STOCHASTIC CHARACTERIZATION OF A CLASS OF 2-STATE SYSTEM AVAILABILITY PROCESSES

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Abstract

The system availability process indicates whether or not the interconnection of components is operating as intended at each time instant. It is shown that a 2-states system availability process that results from a transformation of a Markov chain is not a Markov chain. The probabilistic characterization of the system availability process is given.

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1 Introduction

In fault tolerant systems, the 2-state system availability process indicates whether or not an interconnection of devices can perform its intended operation at a given time [1–3]. When these devices are operating in a harsh environment, the system availability process is induced by the stochastic upsets affecting each device. In general, the system availability process depends on the correct operation of a sufficient number of interconnected devices. Consider a particular operation performed by the fault tolerant interconnection of $N \geq 2$ devices and assume that the devices are affected by N independent upset processes. Let the state of operation at time $k \in \mathbb{Z}^+ \triangleq \{0, 1, \dots\}$ of the i-th device be denoted by $\boldsymbol{\theta}_i(k)$, $i \in \mathscr{L} \triangleq \{1, \dots, N\}$ such that $\boldsymbol{\theta}_i(k) = 0$ denotes that the i-th device is working as intended and $\boldsymbol{\theta}_i(k) = 1$ denotes that it is not working correctly. Boldfaced characters will denote a random variable or process. The process $\boldsymbol{\theta}_i(k)$ represents the state of the i-th device with state space $\mathcal{I} \triangleq \{0, 1\}$ and input given by an homogeneous Markov chain (HMC). The ambient probability space over which these processes are defined is given by $(\Omega, \mathcal{F}, \Pr)$. In this paper, all Markov chains (MC) satisfy the first-order Markov property, that is, if $\boldsymbol{\theta}(k)$ is a MC then

$$\begin{aligned}\Pr\big(\boldsymbol{\theta}(k+1) = \theta(k+1) \mid \boldsymbol{\theta}(k) = \theta(k), \dots, \boldsymbol{\theta}(0) = \theta(0)\big) \\ = \Pr\big(\boldsymbol{\theta}(k+1) = \theta(k+1) \mid \boldsymbol{\theta}(k) = \theta(k)\big),\end{aligned}$$

where $\Pr(\boldsymbol{\theta}(k) = \theta(k), \dots, \boldsymbol{\theta}(0) = \theta(0)) > 0$. The system availability process, denoted by $\boldsymbol{A}(k)$, is given by a memoryless transformation of the N Markov chains $\boldsymbol{\theta}_i(k)$, $i \in \mathscr{L}$, and it is assumed to take values in $\mathcal{I}$. This paper, that collects some results from [4] and [5], gives a probabilistic characterization of $\boldsymbol{A}(k)$. In particular, for a general system availability transformation the state probabilities, $\Pr(\boldsymbol{A}(k) = j), j \in \mathcal{I}$, and the one-step transition probabilities, $\Pr(\boldsymbol{A}(k+1) = j | \boldsymbol{A}(k) = i)$, $i, j \in \mathcal{I}$, are derived. Steady-state values of these probabilities are also given. Based on the extensive literature for transformed Markov processes, two results are presented. First, necessary and sufficient conditions for the system availability process to be an HMC are provided [6], [7]. Second, a result from [8] is given which shows that the system availability process can be a non-homogeneous Markov chain (NHMC) only for a class of initial distributions.

The rest of the paper is organized as follows. Section 2 characterizes the statistical nature of the system availability process. In Section 3, a result for the system availability process to be an HMC as well as sufficient conditions for it to be an ergodic HMC are given. An example is also given to ilustrate the results of the paper. Finally the conclusions are summarized in Section 4.

2 The System Availability Process

Consider a particular operation performed by the interconnection of $N \geq 2$ devices and assume that the devices are affected by N independent upset processes. Let the mode of operation at time $k \in \mathbb{Z}^+ \triangleq \{0, 1, \dots\}$ of the i-th device be modeled by a state of the HMC $\boldsymbol{\theta}_i(k)$, $i \in \mathscr{L}$. For all $i \in \mathscr{L}$, the state space of $\boldsymbol{\theta}_i(k)$ is assumed to be the finite set $\mathcal{I}$. Let $\boldsymbol{\theta}(k) \triangleq (\boldsymbol{\theta}_1(k), \dots, \boldsymbol{\theta}_N(k))$ be the joint process of the HMC's $\boldsymbol{\theta}_i(k)$, $i \in \mathscr{L}$. The statistical nature of $\boldsymbol{\theta}(k)$ is characterized in Lemma 1. Note that the random processes $\boldsymbol{\theta}_1(k), \dots, \boldsymbol{\theta}_N(k)$ are independent if the random variables of these processes at the time k are mutually independent for every $k \in \mathbb{Z}^+$. Recall that the HMC $\boldsymbol{\theta}(k)$ with transition probability matrix $\Pi = [q_{ij}]$ and k-step transition probability matrix $\Pi^{(k)} = [q_{ij}^{(k)}]$, where

$$q_{ij}^{(k)} \triangleq \Pr(\boldsymbol{\theta}(k) = j | \boldsymbol{\theta}(0) = i)$$

is ergodic if the limits

$$\pi_j = \lim_{k \to \infty} q_{ij}^{(k)}$$

1. exist for all $j \in \mathcal{S}$,

2. are independent of $i \in \mathcal{S}$, and

3. for all $j \in \mathcal{S}$, $\pi_j > 0$ such that $\sum_{j=1}^{N} \pi_j = 1$.

Lemma 1. *Let $\boldsymbol{\theta}_i(k)$, $i \in \mathscr{L}$, be a set of independent HMC's with state space $\mathcal{I}$, initial state probability vector*

$$\pi_{\theta_i}(0) \triangleq [\Pr(\theta_i(0) = 0), \Pr(\theta_i(0) = 1)]$$

and transition probability matrix Π_{θ_i}. Then the joint process $\boldsymbol{\theta}(k)$ is an HMC with state space $\mathcal{I}^N \triangleq \underbrace{\mathcal{I} \times \cdots \times \mathcal{I}}_{N\,times}$, initial state probability vector

$$\pi_{\theta}(0) \triangleq \left[\prod_{i=1}^{N} \Pr\big(\boldsymbol{\theta}_i(0) = 0\big), \ldots, \prod_{i=1}^{N} \Pr\big(\boldsymbol{\theta}_i(0) = 1\big)\right] = \pi_{\theta_1}(0) \otimes \cdots \otimes \pi_{\theta_N}(0)$$

and transition probability matrix

$$\Pi_{\theta} \triangleq \Pi_{\theta_1} \otimes \cdots \otimes \Pi_{\theta_N},$$

where $\otimes$ is the Kronecker product. The joint process $\boldsymbol{\theta}(k)$ is ergodic if each of the Markov chains $\boldsymbol{\theta}_i(k)$ satisfies this property.

Proof. The initial state probability vector $\pi_{\theta}(0)$ follows from the independence of the HMC's $\boldsymbol{\theta}_i$, $i \in \mathscr{L}$. The rest of the theorem is a direct generalization of Lemma 7.19 in [9]. □

Definition 1. *Let $\boldsymbol{\theta}_i(k)$, $i \in \mathscr{L}$, be a set of independent HMC's with state space $\mathcal{I}$ and let $\boldsymbol{\theta}(k)$ be the joint HMC. The onto, memoryless function*

$$\begin{aligned} \phi : &\mathcal{I}^N \rightarrow \mathcal{I} \\ &\boldsymbol{\theta}(k) \mapsto \phi(\boldsymbol{\theta}(k)) = j \end{aligned}$$

is called a structure function.

Clearly ϕ is a measurable mapping, thus for each $k \in \mathbb{Z}^+$ the function ϕ induces a random variable defined by $\boldsymbol{A}(k) \triangleq \phi(\boldsymbol{\theta}(k))$ with state space $\mathcal{I}$. The family of random variables $\{\boldsymbol{A}(k) : k \in \mathbb{Z}^+\}$ will be simply denoted as the stochastic process $\boldsymbol{A}(k)$ and it is called the (induced) system availability process.

The structure function ϕ partitions the state space of $\boldsymbol{\theta}(k)$ as follows:

$$\mathcal{I}^N = I_0 \bigsqcup I_1, \tag{1}$$

where $\bigsqcup$ denotes disjoint union and for each $j \in \mathcal{I}$, $I_j \triangleq \phi^{-1}(j) = \{\theta \in \mathcal{I}^N : \phi(\theta) = j\}$. This partition is denoted by $\mathcal{P}_{\mathcal{I}^N} \triangleq \{I_0, I_1\}$.

The statistical characterization of $\boldsymbol{A}(k) = \phi(\boldsymbol{\theta}(k))$ is given in this section. It is known that, in general, $\boldsymbol{A}(k)$ will not be an HMC for all initial distribution of $\boldsymbol{\theta}(k)$, but it can be a NHMC for some ones [8]. First, the state probability vector and steady-state probability vector are characterized in Lemma 2 and Theorem 1, respectively. Second, the transition probabilities and their steady-state values are characterized in Lemma 3 and Theorem 2, respectively.

Lemma 2. *Let $\boldsymbol{\theta}_i(k)$, $i \in \mathscr{L}$, be independent HMC's with state space $\mathcal{I}$ and initial distribution $\pi_{\theta_i}(0)$, and let $\boldsymbol{\theta}(k)$ be the joint HMC with transition probability matrix Π_{θ}. Let ϕ be a structure function and $\boldsymbol{A}(k) = \phi(\boldsymbol{\theta}(k))$, the system availability process. Then the state probability vector of $\boldsymbol{A}(k)$, $\pi_A(k) \triangleq \big[\Pr(\boldsymbol{A}(k) = 0), \Pr(\boldsymbol{A}(k) = 1)\big]$, is characterized by*

$$\Pr(\boldsymbol{A}(k) = j) = \sum_{\theta \in I_j} \prod_{i=1}^{N} \pi_{\theta_i}(0) \Pi_{\theta_i}^k \begin{bmatrix} 1_{\{\theta_i = 0\}} \\ 1_{\{\theta_i = 1\}} \end{bmatrix}, \quad j \in \mathcal{I}, \tag{2}$$

where $1_{\{\cdot\}}$ is the indicator function of $\{\cdot\}$, and $\theta_i(k)$ is the i-th component of $\theta(k)$.

Proof. Since ϕ is a measurable mapping, for each $j \in \mathcal{I}$ it follows that

$$\Pr(\boldsymbol{A}(k) = j) = \sum_{\theta \in I_j} \Pr(\boldsymbol{\theta}(k) = \theta). \tag{3}$$

From the assumption that the processes $\boldsymbol{\theta}_i(k)$ are independent HMCs, the following equalities hold

$$\begin{aligned}\Pr(\boldsymbol{A}(k)=j) &= \sum_{\theta\in I_j}\prod_{i=1}^{N}\Pr(\boldsymbol{\theta}_i(k)=\theta_i)\\ &= \sum_{\theta\in I_j}\prod_{i=1}^{N}\pi_{\theta_i}(k)\begin{bmatrix}1_{\{\theta_i=0\}}\\ 1_{\{\theta_i=1\}}\end{bmatrix}.\end{aligned}$$

Since $\boldsymbol{\theta}_i(k)$, $i\in\mathscr{L}$, is an HMC, it follows that

$$\Pr(\boldsymbol{A}(k)=j)=\sum_{\theta\in I_j}\prod_{i=1}^{N}\pi_{\theta_i}(0)\Pi_{\theta_i}^{k}\begin{bmatrix}1_{\{\theta_i=0\}}\\ 1_{\{\theta_i=1\}}\end{bmatrix}.$$

Finally, the partition in (1) and Equation (3) show that $\sum_{j=0}^{1}\Pr(\boldsymbol{A}(k)=j)=1$. □

The probability $\lim_{k\to\infty}\Pr(\boldsymbol{A}(k)=0)$ is called the *availability* of the system (at steady-state). It is computed in Theorem 1 and shown to be constant under the additional assumptions that the independent HMC's $\boldsymbol{\theta}_i(k)$ are ergodic. Notice that under these conditions the joint process $\boldsymbol{\theta}(k)$ is also ergodic [9].

Theorem 1. *Let $\boldsymbol{\theta}_i(k)$, $i\in\mathscr{L}$, be ergodic HMC's with stationary probability vectors π_{θ_i}. Then the availability (at steady-state) of the system is*

$$\lim_{k\to\infty}\Pr(\boldsymbol{A}(k)=0)=\sum_{\theta\in I_j}\prod_{i=1}^{N}\pi_{\theta_i}\begin{bmatrix}1_{\{\theta_i=0\}}\\ 1_{\{\theta_i=1\}}\end{bmatrix}. \qquad (4)$$

Proof. Under the given assumptions, the limit exists and (4) follows directly from Equation (2). □

Since ϕ reduces the 2^N states of the HMC $\boldsymbol{\theta}(k)$ down to 2, it is a type of a lumping Markov transformation that has been extensively studied since the 1950's (cf. [6–8, 10, 11]). Thus, necessary and sufficient conditions are well known for: $\boldsymbol{A}(k)$ to be an HMC, and a NHMC. In general the process $\boldsymbol{A}(k) = \phi(\boldsymbol{\theta}(k))$ is called a lumped process. To simplify the presentation, the 2^N possible states of $\boldsymbol{\theta}(k)$, labeled in their natural last-lexical order [11], are assigned values in $L = \{1, 2, \ldots, 2^N\}$. Let $\xi : \mathcal{I}^N \to L$ denote the bijective function that maps a state to an integer label in L, such as, $\xi((0, 0, \ldots, 0)) = 1$ and $\xi((1, 1, \ldots, 1)) = 2^N$. Thus, ϕ induces through ξ the partition $\mathcal{P}_L \triangleq \{L_0, L_1\}$, where $L = L_0 \bigsqcup L_1$ and $L_j = \{l \in L : l = \xi(\theta), \forall\ \theta \in I_j\}$. The following $2^N \times 2$ matrix characterizes this partition and is useful in the analysis of the lumping operation:

Definition 2. *Let $\mathcal{M} = [m_{ij}]$ be a matrix of dimension $2^N \times 2$ such that for $j \in \mathcal{I}$ and $i \in L$, m_{ij} is defined as follows:*

$$m_{ij} = \begin{cases} 1 & : \text{ whenever } \phi(\xi^{-1}(i)) = j, \\ 0 & : \text{ otherwise.} \end{cases}$$

The matrix $\mathcal{M}$ will be called *lumping matrix*, and its columns will be denoted sequentially from left to right as M_0 and M_1.

The following lemma gives the transition probabilities of the process $\boldsymbol{A}(k)$.

Lemma 3. *Let $\boldsymbol{\theta}_i(k)$, $i \in \mathscr{L}$, be a set of independent HMC's, and let the transition probability matrix of the joint HMC $\boldsymbol{\theta}(k)$ be $\Pi_\theta = [q_{mn}]$, $m, n \in L$. If $\Pr(\boldsymbol{A}(k) = i) > 0$, $i \in \mathcal{I}$, then the diagonal transition probabilities of $\boldsymbol{A}(k)$, $p_{ii}(k) = \Pr(\boldsymbol{A}(k+1) = i | \boldsymbol{A}(k) = i)$, are*

$$p_{ii}(k) = \frac{\pi_\theta(0)\Pi_\theta^k}{\pi_\theta(0)\Pi_\theta^k M_i} \sum_{m,n \in L_i} q_{mn} e_m, \quad i \in \mathcal{I}, \tag{5}$$

where $e_m \in \mathbb{R}^{2^N}$ is the vector of zeros with a single 1 in the m-th position.

Proof. By the definition of the event $\{\boldsymbol{A}(k) = i\}$, and since ϕ is a measurable function it follows that

$$p_{ii}(k) = \Pr(\boldsymbol{A}(k+1) = i \mid \boldsymbol{A}(k) = i)$$

$$= \Pr\big(\boldsymbol{\theta}(k+1) \in \cup_{n\in L_i}\{\xi^{-1}(n)\} \mid \boldsymbol{\theta}(k) \in \cup_{m\in L_i}\{\xi^{-1}(m)\}\big)$$

$$= \frac{\left(\sum\limits_{m,n\in L_i} \Pr\Big(\boldsymbol{\theta}(k+1) = \xi^{-1}(n)\Big|\boldsymbol{\theta}(k) = \xi^{-1}(m)\Big) \Pr\big(\boldsymbol{\theta}(k) = \xi^{-1}(m)\big)\right)}{\sum\limits_{m\in L_i} \Pr\big(\boldsymbol{\theta}(k) = \xi^{-1}(m)\big)}$$

$$= \frac{\sum\limits_{m,n\in L_i} q_{mn} \Pr(\boldsymbol{\theta}(k) = \xi^{-1}(m))}{\sum\limits_{m\in L_i} \Pr(\boldsymbol{\theta}(k) = \xi^{-1}(m))}$$

$$= \frac{\sum\limits_{m,n\in L_i} q_{mn}\pi_\theta(k)e_m}{\pi_\theta(k)M_i}$$

$$= \frac{\sum\limits_{m,n\in L_i} q_{mn}\pi_\theta(0)\Pi_\theta^k e_m}{\pi_\theta(0)\Pi_\theta^k M_i}$$

$$= \frac{\pi_\theta(0)\Pi_\theta^k}{\pi_\theta(0)\Pi_\theta^k M_i} \sum_{m,n\in L_i} q_{mn}e_m$$

□

Observe that the one-step transition probabilities $p_{ij}(k)$ given in Equation (5) do not depend of whether or not the process $\boldsymbol{A}(k)$ is a MC. Furthermore, these transition probabilities are given in terms of the transition probabilities of the joint process $\boldsymbol{\theta}(k)$, which are supposed to be known. Since $\mathcal{P}_{\mathcal{I}^N}$ is a partition, it is clear that the matrix of one-step transition probabilities of $\boldsymbol{A}(k)$, $\Pi_A(k) = \left[p_{ij}(k)\right]$, $i,j \in \mathcal{I}$, is a stochastic matrix. The steady-state of these probabilities are given in the following Theorem.

Theorem 2. *Let $\boldsymbol{\theta}_i(k)$, $i \in \mathscr{L}$, be a set of independent, ergodic HMC's, and let $\boldsymbol{\theta}(k)$ be the joint HMC with transition probability matrix $\Pi_\theta = [q_{mn}]$, $m,n \in L$ and stationary probability vector given by π_θ. Then $\lim_{k\to\infty} \Pi_A(k) = \Pi$, where Π is a stochastic matrix with entries*

$$\overline{p}_{ij} = \lim_{k\to\infty} p_{ij}(k) = \frac{\pi_\theta}{\pi_\theta M_i} \sum_{m,n\in L_i} q_{mn} e_m, \quad i,j \in \mathcal{I} \tag{6}$$

Proof. Since each HMC $\boldsymbol{\theta}_i$, $i \in \mathscr{L}$, is ergodic then by Lemma 1 the joint process $\boldsymbol{\theta}(k)$ is also ergodic. Therefore, for k big enough it follows that $\Pi_\theta^k = \overline{1}\pi_\theta$, where $1 \triangleq \underbrace{(1,\dots,1)}_{2^N \text{times}}$ and the stationary probability vector π_θ has positive components. Now for k big enough it follows that $\Pr(\boldsymbol{A}(k) = i) = \pi_\theta(0)\Pi_\theta^k M_i = \pi_\theta(0)\overline{1}\pi_\theta M_i = \pi_\theta M_i$, which is positive because ϕ is an onto mapping implying that the column M_i, $i \in \mathscr{L}$, has at least one entry equal to 1. Then the claim follows directly by taking limits in equations (5). □

The next section gives conditions under which $\boldsymbol{A}(k)$ is an HMC.

3 HMC Conditions

Strong lumpability is the name given to the property under which a transformation of a finite-state HMC results in another reduced finite-state HMC for any initial state probability vector of the underlying process $\boldsymbol{\theta}(k)$. A result that gives sufficient conditions for an HMC to be strong lumpable was given by Kemeny and Snell in 1960 [6]. Theorem 3 below reformulates these conditions for the lumped process $\boldsymbol{A}(k) = \phi(\boldsymbol{\theta}(k))$. Consider the partition $\mathcal{P}_{\mathcal{I}^N}$. Denote by $\Pr(m, I_r)$, $r \in \mathcal{I}$ the probability of moving from the state θ of $\boldsymbol{\theta}(k)$, labeled by $m \in L$, to the set $I_r \in \mathcal{P}_{\mathcal{I}^N}$, that is, $\Pr(m, I_r) \triangleq \sum_{n \in I_r} q_{mn}$.

Theorem 3. *Let $\boldsymbol{\theta}_i(k)$, $i \in \mathscr{L}$, be a set of independent HMCs with state space $\mathcal{I}$ and let $\boldsymbol{\theta}(k)$ be the joint HMC. Let ϕ be a structure function and $\boldsymbol{A}(k) = \phi(\boldsymbol{\theta}(k))$, a lumped process with state space $\mathcal{I}$. Then $\boldsymbol{A}(k)$ is an HMC for every initial state probability vector $\pi_\theta(0)$ if and only if for every pair of sets I_r and I_t in $\mathcal{P}_{\mathcal{I}^N}$, the probability $\Pr(m, I_t)$ has the same value for any m in I_r. This common value is the one-step transition probability corresponding the process $\boldsymbol{A}(k)$ of moving from the set I_r into the set I_t.*

Proof. It is a direct application of Kemeny-Snell's Theorem 6.3.2 in [6, p. 124]. □

The next result shows that $\boldsymbol{A}(k)$ can be an NHMC only for some but not all initial state probability vectors.

Lemma 4. *Let $\boldsymbol{\theta}_i(k)$, $i \in \mathscr{L}$, be a set of independent HMCs with state space $\mathcal{I}$ and let $\boldsymbol{\theta}(k)$ be the joint HMC with initial state probability vector $\pi_z(0)$. Let ϕ be a structure function and $\boldsymbol{A}(k) = \phi(\boldsymbol{z}(k))$ a lumped process with state space $\mathcal{I}$. If $\boldsymbol{A}(k)$ is an MC for all $\pi_\theta(0)$ then it is an HMC.*

Proof. This follows directly from [8, pp. 105-106]. □

By adding the ergodicity property to the HMCs $\boldsymbol{\theta}_i$, $i \in \mathscr{L}$, in Lemma 4, one can obtain the following result.

Theorem 4. *Let $\boldsymbol{\theta}_i(k)$, $i \in \mathscr{L}$, be a set of independent, ergodic HMCs. Let $\boldsymbol{\theta}(k)$ be the joint HMC with initial state probability vector $\pi_\theta(0)$. Let ϕ be a structure function and $\boldsymbol{A}(k) = \phi(\boldsymbol{\theta}(k))$ a lumped process with state space $\mathcal{I}$. If $\boldsymbol{A}(k)$ is an MC for all $\pi_\theta(0)$ then it is an ergodic HMC.*

Proof. By Lemma 4 $\boldsymbol{A}(k)$ is an HMC. The ergodicity of $\boldsymbol{A}(k)$ follows from Lemma 1. □

Lemma 5 below gives necessary and sufficient conditions under which the process $\boldsymbol{A}(k) = \phi(\boldsymbol{\theta}(k))$ will be an HMC for all initial state probability vectors $\pi_\theta(0)$. It is a reformulation, in terms of the lumpability matrix $\mathcal{M}$ of Theorem 3. The result is similar, but not exactly equal to Lemma 1 given in [7]. Moreover, it is easier to apply, since it does not require relabeling of the states.

Lemma 5. *Let $\boldsymbol{\theta}_i(k)$, $i \in \mathscr{L}$, be a set of independent HMCs with state space $\mathcal{I}$ and let $\boldsymbol{\theta}(k)$ be the joint HMC with transition probability matrix Π_θ. Let ϕ be a lumping transformation and $\boldsymbol{A}(k) = \phi(\boldsymbol{\theta}(k))$ a lumped process with state space $\mathcal{I}$. Then the process $\boldsymbol{A}(k)$ is an HMC for every initial state probability vector $\pi_\theta(0)$ if and only if there exists constants μ_1 and μ_2 in $[0,1]$ satisfying*

$$\Pi_m M_1 = 1 - \mu_0 \;\; \forall\, m \in L_0 \quad \textit{and} \quad \Pi_m M_0 = 1 - \mu_1 \;\; \forall\, m \in L_1,$$

where Π_m is the m-th row of Π_θ. Furthermore, the transition probability matrix of $\boldsymbol{A}(k)$ is $\Pi_{\boldsymbol{A}} = \begin{bmatrix} \mu_0 & 1-\mu_0 \\ 1-\mu_1 & \mu_1 \end{bmatrix}$.

Proof. The set of labels L_0 and L_1 correspond to the set of states I_0 and I_1, respectively, in the partition $\mathcal{P}_{\mathcal{I}^N}$ induced by the structure function ϕ. The claim follows directly from Theorem 3 by observing that $\Pr(m, I_1) = \Pi_m M_1 = 1-\mu_0\; \forall\, m \in L_0$ and $\Pr(m, I_0) = \Pi_m M_0 = 1-\mu_1\; \forall\, m \in L_1$. □

The following is an example of a parallel interconnection known as 1-out-of-2, that is, the interconnection is considered to be working correctly if a least 1 of the devices is working correctly.

Example 1. *Consider an interconnection of $N = 2$ devices with upset processes given by an HMC with transition probability matrices*

$$\Pi_{\theta_i} \triangleq \begin{bmatrix} p_{11}^i & p_{12}^i \\ p_{21}^i & p_{22}^i \end{bmatrix}$$

Table 1: Transformation table for Example 1

$\boldsymbol{\theta}_1(k)$	$\boldsymbol{\theta}_2(k)$	$\boldsymbol{\theta}(k)$	$\xi(\boldsymbol{\theta}(k))$	$\boldsymbol{A}(k) = \phi(\boldsymbol{\theta}(k))$
0	0	(0,0)	1	0
0	1	(0,1)	2	0
1	0	(1,0)	3	0
1	1	(1,1)	4	1

and initial state probability vector $\pi_{\theta_i}(0)$, $i = 1, 2$. If the system availability process is given by the process $\boldsymbol{A}(k)$ defined in Table 1, then the state space of $\boldsymbol{\theta}(k)$ is partitioned as $\mathcal{I}^2 = I_0 \bigcup I_1$, where $I_0 = \{(0,0), (0,1), (1,0)\}$ and $I_1 = \{(1,1)\}$.

The lumping matrix is

$$\mathcal{M} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

By Lemma 2 the probability that the network is working correctly is

$$\begin{aligned}
&\Pr(\boldsymbol{A}(k) = 0) \\
&= \pi_{\theta_1}(0)\Pi^k_{\theta_1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \pi_{\theta_2}(0)\Pi^k_{\theta_2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \pi_{\theta_1}(0)\Pi^k_{\theta_1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pi_{\theta_2}(0)\Pi^k_{\theta_2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \pi_{\theta_1}(0)\Pi^k_{\theta_1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \pi_{\theta_1}(0)\Pi^k_{\theta_1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pi_{\theta_2}(0)\Pi^k_{\theta_2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \pi_{\theta_1}(0)\Pi^k_{\theta_1} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pi_{\theta_2}(0)\Pi^k_{\theta_2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right).
\end{aligned}$$

The stationary probability vector for $\boldsymbol{A}(k)$ exists whenever $\boldsymbol{\theta}_1(k)$ and $\boldsymbol{\theta}_2(k)$ are ergodic. Let the stationary probability vectors of these processes be $\pi_{\theta_1} \triangleq \begin{bmatrix} \pi^1_{\theta_1} & \pi^2_{\theta_1} \end{bmatrix}$ and $\pi_{\theta_2} \triangleq \begin{bmatrix} \pi^1_{\theta_2} & \pi^2_{\theta_2} \end{bmatrix}$, respectively. From

Theorem 1 it follows that

$$\begin{aligned}\lim_{k\to\infty}\Pr(\boldsymbol{A}(k)=0) &= \pi_{\theta_1}\begin{bmatrix}1\\0\end{bmatrix}\pi_{\theta_2}\begin{bmatrix}1\\1\end{bmatrix}+\pi_{\theta_1}\begin{bmatrix}0\\1\end{bmatrix}\pi_{\theta_2}\begin{bmatrix}1\\0\end{bmatrix}\\ &= \pi_{\theta_1}\begin{bmatrix}1\\0\end{bmatrix}+\pi_{\theta_1}\begin{bmatrix}0\\1\end{bmatrix}\pi_{\theta_2}\begin{bmatrix}1\\0\end{bmatrix}\\ &= \pi^1_{\theta_1}+\pi^2_{\theta_1}\pi^1_{\theta_2}\\ &= 1-\pi^2_{\theta_1}(1-\pi^1_{\theta_2}).\end{aligned}$$

To calculate the one-step transition probabilities $p_{00}(k)$ and $p_{11}(k)$ observe first that ϕ partitions the set of labels as $L = L_0 \bigcup L_1$, where $L_0 = \{1,2,3\}$ and $L_1 = \{4\}$. Thus,

$$p_{00}(k) = \frac{1}{\pi_\theta(0)\Pi^k_\theta M_0}\Big(\big(q_{11}+q_{12}+q_{13}\big)e_1+\big(q_{21}+q_{22}+q_{23}\big)e_2+ \big(q_{31}+q_{32}+q_{33}\big)e_3\Big)\pi_\theta(0)\Pi^k_\theta$$

and

$$p_{11}(k) = \frac{1}{\pi_\theta(0)\Pi^k_\theta M_1}q_{44}\pi_\theta(0)\Pi^k_\theta,$$

where $M_0 = [1\ 1\ 1\ 0]^T$ and $M_1 = [0\ 0\ 0\ 1]^T$.

The transition probability matrix of the joint process $\boldsymbol{\theta}(k)$ is

$$\Pi_\theta = \begin{bmatrix} p^1_{11}\times p^2_{11} & p^1_{11}\times p^2_{12} & p^1_{12}\times p^2_{11} & p^1_{12}\times p^2_{12}\\ p^1_{11}\times p^2_{21} & p^1_{11}\times p^2_{22} & p^1_{12}\times p^2_{21} & p^1_{12}\times p^2_{22}\\ p^1_{21}\times p^2_{11} & p^1_{21}\times p^2_{12} & p^1_{22}\times p^2_{11} & p^1_{22}\times p^2_{12}\\ p^1_{21}\times p^2_{21} & p^1_{21}\times p^2_{22} & p^1_{22}\times p^2_{21} & p^1_{22}\times p^2_{22}\end{bmatrix}.$$

Lemma 5 is used to determine the conditions for $\boldsymbol{A}(k) = \phi(\boldsymbol{\theta}(k))$ to be an HMC. The process $\boldsymbol{A}(k) = \phi(\boldsymbol{\theta}(k))$ will be an HMC if and only if the following equalities are satisfied:

$$\begin{aligned}\Pi_1 M_1 &= 1-\mu_0 = p^1_{12}\times p^2_{12}\\ \Pi_2 M_1 &= 1-\mu_0 = p^1_{12}\times p^2_{22}\\ \Pi_3 M_1 &= 1-\mu_0 = p^1_{22}\times p^2_{12}.\end{aligned}$$

Lemma 5 gives a fourth equation, $\Pi_4 M_0 = 1 - \mu_1 = 1 - p^1_{22} \times p^2_{22}$, which is not needed since it is dependent on the first three equations. These relations imply that

$$p^1_{12} \times p^2_{12} = p^1_{12} \times p^2_{22} = p^1_{22} \times p^2_{12}. \tag{7}$$

If these equalities do not hold, then $\boldsymbol{A}(k)$ will not be an HMC for all initial state probability vectors $\pi_\theta(0)$. By Lemma 4, however, $\boldsymbol{A}(k)$ could be an NHMC for some but not all $\pi_\theta(0)$. Whenever the stationary probability vector for $\boldsymbol{\theta}(k)$ exists then as $k \to \infty$, $\boldsymbol{A}(k)$ is characterized by a constant transition probability matrix as shown in Theorem 1. Assume the 2-state HMCs $\boldsymbol{\theta}_1(k)$ and $\boldsymbol{\theta}_2(k)$ have transition probability matrices Π_{θ_1} and Π_{θ_2} with positive entries. Then from (7) the necessary and sufficient conditions for $\boldsymbol{A}(k)$ to be an HMC are

$$p^1_{12} = p^1_{22} \text{ and } p^2_{12} = p^2_{22}.$$

In this case, Π_{θ_1} and Π_{θ_2} have the form

$$\Pi_{\theta_1} \triangleq \begin{bmatrix} a & 1-a \\ a & 1-a \end{bmatrix}, \quad \Pi_{\theta_2} \triangleq \begin{bmatrix} b & 1-b \\ b & 1-b \end{bmatrix}, \tag{8}$$

where $a = 1 - p^1_{12}$ and $b = 1 - p^2_{12}$ with $a, b \in]0,1[$. If the initial state probability vectors are $\pi_{\theta_1}(0) = [a,\ 1-a]$ and $\pi_{\theta_2}(0) = [b,\ 1-b]$, then the processes $\boldsymbol{\theta}_1(k)$ and $\boldsymbol{\theta}_2(k)$ with transition probability matrices given in (8) are i.i.d. processes. Since $\mu_0 = 1 - \mu_1$, then Π_A has equal rows, and $\pi_A(0) = [\mu_0\ 1 - \mu_0]$. Thus, $\boldsymbol{A}(k)$ is an i.i.d process for $k \geq 1$.

This example shows, in particular, that for the 2-state MCs $\boldsymbol{\theta}_1(k)$ and $\boldsymbol{\theta}_2(k)$ with positive entries in their transition probabilities, the 2-state system availability process $\boldsymbol{A}(k) = \phi(\boldsymbol{\theta}(k))$, where ϕ is the 1-out-of-2 structure function, can not be an HMC for all $\pi_{\theta_1}(0)$ and $\pi_{\theta_2}(0)$.

4 Conclusions

A characterization of a class of 2-state system availability process has been given for a general transformation $\boldsymbol{A}(k)$ of Markov upset processes affecting the interconnected fault tolerant devices. It has been shown that this process is not necessarily a MC and, therefore, it is called a lumped process. The transition probabilities and their steady-states values have been given under this general situation. Finally an example that shows that a transformation of two HMC's with positive entries in their transition probability matrices never yield an HMC system availability process was also given.

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Resumen
El proceso de disponibilidad de un sistema indica si la interconexión de componentes está operando como se requiere en cada instante del tiempo. En este artículo se prueba que un proceso de disponibilidad

con dos estados que resulta de una transformación de una cadena de Markov no es una cadena de Markov. Además, se da la caracterización probabilística de este tipo de procesos de disponibilidad.

Palabras clave: Proceso de disponibilidad, Procesos agregados de Markov.

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