\partial s} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial s} \tag{2.4}$$

$$\frac{\partial F}{\partial y} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} \tag{2.5}$$

$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial t} \tag{2.6}$$

Substituting the values of partial derivatives of $u$ and $v$ the above equations reduces to

$$\frac{\partial F}{\partial x} = 2s\frac{\partial F}{\partial u} \tag{2.7}$$

$$\frac{\partial F}{\partial s} = 2(x-s)\frac{\partial F}{\partial u} \tag{2.8}$$

$$\frac{\partial F}{\partial y} = 2t\frac{\partial F}{\partial v} \tag{2.9}$$

$$\frac{\partial F}{\partial t} = 2(y-t)\frac{\partial F}{\partial v} \tag{2.10}$$

Multiplying (2.7) by $(x-s)$ and (2.8) by $s$ and subtracting, we get

$$(x-s)\frac{\partial F}{\partial x} - s\frac{\partial F}{\partial s} = 0 \tag{2.11}$$

Similarly from (2.9) and (2.10), we get

$$(y-s)\frac{\partial F}{\partial x} - t\frac{\partial F}{\partial t} = 0 \tag{2.12}$$

Since $F = (2xs - s^2,\ 2yt - t^2) = \sum\limits_{n=0}^{\infty}\sum\limits_{k=0}^{\infty} g_{n,k}(x,y)s^n t^k$, it follows from (2.11) that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} x\ \frac{\partial}{\partial x}P_{n,k}(x,y)s^n t^k - \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\partial}{\partial x}P_{n,k}(x,y)s^{n+1}t^k$$
$$-\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} n\ \frac{\partial}{\partial x}P_{n,k}(x,y)s^n t^k = 0$$

or

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} x\,\frac{\partial}{\partial x}P_{n,k}(x,y)s^n t^k - \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\partial}{\partial x}P_{n,k}(x,y)s^n t^k$$
$$= \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{\partial}{\partial x}P_{n-1,k}(x,y)s^n t^k \qquad (2.13)$$

Similarly from (2.12), we obtain

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} y\,\frac{\partial}{\partial y}P_{n,k}(x,y)s^n t^k - \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} k\,P_{n,k}(x,y)s^n t^k$$
$$= \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{\partial}{\partial y}P_{n,k-1}(x,y)s^n t^k \qquad (2.14)$$

Equating the coefficients of $s^n t^k$ in (2.13) and (2.14) we obtain the following results:

**Theorem 1.**

$$G(2xs - s^2,\ 2yt - t^2) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} g_{n,k}(x,y)s^n t^k$$

it follows that $\frac{\partial}{\partial x}g_{0,k}(x,y) = 0,\ k \geq 0,\ \frac{\partial}{\partial y}g_{n,0}(x,y) = 0,\ n \geq 0$ and for $n, k \geq 1$,

$$x\,\frac{\partial}{\partial x}\,g_{n,k}(x,y) - n\,g_{n,k}(x,y) = \frac{\partial}{\partial x}\,g_{n-1,k}(x,y) \qquad (2.15)$$

and

$$y\,\frac{\partial}{\partial y}\,g_{n,k}(x,y) - k\,g_{n,k}(x,y) = \frac{\partial}{\partial y}\,g_{n,k-1}(x,y) \qquad (2.16)$$

Adding (2.15) & (2.16), we obtain

$$\left(x\,\frac{\partial}{\partial x}+y\,\frac{\partial}{\partial y}\right)\,g_{n,k}(x,y)-(n+k)g_{n,k}(x,y)$$
$$=\frac{\partial}{\partial x}\,g_{n-1,k}(x,y)+\frac{\partial}{\partial y}g_{n,k-1}(x,y) \qquad (2.17)$$

The differential recurrence relations (2.15), (2.16) & (2.17) are common to all sets $g_{n,k}(x,y)$ possessing a generating function of the form used in (2.1). In this paper we shall consider the polynomials $g_{n,k}(x,y)$ for the choice $G(u,v)=(1-u-v)^{-\frac{1}{2}}$.

# 3 The Legendre Polynomials of Two Variables

We define the Legendre polynomials of two variables, denoted by $P_{n,k}(x,y)$, by the double generating relation

$$(1-2xs+s^2-2yt+t^2)^{-\frac{1}{2}}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}P_{n,k}(x,y)\ s^n\ t^k \qquad (3.1)$$

in which $(1-2xs+s^2-2yt+t^2)^{-\frac{1}{2}}$ denotes the particular branch which $\to 1$ as s $\to 0$ & t $\to 0$. We shall first show that $P_{n,k}(x,y)$ is a polynomial of degree precisely $n$ in $x$ and $k$ in $y$.

Since $(1-u-v)_{-\alpha}=\sum\limits_{n=0}^{\infty}\sum\limits_{k=0}^{\infty}\frac{(\alpha)_{n+k}u^n v^k}{n!\ k!}$, we may write

$$(1-2xs+s^2-2yt+t^2)^{-\frac{1}{2}}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\frac{1}{2})_{n+k}(2xs-s^2)^n(2yt-t^2)^k}{n!\ k!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{j=0}^{\infty} \frac{(\frac{1}{2})_{n+k}(2x)^{n-r}(2y)^{k-j}(-1)^{r+j}s^{n+r}t^{k+j}}{r!\ j!\ (n-r)!\ (k-j)!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{[\frac{n}{2}]}\sum_{j=0}^{[\frac{k}{2}]} \frac{(\frac{1}{2})_{n+k-r-j}(2x)^{n-2r}(2y)^{k-2j}(-1)^{r+j}s^{n}t^{k}}{r!\ j!\ (n-2r)!\ (k-2j)!}$$

We thus obtain

$$P_{n,k}(x,y) = \sum_{r=0}^{[\frac{n}{2}]}\sum_{j=0}^{[\frac{k}{2}]} \frac{(-1)^{r+j}(\frac{1}{2})_{n+k-r-j}(2x)^{n-2r}(2y)^{k-2j}}{r!\ j!\ (n-2r)!\ (k-2j)!} \tag{3.2}$$

from which it follows that $P_n(x,y)$ is a polynomial in two variables $x$ and $y$ of degree precisely $n$ in $x$ and $k$ in $y$.

Thus $P_n(x,y)$ is a polynomial in two variables $x$ and $y$ of degree $n+k$. Equation (3.2) also yields

$$P_{n,k}(x,y) = \frac{2^{n+k}(\frac{1}{2})_{n+k}x^n y^k}{n!\ k!} + \pi, \tag{3.3}$$

Where $\pi$ is a polynomial in two variables $x$ and $y$ of degree $n+k$-2.

If in (3.1), we replace $x$ by $-x$ and $s$ by $-s$, the left member does not change. Hence

$$P_n(-x,y) = (-1)^n\ P_n(x,y) \tag{3.4}$$

Similarly by replacing $y$ by $-y$ and $t$ by $-t$ in (3.1), we obtain

$$P_n(x,-y) = (-1)^k\ P_n(x,y) \tag{3.5}$$

So that $P_n(x,y)$ is an odd function of $x$ for $n$ odd, an even function of $x$ for $n$ even. Similarly $P_n(x,y)$ is an odd function of $y$ for $k$ odd, an even function of $y$ for $k$ even.

Similarly replacing $x$ by $-x$, $y$ by $-y$, $s$ by $-s$ and $t$ by $-t$ in (3.1), we obtain

$$P_n(-x,-y) = (-1)^{n+k}\, P_n(x,y) \tag{3.6}$$

Putting $t=0$ in (3.1), we get

$$P_{n,0}(x,y) = P_n(x) \tag{3.7}$$

Where $P_n(x)$ is a well known Legendre polynomial.
Similarly by putting $s=0$ in (3.1), we get

$$P_{0,k}(x,y) = P_k(y) \tag{3.8}$$

From (3.1) with $x$=0 and $y$=0, we get

$$(1+s^2+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} P_{n,k}(x,y)\ s^n\ t^k$$

But

$$(1+s^2+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{n+k}\left(\frac{1}{2}\right)_{n+k} s^{2n} y^{2k}}{n!\ k!} \tag{3.9}$$

Hence

$$\left.\begin{aligned} &P_{2n+1,2k}(0,0)=0,\ P_{2n,2k+1}(0,0)=0,\ P_{2n+1,2k+1}(0,0)=0 \\ &P_{2n,2k}(0,0) = \frac{(-1)^{n+k}\left(\frac{1}{2}\right)_{n+k}}{n!\ k!} \end{aligned}\right\} \tag{3.10}$$

Equation (3.2) yields

$$\frac{\partial}{\partial x}\, P_{n,k}(x,y) = \sum_{r=0}^{\left[\frac{n-1}{2}\right]} \sum_{j=0}^{\left[\frac{k}{2}\right]} \frac{(-1)^{r+j}(\frac{1}{2})_{n+k-r-j}2(2x)^{n-1-2r}(2y)^{k-2j}}{r!\; j!\; (n-1-2r)!\; (k-2j)!} \tag{3.11}$$

$$\frac{\partial}{\partial y}\, P_{n,k}(x,y) = \sum_{r=0}^{\left[\frac{n}{2}\right]} \sum_{j=0}^{\left[\frac{k-1}{2}\right]} \frac{(-1)^{r+j}(\frac{1}{2})_{n+k-r-j}2(2x)^{n-2r}2(2y)^{k-1-2j}}{r!\; j!\; (n-2r)!\; (k-1-2j)!} \tag{3.12}$$

$$\left.\begin{aligned}
&\left[\tfrac{\partial}{\partial x}\, P_{2n+1,2k}(x,y)\right]_{x=0,y=0} = \tfrac{(-1)^{n+k}2(\frac{1}{2})_{n+k+1}}{n!\; k!} = \tfrac{(-1)^{n+k}(\frac{3}{2})_{n+k}}{n!\; k!}\\
&\left[\tfrac{\partial}{\partial x}\, P_{2n,2k}(x,y)\right]_{x=0,y=0} = 0,\\
&\left[\tfrac{\partial}{\partial x}\, P_{2n,2k+1}(x,y)\right]_{x=0,y=0} = 0,\\
&\left[\tfrac{\partial}{\partial x}\, P_{2n+1,2k+1}(x,y)\right]_{x=0,y=0} = 0,
\end{aligned}\right\} \tag{3.13}$$

Similarly,

$$\left.\begin{aligned}
&\left[\frac{\partial}{\partial y}\, P_{2n,2k}(x,y)\right]_{x=0,y=0} = 0,\\
&\left[\frac{\partial}{\partial y}\, P_{2n+1,2k}(x,y)\right]_{x=0,y=0} = 0,\\
&\left[\frac{\partial}{\partial y}\, P_{2n+1,2k+1}(x,y)\right]_{x=0,y=0} = 0,\ \textit{and}\\
&\left[\frac{\partial}{\partial y}\, P_{2n,2k+1}(x,y)\right]_{x=0,y=0} = \tfrac{(-1)^{n+k}2(\frac{1}{2})_{n+k+1}}{n!\; k!} = \tfrac{(-1)^{n+k}(\frac{3}{2})_{n+k}}{n!\; k!}
\end{aligned}\right\} \tag{3.14}$$

# 4 Differential Recurrence Relations

From theorem 1, it is evident that the generating relation

$$(1-2xs+s^2-2yt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} P_{n,k}(x,y)\ s^n\ t^k \tag{4.1}$$

implies the differential recurrence relations

$$x\ \frac{\partial}{\partial x}P_{n,k}(x,y) - n\ P_{n,k}(x,y) = \frac{\partial}{\partial x}P_{n-1,k}(x,y), \tag{4.2}$$

$$y\ \frac{\partial}{\partial y}P_{n,k}(x,y) - k\ P_{n,k}(x,y) = \frac{\partial}{\partial y}P_{n,k-1}(x,y), \tag{4.3}$$

and

$$\begin{aligned}\left(x\ \frac{\partial}{\partial x} + y\ \frac{\partial}{\partial y}\right) P_{n,k}(x,y) - (n+k)P_{n,k}(x,y)\\ = \frac{\partial}{\partial x}P_{n-1,k}(x,y) + \frac{\partial}{\partial y}P_{n,k-1}(x,y)\end{aligned} \tag{4.4}$$

From(4.1) it follows by the usual method (differentiation) that

$$(1-2xs+s^2-2yt+t^2)^{-\frac{3}{2}} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{\partial}{\partial x}P_{n,k}(x,y)\ s^{n-1}\ t^k \tag{4.5}$$

$$(1-2xs+s^2-2yt+t^2)^{-\frac{3}{2}} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{\partial}{\partial y}P_{n,k}(x,y)\ s^n\ t^{k-1} \tag{4.6}$$

$$(x-s)(1-2xs+s^2-2yt+t^2)^{-\frac{3}{2}} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} n\ P_{n,k}(x,y)\ s^{n-1}\ t^k \tag{4.7}$$

$$(y-t)(1-2xs+s^2-2yt+t^2)^{-\frac{3}{2}} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} k\ P_{n,k}(x,y)\ s^n\ t^{k-1} \tag{4.8}$$

Since $1-s^2-t^2-2s(x-s)-2t(y-t) = 1-2xs+s^2-2yt+t^2$, we may multiply the left member of (4.5) by $1-s^2$, the left member of (4.6) by $-t^2$, the left member of (4.7) by $-2s$, the left member of (4.8) by $-2t$ and add and obtain the left member of (4.1). In this way we find that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\partial}{\partial x}\ P_{n,k}(x,y)\ s^{n-1}\ t^k - \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\ \frac{\partial}{\partial x}\ P_{n,k}(x,y)\ s^{n+1}\ t^k$$

$$-\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\partial}{\partial y}\ P_{n,k}(x,y)\ s^n\ t^{k+1} - \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\ 2n\ P_{n,k}(x,y)\ s^n\ t^k$$

$$-\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\ 2k\ P_{n,k}(x,y)\ s^n\ t^k = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\ P_{n,k}(x,y)\ s^n\ t^k$$

or

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\partial}{\partial x}\ P_{n,k}(x,y)\ s^{n-1}\ t^k - \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\ \frac{\partial}{\partial x}\ P_{n,k}(x,y)\ s^{n+1}\ t^k$$

$$-\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\partial}{\partial y}\ P_{n,k}(x,y)\ s^n\ t^{k+1} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(2n+2k+1)\ P_{n,k}(x,y)\ s^n\ t^k$$

We thus obtain another differential recurrence relation

$$\begin{aligned}&(2n+2k+1)\ P_{n,k}(x,y)\\ &= \tfrac{\partial}{\partial x}\ P_{n+1,k}(x,y) - \tfrac{\partial}{\partial x}\ P_{n-1,k}(x,y) - \tfrac{\partial}{\partial y}\ P_{n,k-1}(x,y)\end{aligned} \tag{4.9}$$

Similarly, we can get

$$\begin{aligned}&(2n+2k+1)\ P_{n,k}(x,y)\\ &= \tfrac{\partial}{\partial y}\ P_{n,k+1}(x,y) - \tfrac{\partial}{\partial x}\ P_{n-1,k}(x,y) - \tfrac{\partial}{\partial y}\ P_{n,k-1}(x,y)\end{aligned} \tag{4.10}$$

Adding (4.9) successively to (4.2), (4.3) and (4.4), we get

$$x\tfrac{\partial}{\partial x}P_{n,k}(x,y)$$
$$=\tfrac{\partial}{\partial x}P_{n+1,k}(x,y)-\tfrac{\partial}{\partial y}\ P_{n,k-1}(x,y)-(n+2k+1)\ P_{n,k}(x,y) \tag{4.11}$$

$$y\tfrac{\partial}{\partial y}P_{n,k}(x,y)$$
$$=\tfrac{\partial}{\partial x}P_{n+1,k}(x,y)-\tfrac{\partial}{\partial x}P_{n-1,k}(x,y)-(2n+k+1)\ P_{n,k}(x,y) \tag{4.12}$$

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)P_{n,k}(x,y)=\frac{\partial}{\partial x}P_{n+1,k}(x,y)-(n+k+1)\ P_{n,k}(x,y) \tag{4.13}$$

Adding (4.10) successively to (4.2), (4.3) and (4.4), we obtain

$$x\frac{\partial}{\partial x}P_{n,k}(x,y)=\frac{\partial}{\partial y}P_{n,k+1}(x,y)-\frac{\partial}{\partial y}P_{n,k-1}(x,y)-(n+2k+1)\ P_{n,k}(x,y) \tag{4.14}$$

$$y\frac{\partial}{\partial y}P_{n,k}(x,y)=\frac{\partial}{\partial y}P_{n,k+1}(x,y)-\frac{\partial}{\partial x}P_{n-1,k}(x,y)-(2n+k+1)\ P_{n,k}(x,y) \tag{4.15}$$

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)P_{n,k}(x,y)=\frac{\partial}{\partial y}P_{n,k+1}(x,y)-(n+k+1)\ P_{n,k}(x,y) \tag{4.16}$$

Shifting the index form $n$ to $n-1$ in (4.11) and using (4.2), we get

$$(x^2-1)\frac{\partial}{\partial x}P_{n,k}(x,y)$$
$$= n\ x\ P_{n,k}(x,y) - \frac{\partial}{\partial y}P_{n-1,k-1}(x,y) - (n+2k)\ P_{n-1,k}(x,y) \tag{4.17}$$

Similarly shifting the index form $k$ to $k-1$ in (4.15) and using (4.3), we get

$$(y^2-1)\frac{\partial}{\partial y}P_{n,k}(x,y)$$
$$= k\ y\ P_{n,k}(x,y) - \frac{\partial}{\partial x}P_{n-1,k-1}(x,y) - (2n+k)\ P_{n,k-1}(x,y) \tag{4.18}$$

Adding (4.17) and (4.18), we obtain

$$\left\{(x^2-1)\frac{\partial}{\partial x} + (y^2-1)\frac{\partial}{\partial y}\right\}P_{n,k}(x,y) = (nx+ky)P_{n,k}(x,y)$$
$$-\left\{\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right\}P_{n-1,k-1}(x,y) - (n+2k)P_{n-1,k}(x,y)$$
$$-(2n+k)P_{n,k-1}(x,y) \tag{4.19}$$

# 5 Partial Differential Equation of $P_{n,k}(x,y)$:

From (4.2) and (4.3), we have

$$\left.\begin{aligned}
&\frac{\partial}{\partial x}P_{n-1,k}(x,y) = x\ \frac{\partial}{\partial x}P_{n,k}(x,y) - n\ P_{n,k}(x,y),\\
&\frac{\partial^2}{\partial x^2}P_{n-1,k}(x,y) = x\ \frac{\partial^2}{\partial x^2}P_{n,k}(x,y) + (1-n)\ \frac{\partial}{\partial x}P_{n,k}(x,y),
\end{aligned}\right\} \tag{5.1}$$

$$\left.\begin{aligned}
&\tfrac{\partial}{\partial y}P_{n-1,k}(x,y) = y\ \tfrac{\partial}{\partial y}P_{n,k}(x,y) - k\ P_{n,k}(x,y),\\
&\tfrac{\partial^2}{\partial y^2}P_{n-1,k}(x,y) = y\ \tfrac{\partial^2}{\partial y^2}P_{n,k}(x,y) + (1-k)\ \tfrac{\partial}{\partial y}P_{n,k}(x,y),
\end{aligned}\right\} \quad (5.2)$$

Shifting the index form $n$ to $n-1$ in (4.11) and form $k$ to $k-1$ in (4.15), we get

$$\begin{aligned}
&x\tfrac{\partial}{\partial x}P_{n-1,k}(x,y)\\
&= \tfrac{\partial}{\partial x}P_{n,k}(x,y) - (n+2k)\ P_{n-1,k}(x,y) - \tfrac{\partial}{\partial y}P_{n-1,k-1}(x,y)
\end{aligned} \quad (5.3)$$

$$\begin{aligned}
&y\tfrac{\partial}{\partial y}P_{n,k-1}(x,y)\\
&= \tfrac{\partial}{\partial y}P_{n,k}(x,y) - (2n+k)\ P_{n,k-1}(x,y) - \tfrac{\partial}{\partial x}P_{n-1,k-1}(x,y)
\end{aligned} \quad (5.4)$$

Differentiating (5.3) partially w.r.t. $x$ and (5.4) w.r.t. $y$, we get

$$\begin{aligned}
&x\tfrac{\partial^2}{\partial x^2}P_{n-1,k}(x,y)\\
&= \tfrac{\partial^2}{\partial x^2}P_{n,k}(x,y) - (n+2k+1)\tfrac{\partial}{\partial x}\ P_{n-1,k}(x,y) - \tfrac{\partial^2}{\partial x\partial y}P_{n-1,k-1}(x,y)
\end{aligned} \quad (5.5)$$

$$\begin{aligned}
&y\tfrac{\partial^2}{\partial y^2}P_{n,k-1}(x,y)\\
&= \tfrac{\partial^2}{\partial y^2}P_{n,k}(x,y) - (2n+k+1)\tfrac{\partial}{\partial y}\ P_{n,k-1}(x,y) - \tfrac{\partial^2}{\partial x\partial y}P_{n-1,k-1}(x,y)
\end{aligned} \quad (5.6)$$

Using (5.1) in (5.5) and (5.2) in (5.6), we obtain

$$\begin{aligned}
&(1-x^2)\tfrac{\partial^2}{\partial x^2}P_{n,k}(x,y) - 2\ (1+k)\ x\ \tfrac{\partial}{\partial x}P_{n,k}(x,y)\\
&+n(n+2k+1)P_{n,k}(x,y) - \tfrac{\partial^2}{\partial x\partial y}P_{n-1,k-1}(x,y)
\end{aligned} \quad (5.7)$$

$$
\begin{aligned}
&(1-y^2)\tfrac{\partial^2}{\partial y^2}P_{n,k}(x,y) - 2\,(1+n)\;y\;\tfrac{\partial}{\partial y}P_{n,k}(x,y) \\
&+k(2n+k+1)P_{n,k}(x,y) - \tfrac{\partial^2}{\partial x \partial y}P_{n-1,k-1}(x,y)
\end{aligned} \tag{5.8}
$$

Subtracting (5.8) from (5.7), we obtain

$$
\begin{aligned}
&\left\{(1-x^2)\tfrac{\partial^2}{\partial x^2} - (1-y^2)\tfrac{\partial^2}{\partial y^2}\right\}P_{n,k}(x,y) \\
&-2\left\{(1+k)\;x\;\tfrac{\partial}{\partial x} - (1+n)y\tfrac{\partial}{\partial y}\right\}P_{n,k}(x,y) \\
&+(n-k)(n+k+1)P_{n,k}(x,y) = 0
\end{aligned} \tag{5.9}
$$

Here (5.9) is the partial differential equation satisfied by $P_{n,k}(x,y)$

# 6 Additional Double Generating Functions

The generating function $(1-2xs+s^2-2yt+t^2)^{-\frac{1}{2}}$ used to define a polynomial $P_{n,k}(x,y)$ in two variables $x$ & $y$ analogues to Legendre polynomials $P_{n,k}(x,y)$ in a single variable $x$ can be expanded in powers of $s$ and $t$ in new ways, thus yielding the additional results. For instance

$$
\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} P_{n,k}(x,y)\;s^n\;t^k = (1-2xs+s^2-2yt+t^2)^{-\frac{1}{2}}
$$

$$
= \left[(1-xs-yt)^2 - s^2(x^2-1) - t^2(y^2-1) - 2xyst\right]^{-\frac{1}{2}}
$$

$$
= (1-xs-yt)^{-1}\left[1 - \tfrac{s^2(x^2-1)}{(1-xs-yt)^2} - \tfrac{t^2(y^2-1)}{(1-xs-yt)^2} - \tfrac{2xyst}{(1-xs-yt)^2}\right]^{-\frac{1}{2}}
$$

$$
= \sum_{j=0}^{\infty}\sum_{p=0}^{\infty}\sum_{r=0}^{\infty} \frac{(\frac{1}{2})_{j+p+r}\;s^{2j}\;(x^2-1)^j\;t^{2p}\;(y^2-1)^p\;(2xyst)^r}{j!\;p!\;r!\;(1-xs-yt)^{2j+2p+2r+1}}
$$

$$= \sum_{j=0}^{\infty} \sum_{p=0}^{\infty} \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_{j+p+r}\ (1+2j+2p+2r)_{n+k}\ s^{n+2j+r}\ (x^2-1)^j\ t^{k+2p+r}\ (y^2-1)^p\ 2^r\ x^{n+r}\ y^{k+r}}{j!\ p!\ r!\ n!\ k!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{[\frac{n}{2}]} \sum_{p=0}^{[\frac{k}{2}]} \sum_{r=0}^{min(n,k)} \frac{(1)_{n+k}\ (x^2-1)^j\ (y^2-1)^p\ s^n\ t^k\ x^{n-2j} y^{k-2p}}{2^{2j+2p+r}\ (1)_{j+p+r}\ j!\ p!\ r!\ (n-r-2j)!\ (k-r-2p)!}$$

Equating the coefficient of $s^n\ t^k$, we obtain

$$P_{n,k}(x,y)$$

$$= \sum_{j=0}^{[\frac{n}{2}]} \sum_{p=0}^{[\frac{k}{2}]} \sum_{r=0}^{min(n,k)} \frac{(n+k)!\ (x^2-1)^j\ (y^2-1)^p\ x^{n-2j} y^{k-2p}}{2^{2j+2p+r}\ (1)_{j+p+r}\ j!\ p!\ r!\ (n-r-2j)!\ (k-r-2p)!} \tag{6.1}$$

Let us now employ (6.1) to discover new double generating functions for $P_{n,k}(x,y)$. Consider, for arbitrary $c$, the double sum

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(c)_{n+k}\ P_{n,k}(x,y)\ s^n\ t^k}{(n+k)!} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(c)_{n+k}\ s^n\ t^k}{(n+k)!} \sum_{j=0}^{[\frac{n}{2}]} \sum_{p=0}^{[\frac{k}{2}]} \sum_{r=0}^{min(n,k)}$$

$$\frac{(n+k)!\ (x^2-1)^j\ (y^2-1)^p\ x^{n-2j} y^{k-2p}}{2^{2j+2p+r}\ (1)_{j+p+r}\ j!\ p!\ r!\ (n-r-2j)!\ (k-r-2p)!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{[\frac{n}{2}]} \sum_{p=0}^{[\frac{k}{2}]} \sum_{r=0}^{min(n,k)}$$

$$\frac{(c)_{n+k+2j+2p+2r}\ (x^2-1)^j\ (y^2-1)^p\ x^{n+r}\ y^{k+r}\ s^{n+2j+r}\ t^{k+2p+r}}{2^{2j+2p+r}\ (1)_{j+p+r}\ j!\ p!\ r!\ n!\ k!}$$

$$= \sum_{j=0}^{\infty} \sum_{p=0}^{\infty} \sum_{r=0}^{\infty} \frac{(c)_{2j+2p+2r}\ (x^2-1)^j\ (y^2-1)^p\ (xy)^r\ s^{2j+r}\ t^{2p+r}}{2^{2j+2p+r}\ (1)_{j+p+r}\ j!\ p!\ r!} \sum_{n=0}^{\infty}$$

$$\times \sum_{k=0}^{\infty} \frac{(c+2j+2p+2r)_{n+k}\ (xs)^n\ (yt)^k}{n!\ k!}$$

$$= \sum_{j=0}^{\infty}\sum_{p=0}^{\infty}\sum_{r=0}^{\infty} \frac{(c)_{2j+2p+2r}\ (x^2-1)^j\ (y^2-1)^p\ (xy)^r\ s^{2j+r}\ t^{2p+r}}{2^{2j+2p+r}\ (1)_{j+p+r}\ j!\ p!\ r!}$$

$$(1-xs-yt)^{-c-2j-2p-2r}$$

$$= (1-xs-yt)^{-c} \sum_{j=0}^{\infty}\sum_{p=0}^{\infty}\sum_{r=0}^{\infty} \frac{\left(\frac{c}{2}\right)_{j+p+r}\ \left(\frac{c}{2}+\frac{1}{2}\right)_{j+p+r}}{(1)_{j+p+r}\ j!\ p!\ r!} \left\{\frac{s^2(x^2-1)}{(1-xs-yt)^2}\right\}^j$$

$$\times \left\{\frac{t^2(y^2-1)}{(1-xs-yt)^2}\right\}^p \left\{\frac{2xyst}{(1-xs-yt)^2}\right\}^r$$

$$= (1-xs-yt)^{-c}.F^{(3)} \left[ \begin{array}{l} \frac{c}{2}, \frac{c}{2}+\frac{1}{2} :: -;-;- : -;-;-; \\ 1 \qquad\quad :: -;-;- : -;-;-; \end{array} \left\{\frac{s^2(x^2-1)}{(1-xs-yt)^2}\right\}^j \right.$$

$$\left. \left\{\frac{t^2(y^2-1)}{(1-xs-yt)^2}\right\}^p \left\{\frac{2xyst}{(1-xs-yt)^2}\right\}^r \right]$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(c)_{n+k}\ P_{n,k}(x,y)\ s^n\ t^k}{(n+k)!} \tag{6.2}$$

In which $c$ may be any complex number. If $c$ is unity, (6.2) degenerates into the generating relation used to define $P_{n,k}(x,y)$.

Let us now return to (6.1) and consider the double sum

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{P_{n,k}(x,y)\ s^n\ t^k}{(n+k)!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\left[\frac{n}{2}\right]}\sum_{p=0}^{\left[\frac{k}{2}\right]}\sum_{r=0}^{min(n,k)} \frac{(x^2-1)^j\ (y^2-1)^p\ x^{n-2j} y^{k-2p}\ s^n\ t^k}{2^{2j+2p+r}\ (1)_{j+p+r}\ j!\ p!\ r!\ (n-2j-r)!\ (k-2p-r)!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{p=0}^{\infty}\sum_{r=0}^{\infty} \frac{(x^2-1)^j\ (y^2-1)^p\ x^{n+r} y^{k+r}\ s^{n+2j+r} t^{k+2p+r}}{2^{2j+p+r}(1)_{j+p+r}\ j!\ p!\ r!\ n!\ k!}$$

$$= \sum_{j=0}^{\infty}\sum_{p=0}^{\infty}\sum_{r=0}^{\infty} \frac{(x^2-1)^j\ (y^2-1)^p\ (xyst)^r\ s^{2j}\ t^{2p}}{2^{2j+p+r}(1)_{j+p+r}\ j!\ p!\ r!} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(xs)^n\ (yt)^k}{n!\ k!}$$

$$= e^{xs+yt} \sum_{j=0}^{\infty} \sum_{p=0}^{\infty} \sum_{r=0}^{\infty} \frac{1}{j!\; p!\; r!\; (1)_{j+p+r}} \left\{ \frac{s^2(x^2-1)}{4} \right\}^j \left\{ \frac{t^2(y^2-1)}{4} \right\}^p \left\{ \frac{xyst}{2} \right\}^r$$

$$= e^{xs+yt} F^{(3)} \left[ \begin{array}{l} - :: -; -; - : -; -; -; \\ 1 :: -; -; - : -; -; -; \end{array} \frac{s^2(x^2-1)}{4}, \frac{t^2(y^2-1)}{4}, \frac{xyst}{2} \right] \qquad (6.3)$$

# 7 Triple Hypergeometric Forms of $P_{n,k}(x,y)$

We return once more to the original definition of $P_{n,k}(x,y)$:

$$(1 - 2xs + s^2 - 2yt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} P_{n,k}(x,y)\; s^n\; t^k \qquad (7.1)$$

This time we note that

$$\begin{aligned} &(1 - 2xs + s^2 - 2yt + t^2)^{-\frac{1}{2}} \\ &= \left[(1-s-t)^2 - 2s(x-1) - 2t(y-1) - 2st\right]^{-\frac{1}{2}} \\ &= (1-s-t)^{-1} \left[(1 - \tfrac{2s(x-1)}{(1-s-t)^2} - \tfrac{2t(y-1)}{(1-s-t)^2} - \tfrac{2st}{(1-s-t)^2}\right]^{-\frac{1}{2}} \end{aligned}$$

which permits us to write

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} P_{n,k}(x,y)\; s^n\; t^k$$

$$= \sum_{j=0}^{\infty} \sum_{p=0}^{\infty} \sum_{r=0}^{\infty} \frac{2^{j+p+r} \left(\frac{1}{2}\right)_{j+p+r} (x-1)^j (y-1)^p s^{j+r} t^{p+r}}{j!p!r!(1-s-t)^{2j+2p+2r+1}}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \sum_{p=0}^{\infty} \sum_{r=0}^{\infty} \frac{2^{j+p+r} \left(\frac{1}{2}\right)_{j+p+r} (x-1)^j (y-1)^p (1)_{2j+2p+2r+n+k} s^{n+j+r} t^{k+p+r}}{j!\; p!\; r!\; n!\; k!\; (1)_{2j+2p+2r}}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \sum_{p=0}^{k} \sum_{r=0}^{min(n,k)} \frac{2^{j+p+r} \left(\frac{1}{2}\right)_{j+p+r} (x-1)^j (y-1)^p (1)_{n+k+j+p} s^n t^k}{j!\; p!\; r!\; (n-j-r)!\; (k-p-r)!\; (1)_{2j+2p+2r}}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{n} \sum_{p=0}^{k} \sum_{r=0}^{min(n,k)} \frac{(1)_{n+k+j+p} \ (x-1)^j \ (y-1)^p \ s^n t^k (1)_{n+k+j+p}}{2^{j+p+r}(1)_{j+p+r} j! \ p! \ r! \ (n-j-r)! \ (k-p-r)!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{n} \sum_{p=0}^{k} \sum_{r=0}^{min(n,k)} \frac{(n+k)!(1+n+k)_{j+p} \ (-n)_{j+r} \ (-k)_{p+r}(1-x)^j \ (1-y)^p \ s^n t^k}{2^{j+p+r}(1)_{j+p+r} \ j! \ p! \ r! \ n! \ k!}$$

Therefore

$$P_{n,k}(x,y) = \frac{(n+k)!}{n! \ k!}$$

$$F^{(3)} \left[ \begin{array}{l} - :: -n; -k; 1+n+k : -; -; -; \\ 1 :: -; -; - : -; -; - \end{array} \begin{array}{l} \frac{1-x}{2}, \frac{1-y}{2}, \frac{1}{2} \\ ; \end{array} \right] \tag{7.2}$$

Since $P_{n,k}(-x,-y) = (-1)^{n+k} P_{n,k}(x,y)$, it follows from (7.2) that also

$$P_{n,k}(x,y) = \frac{(-1)^{n+k}(n+k)!}{n! \ k!}$$

$$F^{(3)} \left[ \begin{array}{l} - :: -n; -k; 1+n+k : -; -; -; \\ 1 :: -; -; - : -; -; - \end{array} \begin{array}{l} \frac{1+x}{2}, \frac{1+y}{2}, \frac{1}{2} \\ ; \end{array} \right] \tag{7.3}$$

Next, consider (3.2) again

$$P_{n,k}(x,y) = \sum_{n=0}^{\left[\frac{n}{2}\right]} \sum_{j=0}^{\left[\frac{k}{2}\right]} \frac{(-1)^{r+j} \left(\frac{1}{2}\right)_{n+k-r-j} (2x)^{n-2r} (2y)^{k-2j}}{r! \ j! \ (n-2r)! \ (k-2j)!}$$

We may write it as

$$P_{n,k}(x,y) = \sum_{n=0}^{\left[\frac{n}{2}\right]} \sum_{j=0}^{\left[\frac{k}{2}\right]} \frac{(-n)^{2r} (-k)^{2j} \left(\frac{1}{2}\right)_{n+k} (2x)^{n-2r} (2y)^{k-2j}}{r! \ j! \ n! \ k! \ \left(\frac{1}{2}-n-k\right)_{r+j}}$$

$$= \frac{2^{n+k} \left(\frac{1}{2}\right)_{n+k} x^n y^k}{n! \ k!} \sum_{r=0}^{\left[\frac{n}{2}\right]} \sum_{j=0}^{\left[\frac{k}{2}\right]} \frac{\left(-\frac{n}{2}\right)_r \left(-\frac{n}{2}+\frac{1}{2}\right)_r \left(\frac{-k}{2}\right)_j \left(-\frac{k}{2}+\frac{1}{2}\right)_j}{r! \ j! \ \left(\frac{1}{2}-n-k\right)_{r+j} x^{2r} y^{2j}}$$

or in terms of Kampe de Feriet's double hypergeometric function, we have

$$P_{n,k}(x,y) = \frac{2^{n+k}\left(\frac{1}{2}\right)_{n+k} x^n y^k}{n!\ k!}$$
$$F^{(3)}\left[\begin{array}{l} - : -\frac{n}{2}, -\frac{n}{2}+\frac{1}{2}; -\frac{k}{2}, -\frac{k}{2}+\frac{1}{2}; \\ \frac{1}{2}-n-k : -; - \qquad\qquad ; \end{array} \frac{1}{x^2}, \frac{1}{y^2}\right] \tag{7.4}$$

# 8 A Special Property of $P_{n,k}(x,y)$

We now return to the original definition of $P_{n,k}(x,y)$ and for convenience use $\rho = (1-2xs+s^2-2yt+t^2)^{\frac{1}{2}}$. We know that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} P_{n,k}(x,y)s^n t^k = \rho^{-1} \tag{8.1}$$

In (8.1), we replace $x$ by $\frac{x-s}{s}$, $y$ by $\frac{y-t}{\rho}$, $s$ by $\frac{u}{s}$ and $t$ by $\frac{v}{s}$ to get

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} P_{n,k}\left(\tfrac{x-s}{\rho}, \tfrac{y-t}{\rho}\right)\rho^{-n-k}u^n v^k = \left[1 - \frac{2(x-s)u}{\rho^2} + \frac{u^2}{\rho^2} - \frac{2(y-t)v}{\rho^2} + \frac{v^2}{\rho^2}\right]^{-\frac{1}{2}}$$

$$= \rho\left[\rho^2 - 2(x-s)u + u^2 - 2(y-t)v + v^2\right]^{-\frac{1}{2}}$$

We may now write

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} P_{n,k}\left(\tfrac{x-s}{\rho}, \tfrac{y-t}{\rho}\right)\rho^{-n-k-1}u^n v^k$$

$$= \left[1 - 2xs + s^2 - 2yt + t^2 - 2xu + 2us + u^2 - 2yv + 2vt + v^2\right]^{-\frac{1}{2}}$$

$$= \left[1 - 2x(s+u) + (s+u)^2 - 2y(t+v) + (t+v)^2\right]^{-\frac{1}{2}}$$

which by (8.1) yields

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} P_{n,k}(\tfrac{x-s}{\rho}, \tfrac{y-t}{\rho})\rho^{-n-k-1}u^n v^k$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} P_{n,k}(x,y)\ (s+u)^n\ (t+v)^k$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{n}\sum_{j=0}^{k} \frac{n!\ k!\ P_{n,k}(x,y)\ s^r\ u^{n-r}\ t^j\ v^{k-j}}{r!\ j!\ (n-r)!\ (k-j)!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{j=0}^{\infty} \frac{(n+r)!\ (k+j)!\ P_{n+r,k+j}(x,y)\ s^r\ t^j\ u^n\ v^k}{r!\ j!\ n!\ k!}$$

Equating the coefficients $u^n\ v^k$ in the above, we find that

$$P_{n,k}(\tfrac{x-s}{\rho}, \tfrac{y-t}{\rho})\rho^{-n-k-1} = \sum_{r=0}^{\infty}\sum_{j=0}^{\infty} \frac{(n+r)!\ (k+j)!\ P_{n+r,k+j}(x,y)\ s^r\ t^j}{r!\ j!\ n!\ k!} \quad (8.2)$$

In winch $\rho = (1-2xs+s^2-2yt+t^2)^{\frac{1}{2}}$.

# 9 More Generating Functions

As an example of the use of equation (8.2), we shall apply (8.2) to the generating relation

$$e^{xs+yt}F^{(3)}\left[\begin{array}{l} -::-;-;-:-;-;- \\ 1::-;-;-:-;-;- \end{array} \frac{s^2(x^2-1)}{4}, \frac{t^2(y^2-1)}{4}, \frac{xyst}{2}\right] \quad (7.3)$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{P_{n,k}(x,y)s^n t^k}{(n+k)!} \quad (9.1)$$

In (8.1), we replace $x$ by $\frac{x-s}{\rho}$, $y$ by $\frac{y-t}{\rho}$, $s$ by $-\frac{su}{\rho}$ and $t$ by $-\frac{tv}{\rho}$ and multiply each member by $\rho^{-1}$ were $\rho = (1-2xs+s^2-2yt+t^2)6\frac{1}{2}$, we obtain

$$\rho^{-1}\, exp\left[-\frac{su(x-s)+tv(y-t)}{\rho^2}\right]$$

$$\times F^{(3)}\left[\begin{array}{l} -::-;-;-:-;-;- \\ 1::-;-;-:-;-;-; \end{array} \frac{u^2s^2(x^2-1+2yt-t^2)}{4\rho^2},\right.$$

$$\left.\frac{v^2t^2(y^2-1+2xs-s^2)}{4\rho^2}, \frac{uvst(x-s)(y-t)}{2\rho^4}\right]$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{n+k}\ \rho^{-n-k-1}\ P_{n,k}(\frac{x-s}{\rho}, \frac{y-t}{\rho})\ s^n\ u^n\ t^k\ v^k}{(n+k)!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{n+k}\ (n+r)!\ (k+j)!\ P_{n+r,k+j}(x,y)\ s^{n+r}\ u^n\ t^{k+j}\ v^k}{n!\ k!\ r!\ j!\ (n+k)!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{n}\sum_{j=0}^{k} \frac{(-1)^{n+k-r-j}\ n!\ k!\ P_{n,k}(x,y)\ u^{n-r}\ v^{k-j}\ s^n\ t^k}{(n-r)!\ (k-j)!\ r!\ j!\ (n+k-r-j)!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{n}\sum_{j=0}^{k} \frac{(-1)^{r+j}\ n!\ k!\ P_{n,k}(x,y)\ u^r\ v^j\ s^n\ t^k}{r!\ j!\ (n-r)!\ (k-j)!\ (r+j)!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{n}\sum_{j=0}^{k} \frac{(-n)_r\ (-k)_j\ P_{n,k}(x,y)\ u^r\ v^j\ s^n\ t^k}{r!\ j!\ (r+j)!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \Phi_2[-n,\ -k;\ 1,\ u,\ v]\ P_{n,k}(x,y)\ s^n\ t^k$$

were $\Phi_2$ is one of the seven confluent forms of the four Appell series defined by Humbert (see [3], pp. 25).

This gives a bilinear double generating relation.

# References

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**Resumen**

El presente artículo aborda el estudio de un polinomio $P_n(x)$ de dos variables análogo al polinomio de Legendre $P_n(x)$. El artículo contiene las relaciones de recurrencia diferenciales, una ecuación diferencial parcial, funciones o generadoras dobles, formas hipergeométricas dobles y triples, una propiedad especial y una función generadora bilineal doble para los nuevos polinomios $P_{n,k}(x, y)$ que se acaban de definir.

**Palabras Clave:** Polinomios de Legendre de dos variables, Representaciones integrales, Funciones generadoras y Relaciones de recurrencia.

Mumtaz Ahmad Khan
Department of Applied Mathematics,
Faculty of Engineering and Technology,
Aligarh Muslim University. Aligarh - 202002, U.P., India.
mumtaz_ahmad_khan_2008@yahoo.com

Mukesh Pal Singh
Department of Mathematics,
Accurate Institute of Management and Technology
49, Knowledge Park III Greater Noida-201306 (U.P.) India.