ON SOMEWHAT e-CONTINUITY

Miguel Caldas [1]

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Abstract

A new class of functions, called somewhat e-continuous functions, has been defined and studied by making use of e-open sets. Characterizations and properties of somewhat e-continuous functions are presented.

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1. *Departamento de Matemática Aplicada, UFF, Brasil.*

1. Introduction and Preliminaries

Recent progress in the study of characterizations and generalizations of continuity has been done by means of several generalized closed sets. The first step of generalizing closed set was done by Levine in 1970 [6]. The notion of generalized closed sets has been studied extensively in recent years by many topologist because generalized closed sets are the only natural generalization of closed sets. More importantly, they also suggest several new properties of topological spaces. As a generalization of closed sets, e-closed sets were introduced and studied by E. Ekici ([1], [2], [3]). In this paper somewhat e-continuous and somewhat e-open are introduced and get results which similar the results for somewhat continuous functions.

Throughout the paper (X, τ) and (Y, σ) (or simply X and Y) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), $cl(A)$, $int(A)$ and $X \backslash A$ denote the closure of A, the interior of A and the complement of A in X, respectively. A point $x \in X$ is called a δ-cluster point of A [7] if $int(cl(V)) \cap A \neq \emptyset$ for every open set V of X containing x. The set of all δ-cluster points of A is called the δ-closure of A and is denoted by $\delta cl(A)$. If $A = \delta cl(A))$, then A is said to be δ-closed. The complement of a δ-closed set is said to be δ-open). The union of all δ-open sets contained in a subset A is called the δ-interior of A and is denoted by $\delta int(A)$. A subset A of a topological space X is said to be e-open [1] if $A \subset int(\delta cl(A)) \cup cl(\delta int(A))$. The complement of an e-open set is said to be e-closed. The intersection (union) of all e-closed (e-open) sets containing (contained in) A in X is called the e-closure (e-interior) of A and is denoted by $cl_e(A)$ (resp. $int_e(A)$). By $eO(X)$ or $eO(X, \tau)$, we denote the collection of all e-open sets of X.

Lemma 1.1. *([1], [3]). The following properties holds for the e-closure of a set in a space X:*

1. Arbitrary union (intersection) of e-open (e-closed) sets in X is *e-open (resp. e-closed).*

2. *A is e-closed in X if and only if $A = cl_e(A)$.*

3. *$cl_e(A) \subset cl_e(B)$ whenever $A \subset B(\subset X)$.*

4. *$cl_e(A)$ is e-closed in X.*

5. *$cl_e(cl_e(A)) = cl_e(A)$.*

6. *$cl_e(A) = \{x \in X \mid U \cap A \neq \emptyset$ for every e-open set U containing $x\}$.*

7. *If A is e-open, then $cl_e(A) = cl(A)$.*

2. Somewhat e-Continuous Functions

Definition 1. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be somewhat e-continuous provided that if for $V \in \sigma$ and $f^{-1}(V) \neq \emptyset$ then there is $U \in eO(X, \tau)$ of X such that $U \neq \emptyset$ and $U \subset f^{-1}(V)$.*

Definition 2. *A function $f : (X, \tau) \to (Y, \sigma)$ is called somewhat continuous [4], if for $V \in \sigma$ and $f^{-1}(V) \neq \emptyset$ there exists an open set U of X such that $U \neq \emptyset$ and $U \subset f^{-1}(V)$.*

It is clear that every continuous function is somewhat continuous and every somewhat continuous is somewhat e-continuous. But the converses are not true as shows the following examples.

Example 2.1. *([4], Example 1) Let $X = \{a, b, c\}$ with topologies $\tau = \{\emptyset, \{b\}, \{a, c\}, X\}$ and $\sigma = \{\emptyset, \{a, b\}, X\}$. Then the identity function $f : (X, \tau) \to (Y, \sigma)$ is somewhat continuous. But f is not continuous.*

Example 2.2. *Let $X = \{a, b, c\}$ with topologies $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, X\}$. Define a function $f : (X, \tau) \to (X, \sigma)$ by $f(a) = c$, $f(b) = a$ and $f(c) = b$. Then f is somewhat e-continuous. But f is not somewhat continuous.*

Recall that, a subset E of a topological space (X, τ) is said to be e-dense in X [2] if $cl_e(E) = X$, equivalently if there is no proper e-closed set C in X such that $E \subset C \subset X$.

Theorem 2.3. *For a surjective function $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*
(i) f is somewhat e-continuous;
(ii) If C is a closed subset of Y such that $f^{-1}(C) \neq X$, then there is a proper e-closed subset F of X such that $f^{-1}(C) \subset F$;
iii) if E is an e-dense subset of X, then $f(E)$ is a dense subset of Y.

Proof.

(i) $\Rightarrow$ *(ii)* : Let C be a closed subset of Y such that $f^{-1}(C) \neq X$. Then $Y \backslash C$ is an open set in Y such that $f^{-1}(Y \backslash C) = X \backslash f^{-1}(C) \neq \emptyset$. By (i), there exists an e-open set U in X such that $U \neq \emptyset$ and $U \subset f^{-1}(Y \backslash C) = X \backslash f^{-1}(C)$. This means that $f^{-1}(C) \subset X \backslash U$ and $X \backslash U = F$ is a proper e-closed set in X.

(ii) $\Rightarrow$ *(i)* : Let $V \in \sigma$ and $f^{-1}(V) \neq \emptyset$. Then $Y \backslash V$ is closed and $f^{-1}(Y \backslash V) = X \backslash f^{-1}(V) \neq X$. By (ii), there exists a proper e-closed set F of X such that $f^{-1}(Y \backslash V) \subset F$. This implies that $X \backslash F \subset f^{-1}(V)$ and $X \backslash F \in eO(X)$ with $X \backslash F \neq \emptyset$.

(ii) $\Rightarrow$ *(iii)* : Let E be an e-dense set in X. Suppose that $f(E)$ is not dense in Y. Then there exists a proper closed set C in Y such that $f(E) \subset C \subset Y$. Clearly $f^{-1}(C) \neq X$. By (ii), there exists a proper e-closed subset F such that $E \subset f^{-1}(C) \subset F \subset X$. This is a contradiction to the fact that E is e-dense in X.

(iii) $\Rightarrow$ *(ii)* : Suppose (ii) is not true. This means that there exists a closed set C in Y such that $f^{-1}(C) \neq X$ but there is not proper e-closed set F in X such that $f^{-1}(C) \subset F$. This means that $f^{-1}(C)$ is e-dense in X. But by (iii), $f(f^{-1}(C)) = C$ must be dense in Y, which is a contradiction to the choice of C.

Definition 3. *If X is a set and τ and τ^* are topologies on X, then τ is said to be e-equivalent (resp. equivalent [4]) to τ^* provided if $U \in \tau$ and $U \neq \emptyset$ then there is an e-open (resp. open) set V in (X, τ^*) such that $V \neq \emptyset$ and $V \subset U$ and if $U \in \tau^*$ and $U \neq \emptyset$ then there is an e-open (resp. open) set V in (X, τ) such that $V \neq \emptyset$ and $V \subset U$*

Now consider the identity function $f : (X, \tau) \to (X, \tau^*)$ and assume that τ and τ^* are e-equivalent. Then $f : (X, \tau) \to (X, \tau^*)$ and $f^{-1} : (X, \tau^*) \to (X, \tau)$ are somewhat e-continuous. Conversely, if the identity function $f : (X, \tau) \to (X, \tau^*)$ is somewhat e-continuous in both directions, then τ and τ^* are e-equivalent.

Theorem 2.4. *If $f : (X, \tau) \to (Y, \sigma)$ is a surjection somewhat e-continuous and τ^* is a topology for X, which is e-equivalent to τ, then $f : (X, \tau^*) \to (Y, \sigma)$ is somewhat e-continuous.*

Proof. Let V be an open subset of Y such that $f^{-1}(V) \neq \emptyset$. Since $f : (X, \tau) \to (Y, \sigma)$ is somewhat e-continuous, there exists a nonempty e-open set U in (X, τ) such that $U \subset f^{-1}(V)$. But by hypothesis τ^* is e-equivalent to τ. Therefore, there exists an e-open set U^* in (X, τ^*) such that $U^* \subset U$. But $U \subset f^{-1}(V)$. Then $U^* \subset f^{-1}(V)$; hence $f : (X, \tau^*) \to (Y, \sigma)$ is somewhat e-continuous.

Theorem 2.5. *Let $f : (X, \tau) \to (Y, \sigma)$ be a somewhat e-continuous surjection and σ^* be a topology for Y, which is equivalent to σ. Then $f : (X, \tau) \to (Y, \sigma^*)$ is somewhat e-continuous.*

Proof. Let V^* be an open subset of (Y, σ^*) such that $f^{-1}(V^*) \neq \emptyset$. Since σ^* is equivalent to σ, there exists a nonempty open set V in (Y, σ) such that $V \subset V^*$. Now $\emptyset \neq f^{-1}(V) \subset f^{-1}(V^*)$. Since $f : (X, \tau) \to (Y, \sigma)$ is somewhat e-continuous, there exists a nonempty e-open set U in (X, τ) such that $U \subset f^{-1}(V)$. Then $U \subset f^{-1}(V^*)$; hence $f : (X, \tau) \to (Y, \sigma^*)$ is somewhat e-continuous.

Theorem 2.6. *If $f : (X, \tau) \to (X, \sigma)$ is somewhat e-continuous and $g : (X, \sigma) \to (X, \eta)$ is continuous, then $fog : (X, \tau) \to (Z, \eta)$ is somewhat e-continuous.*

Definition 4. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be somewhat e-open provided that if $U \in \tau$ and $U \neq \emptyset$, then there exists an e-open set V in Y such that $V \neq \emptyset$ and $V \subset f(U)$.*

We have the following obvious characterization of somewhat e-openness.

Theorem 2.7. *A function $f : (X, \tau) \to (Y, \sigma)$ is somewhat e-open if and only if for any $A \subset X$, $int(A) \neq \emptyset$ implies that $int_e(f(A)) \neq \emptyset$.*

Theorem 2.8. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*
(i) f is somewhat e-open;
(ii) If D is an e-dense subset of Y, then $f^{-1}(D)$ is a dense subset of X.

Proof.

$(i) \Rightarrow (ii)$: Suppose D is an e-dense set in Y. We want to show that $f^{-1}(D)$ is a dense subset of X. Suppose that $f^{-1}(D)$ is not dense in X. Then there exists a closed set B in X such that $f^{-1}(D) \subset B \subset X$. By (i) and since that $X \backslash B$ is open, there exists a nonempty e-open subset E in Y such that $E \subset f(X \backslash B)$. Therefore $E \subset f(X \backslash B) \subset f(f^{-1}(Y \backslash D)) \subset Y \backslash D$. That is, $D \subset Y \backslash E \subset Y$. Now, $Y \backslash E$ is an e-closed set and $D \subset Y \backslash E \subset Y$. This implies that D is not an e-dense set in Y, which is a contradiction to the fact that D is e-dense in Y. Therefore, $f^{-1}(D)$ is a dense subset of X.

$(ii) \Rightarrow (i)$: Suppose D is a nonempty open subset of X. We want to show that $int_e(f(D)) \neq \emptyset$. Suppose $int_e(f(D)) = \emptyset$. Then $cl_e(Y \backslash f(D)) = Y$. Therefore, by (ii) $f^{-1}(Y \backslash f(D))$ is dense in X. But $f^{-1}(Y \backslash f(D)) \subset X \backslash D$. Now $X \backslash D$ is closed. Therefore $f^{-1}(Y \backslash f(D)) \subset X \backslash D$ gives $X = cl(f^{-1}((Y \backslash f(D)) \subset X \backslash D$. This implies that $D = \emptyset$ which is contrary to $D \neq \emptyset$. Therefore $int_e(f(D)) \neq \emptyset$. This proves that f is somewhat e-open.

Theorem 2.9. *For a bijective function $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*
(i) f is somewhat e-open;
(ii) If C is a closed subset of X such that $f(C) \neq Y$, then there is an e-closed subset F of Y such that $F \neq Y$ and $f(C) \subset F$.

Proof.

$(i) \Rightarrow (ii)$: Let C be any closed subset of X such that $f(C) \neq Y$. Then $X \backslash C$ is an open set in X and $X \backslash C \neq \emptyset$. Since f is somewhat e-open there exists an e-open set V in Y such that $V \neq \emptyset$ and $V \subset f(X \backslash C)$. Put $F = Y \backslash V$. Clearly F is e-closed in Y and we claim $F \neq Y$. If $F = Y$, then $V = \emptyset$ which is a contradiction. Since $V \subset f(X \backslash C)$, $f(C) = (Y \backslash f(X \backslash C)) \subset Y \backslash V = F$.

$(ii) \Rightarrow (i)$: Let U be any nonempty open subset of X. Then $C = X \backslash U$ is closed set in X and $f(X \backslash U) = f(C) = Y \backslash f(U)$ implies $f(C) \neq Y$. Therefore, by (ii), there is an e-closed set F of Y such that $F \neq Y$ and $f(C) \subset F$. Clearly $V = Y \backslash F \in eO(Y, \sigma)$ and $V \neq \emptyset$. Also $V = Y \backslash F \subset Y \backslash f(C) = Y \backslash f(X \backslash U) = f(U)$.

3. e-Resolvable Spaces and e-Irresolvable Spaces

Definition 5. *A topological space (X, τ) is said to be e-resolvable (resp. resolvable [5]) if there exists an e-dense (resp. dense) set A in (X, τ) such that $X \backslash A$ is also e-dense (resp. dense) in (X, τ). A space (X, τ) is called e-irresolvable (resp. irresolvable) if it is not e-resolvable (resp. resolvable).*

Theorem 3.1. *For a topological space (X, τ), the following statements are equivalent:*
(i) (X, τ) is e-resolvable;
(ii) (X, τ) has a pair of e-dense sets A and B such that $A \subset (X \backslash B)$.

Proof.

$(i) \Rightarrow (ii)$: Suppose that (X,τ) is e-resolvable. There exists an e-dense set A in (X,τ) such that $X\backslash A$ is e-dense in (X,τ). Set $B = X\backslash A$, then we have $A = X\backslash B$.

$(ii) \Rightarrow (i)$: Suppose that the statement (ii) holds. Let (X,τ) be e-irresolvable. Then $X\backslash B$ is not e-dense and $cl_e(A) \subset cl_e(X\backslash B) \neq X$. Hence A is not e-dense. This contradicts the assumption.

Theorem 3.2. *For a topological space (X,τ), the following statements are equivalent:*
(i) (X,τ) is e-irresolvable (resp. irresolvable);
(ii) For any e-dense (resp. dense) set A in X, $int_e(A) \neq \emptyset$ (resp $int(A) \neq \emptyset$).

Proof. (We prove the first statement since the proof of the second is similar).

$(i) \Rightarrow (ii)$: Let A be any e-dense set of X. Then we have $cl_e(X\backslash A) \neq X$, hence $int_e(A) \neq \emptyset$.

$(ii) \Rightarrow (i)$: Suppose that (X,τ) is an e-resolvable space. There exists an e-dense set A in (X,τ) such that $X\backslash A$ is also e-dense in (X,τ). It follows that $int_e(A) = \emptyset$, which is a contradiction; hence (X,τ) is e-irresolvable.

Definition 6. *A topological space (X,τ) is said to be strongly e-irresolvable if for a nonempty set A in X $int_e(A) = \emptyset$ implies $int_e(cl_e(A)) = \emptyset$.*

Theorem 3.3. *If (X,τ) is an strongly e-irresolvable space and $cl_e(A) = X$ for a nonempty subset A of X, then $cl_e(int_e(A)) = X$.*

Proof. The proof is clear.

Theorem 3.4. *If (X, τ) is an strongly e-irresolvable space and $int_e(Cl_e(A)) \neq \emptyset$ for a nonempty subset A of X, then $int_e(A) \neq \emptyset$.*

Proof. The proof is clear.

Theorem 3.5. *Every strongly e-irresolvable is e-irresolvable.*

Proof. This follows from Theorems 3.2 and 3.3.

Theorem 3.6. *If $f : (X, \tau) \to (Y, \sigma)$ is a somewhat e-open function and $int_e(B) = \emptyset$ for a nonempty subset B of Y, then $int(f^{-1}(B)) = \emptyset$.*

Proof. Let B be a nonempty set in Y such that $int_e(B) = \emptyset$. Then $cl_e(Y \backslash B) = Y$. Since f is somewhat e-open and $Y \backslash B$ is e-dense in Y, by Theorem 2.8 $f^{-1}(Y \backslash B)$ is dense in X. Then $cl(X \backslash f^{-1}(B)) = X$. Hence $int(f^{-1}(B)) = \emptyset$.

Theorem 3.7. *If $f : (X, \tau) \to (Y, \sigma)$ be a somewhat e-open function. If X is irresolvable, then Y is e-irresolvable.*

Proof. Let B be a nonempty set in Y such that $cl_e(B) = Y$. We show that $int(B) \neq \emptyset$. Suppose not, then $cl_e(Y \backslash B) = Y$. Since f is somewhat e-open and $Y \backslash B$ is e-dense in Y, we have by Theorem 2.8 $f^{-1}(Y \backslash B)$ is dense in X. Then $int(f^{-1}(B)) = \emptyset$. Now, since B is e-dense in Y and using again Theorem 2.8 $f^{-1}(B)$ is dense in X. Therefore we have that for the dense set $f^{-1}(B)$ that $int(f^{-1}(B)) = \emptyset$, which is a contradiction to Theorem 3.2. Hence we must have $int(B) \neq \emptyset$ for all e-dense sets B in Y. Hence by Theorem 3.2, Y is e-irresolvable.

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Resumen

Se define y estudia una nueva clase de funciones, denominadas funciones casi e-continuas haciendo uso de conjuntos e-abiertos. Se presentan caracterizaciones y propiedades de dichas funciones consideradas como si fueran casi e-continuas.

Palabras clave: Espacios topológicos, Conjuntos e-abiertos, e-continuidad, casi e-continuidad.

Miguel Caldas
Departamento de Matemática Aplicada,
Universidade Federal Fluminense,
Rua Mário Santos Braga, s/n^o
24020-140, Niterói, RJ BRAZIL.
gmamccs@vm.uff.br