# A STUDY OF MODIFIED HERMITE POLYNOMIALS OF TWO VARIABLES

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***Abstract***

*The present paper is a study of modified Hermite polynomials of two variables $H_n(x, y; a)$ which for $a = e$ reduces to Hermite polynomials of two variables $H_n(x, y)$ due to M.A. Khan and G.S. Abukhammash [2]*

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# 1. Introduction

Hermite polynomials of two variable $H_n(x,y)$ were defined and studied by M.A. Khan and G.S. Abukhammash [2]. They defined them by means of the following generating relation

$$e^{2xt-(y+1)t^2} = \sum_{n=0}^{\infty} \frac{H_n(x,y)t^n}{n!} \tag{1.1}$$

The aim of the present paper is to modify the definition (1.1) and to obtain generating functions, recurrence relations, Rodrigues formula, relationship with Legendre polynomials, expansion of polynomials and other properties for the modified Hermite polynomials of two variables $H_n(x,y;a)$.

# 2. Definition

The modified Hermite polynomials $H_n(x,y;a)$ of two variables are defined by means of the generating relation:

$$a^{2xt-(y+1)t^2} = \sum_{n=0}^{\infty} \frac{H_n(x,y;a)t^n}{n!}, \qquad a > 0,\ a \neq 1 \tag{2.1}$$

It follows from (2.1) that

$$H_n(x,y;a) = \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{n! H_{n-2r}(x;a)(-y)^r (loga)^r}{r!(n-2r)!} \tag{2.2}$$

where $H_n(x;a)$ stands for the modified Hermite polynomial of one variable [4].

The definition (2.2) is equivalent to the following explicit representation of $H_n(x,y;a)$

$$H_n(x,y;a) = \sum_{r=0}^{\left[\frac{n}{2}\right]} \sum_{s=0}^{\left[\frac{n}{2}-r\right]} \frac{(-n)_{2r+2s}(2x)^{n-2r-2s}(-y)^r(-1)^s(loga)^{n-r-s}}{r!s!} \tag{2.3}$$

In terms of double hypergeometric function, modified Hermite polynomials of two variables can be written as

$$H_n(x,y;a) = (2xloga)^n F^{2:0;0}_{0:0;0}$$
$$\left[ \begin{array}{ll} -\frac{n}{2}, -\frac{n}{2}+\frac{1}{2} : -; - & ; \\ - : -; - & ; \end{array} -\frac{y}{x^2loga}, -\frac{1}{y^2loga} \right] \tag{2.4}$$

For $a = e$ (2.2), (2.3) and (2.4) reduces to Hermite polynomials of two variables $H_n(x,y)$ due to M.A. Khan and G.S. Abukhammash [2].

It may be remarked that $H_n(x,y;a)$ is an even function of $x$ for even $n$, an odd function of $x$ for odd $n$.

$$H_n(-x,y;a) = (-1)^n \; H_n(x,y;a)$$

Also,

$$H_{2n}(0,y;a) = (-1)^n(y+1)^{\frac{n}{2}}2^{2n}\left(\frac{1}{2}\right)_n (loga)^n, \qquad H_{2n+1}(0,y;a) = 0$$

$$H_{2n}(0,0;a) = (-1)^n2^{2n}\left(\frac{1}{2}\right)_n (loga)^n, \qquad H_{2n+1}(0,0;a) = 0$$

and

$$\begin{aligned}\frac{\partial}{\partial x}H_{2n}(0,y;a) &= 0,\\ \frac{\partial}{\partial x}H_{2n+1}(0,y;a) &= (-1)^n(2loga)^{n+1}\left(\frac{3}{2}\right)_n(y+1)^{\frac{n+1}{2}}\end{aligned}$$

# 3. Recurrence Relations

Following recurrence relations hold for $H_n(x,y;a)$

$$\frac{\partial}{\partial x}H_n(x,y;a) = 2nloga\ H_{n-1}(x,y;a) \tag{3.1}$$

Iteration of (3.1) gives

$$\frac{\partial^s}{\partial x^s}H_n(x,y;a) = \frac{(2loga)^s n! H_{n-s}(x,y;a)}{(n-s)!} \tag{3.2}$$

$$\frac{\partial}{\partial y}H_n(x,y;a) = -n(n-1)\ loga\ H_{n-2}(x,y;a) \tag{3.3}$$

Iteration of (3.3) gives

$$\frac{\partial^r}{\partial y^r}H_n(x,y;a) = \frac{(-1)^r n!}{(n-2r)!}H_{n-2r}(x,y;a) \tag{3.4}$$

$$H_{n+1}(x,y;a) = (2xloga)\ \{xH_n(x,y;a) - n(y-1)H_{n-1}(x,y;a)\} \tag{3.5}$$

$$x\frac{\partial}{\partial x}H_n(x,y;a)-nH_n(x,y;a)+2y\frac{\partial}{\partial y}H_n(x,y;a)=n\frac{\partial}{\partial y}H_{n-1}(x,y;a). \tag{3.6}$$

$$2nxloga\ H_{n-1}(x,y;a)-nH_n(x,y;a)-2n(n-1)yloga\ H_{n-2}(x,y;a)$$

$$=n\frac{\partial}{\partial x}H_{n-1}(x,y;a) \tag{3.7}$$

Also we have

$$2nxloga\ H_{n-1}(x,y;a)-nH_n(x,y;a)=2n(n-1)(y+1)H_{n-2}(x,y;a) \tag{3.8}$$

which is a pure recurrence relation.

# 4. Relation Between $H_n(x,y;a)$ and $H_n(x;a)$

A relation between $H_n(x,y;a)$ and $H_n(x;a)$ is as given below:

$$H_n(x,y;a)=(y+1)^{\frac{n}{2}}\ H_n(\frac{x}{\sqrt{y+1}};a) \tag{4.1}$$

# 5. Other Generating Function for $H_n(x,y;a)$

Some other generatig functions for $H_n(x,y;a)$ are as follows:

$$\sum_{n=0}^{\infty} \frac{(C)_n H_n(x,y;a)t^n}{n!}$$

$$\cong (1-2xtloga)^{-c} F^{2:0;0}_{0:0;0}$$

$$\left[ \begin{array}{ccc} \frac{C}{2}, \frac{C}{2} + \frac{1}{2} : -; - & ; & -\dfrac{4yt^2loga}{(1-2xtloga)^2}, -\dfrac{4t^2loga}{(1-2xtloga)^2} \\ - : -; - & ; & \end{array} \right] \tag{5.1}$$

$$\sum_{n=0}^{\infty} \frac{H_{n+k}(x,y;a)t^n}{n!} = a^{2xt-(y+1)t^2} H_k(x-ty-t, y; a) \tag{5.2}$$

$$\sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{H_{n+r+s}(x,y;a)t^n u^r}{n! r!}$$

$$= a^{2xt-(y+1)t^2} . a^{2(x-ty-t)u-(y+1)u^2} H_r(x-ty-t-yu-u, y; a) \tag{5.3}$$

# 6. Rodrigues Formula

A Rodrigues formula for $H_n(x,y;a)$ is given by

$$H_n(x,y;a) = (-1)^n (y+1)^{\frac{n}{2}} \; a^{\frac{x^2}{(y+1)}} \; D^n \; a^{-\frac{x^2}{(y+1)}}, \; D \equiv \frac{d}{dx} \tag{6.1}$$

a formula of the same nature as Rodrigues's formula for modified Hermite polynomial of one variable $H_n(x)$.

# 7. Special Properties

Consider the identity

$$a^{2xt-(y+1)t^2} = a^{2(xt)-(y+1)(xt)^2}.a^{(y+1)(x^2-1)t^2}$$

or,

$$\sum_{n=0}^{\infty} \frac{H_n(x,y;a)t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{H_n(1,y;a)(xt)^n}{n!} \sum_{k=0}^{\infty} \frac{(y+1)^k \left(x^2-1\right)^k (loga)^k t^{2k}}{k!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{H_{n-2k}(1,y;a)x^{n-2k}(y+1)^k \left(x^2-1\right)^k (loga)^k t^n}{k!(n-2k)!}$$

Equating the coefficients of $t^n$, we get

$$H_n(x,y;a) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{n!H_{n-2k}(1,y;a)x^{n-2k}(y+1)^k \left(x^2-1\right)^k (loga)^k}{k!(n-2k)!}. \tag{7.1}$$

Next by considering the identity

$$a^{2(x_1+x_2)t-(y+1)t^2} = a^{2x_1t-(y+1)t^2}.a^{2x_2t-(y+1)t^2}.a^{(y+1)t^2}$$

we obtain

$$H_n(x_1+x_2,y;a)=\sum_{r=0}^{n}\sum_{s=0}^{\left[\frac{n-r}{2}\right]}\frac{n!H_{n-r-2s}(x_1,y)H_r(x_2,y)(y+1)^s(loga)^s}{r!s!(n-r-2s)!} \tag{7.2}$$

Now by considering the identity

$$a^{2xt-(y_1+y_2+1)t^2}.a^{2xt-t^2}=a^{2xt-(y_1+1)t^2}.a^{2xt-(y_2+1)t^2}$$

we get

$$H_n(x,y_1+y_2;a)H_k(x;a)=H_n(x,y_1;a)H_k(x,y_2;a) \tag{7.3}$$

where $H_k(x;a)$ is modified Hermite polynomial of one variable [4].

Similarly

$$H_n(\lambda x,y;a)=\sum_{k=0}^{n}\frac{H_{n-k}(x,y;a)2^k(\lambda-1)^k x^k(loga)^k}{k!(n-k)!} \tag{7.4}$$

$$H_n(x,\lambda y;a)=\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n!H_{n-2k}(x,y;a)\left[(1-\lambda)yloga\right]^k}{k!(n-2k)!} \tag{7.5}$$

$$H_n(\lambda x,\mu y;a)$$
$$=\sum_{r=0}^{n}\sum_{s=0}^{\left[\frac{n-r}{2}\right]}\frac{n!H_{n-r-2s}(x,y;a)H_r\left\{(\lambda-1)x,(1-\mu)y;a\right\}(loga)^s}{r!s!(n-r-2s)!} \tag{7.6}$$

# 8. Expansion of Polynomials

Here we expand the Legendre polynomials of one and two variables in a series of modified Hermite polynomials of two variable. These expansions are as given below:

$$P_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]}$$

$$\frac{{}_2F_0\left[-k, \frac{1}{2}+n-k, -; \frac{y+1}{\log a}\right](-1)^k \left(\frac{1}{2}\right)_{n-k} H_{n-2k}(x,y;a)(\log a)^{2k-n}}{k!(n-2k!)} \tag{8.1}$$

$$P_n(x,y) = \sum_{k=0}^{\infty}$$

$$\frac{{}_2F_0\left[\begin{array}{rr} -k, \frac{1}{2}+n-k; & \frac{1}{\log a} \\ -; & \end{array}\right](-1)^k \left(\frac{1}{2}\right)_{n-k} (1+y)^k (\log a)^{2k-n} H_{n-2k}(x,y;a)}{k!(n-2k)!}. \tag{8.2}$$

Where

$$P_n(x,y) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k \left(\frac{1}{2}\right)_{n-k} (2x)^{n-2k} (y+1)^k}{k!\ (n-2k)!}$$

is the Legendre polynomials of two variables defined and studied by Khan, M.A. and Ahmed, S. [3].

# 9. Binomial and Trinomial Operator Representations

In a recent paper in 2008, M.A. Khan and A.K. Shukla [5] obtained binomial and trinomial operator representations of certain polynomials. Using their technique we have obtained certain results of binomial and trinomial operator representation type for modified two variables Hermite polynomials $H_n(x, y; a)$ by using their Rodrigues formula. Here we need the following results of [5]:

$$(D_x+D_y)^n \{f(x)g(y)\} = \sum_{r=0}^{n} \binom{n}{r} D_x^{n-r}\, f(x)\, D_y^r\, g(y) \tag{9.1}$$

$$(D_x+D_y+D_z)^n \{f(x)g(y)h(z)\}$$
$$= \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s}(-1)^{r+s}}{r!s!} D_x^{n-r-s}\; f(x)\; D_y^r\; g(y)\; D_z^s\; h(z) \tag{9.2}$$

and

$$(D_xD_y + D_xD_z + D_yD_z)^n \{f(x)g(y)h(z)\}$$
$$= \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s}(-1)^{r+s}}{r!s!} D_x^{n-s}\; f(x)\; D_y^{n-r}\; g(y)\; D_z^{r+s}\; h(z) \tag{9.3}$$

The results obtained are as follows:

Let $\frac{d}{d\left(\frac{x}{\sqrt{y+1}}\right)} \equiv D_1$ and $\frac{d}{d\left(\frac{w}{\sqrt{z+1}}\right)} \equiv D_2$, then

$$(D_1+D_2)^n\; a^{-\frac{x^2}{y+1}}\; b^{-\frac{w^2}{z+1}}$$

$$= (-1)^n (y+1)^{-\frac{n}{2}}\ a^{-\frac{x^2}{y+1}}\ b^{-\frac{w^2}{z+1}}$$
$$\sum_{r=0}^{n} \binom{n}{r} H_{n-r}(x,y;a) H_r(w,z;b) \left(\sqrt{\frac{y+1}{z+1}}\right)^r \tag{9.4}$$

Again let $\frac{d}{d\left(\frac{u}{\sqrt{v+1}}\right)} \equiv D_1$, $\frac{d}{d\left(\frac{w}{\sqrt{x+1}}\right)} \equiv D_2$ and $\frac{d}{d\left(\frac{y}{\sqrt{z+1}}\right)} \equiv D_3$, then

$$(D_1+D_2+D_3)^n\ a^{-\frac{u^2}{v+1}}\ b^{-\frac{w^2}{x+1}}\ c^{-\frac{y^2}{z+1}}$$
$$= (-1)^n (v+1)^{-\frac{n}{2}}\ a^{-\frac{u^2}{v+1}}\ b^{-\frac{w^2}{x+1}}\ c^{-\frac{y^2}{z+1}} \sum_{r=0}^{n}\sum_{s=0}^{n-r} \frac{(-n)_{r+s}(-1)^{r+s}}{r!s!}$$
$$\times H_{n-r-s}(u,v;a) H_r(w,x;b) H_s(y,z;c) \left(\sqrt{\frac{v+1}{x+1}}\right)^r \left(\sqrt{\frac{v+1}{z+1}}\right)^s \tag{9.5}$$

and

$$(D_1D_2+D_1D_3+D_2D_3)^n\ a^{-\frac{u^2}{v+1}}\ b^{-\frac{w^2}{x+1}}\ c^{-\frac{y^2}{z+1}}$$
$$= (v+1)^{-\frac{n}{2}} (x+1)^{-\frac{n}{2}}\ a^{-\frac{u^2}{v+1}}\ b^{-\frac{w^2}{x+1}}\ c^{-\frac{y^2}{z+1}} \sum_{r=0}^{n}\sum_{s=0}^{n-r} \frac{(-n)_{r+s}(-1)^{r+s}}{r!s!}$$
$$\times H_{n-s}(u,v;a) H_{n-r}(w,x;b) H_{r+s}(y,z;c) \left(\sqrt{\frac{v+1}{z+1}}\right)^s \left(\sqrt{\frac{x+1}{z+1}}\right)^r \tag{9.6}$$

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**Resumen**

El presente artículo se estudian polinomios modificados de Hermite de dos variables $H_n(x, y; a)$ que para $a = e$ se reducen a los polinomios de

Hermite de dos variables $H_n(x, y)$ introducidos por M.A. Khan y G.S. Abukhammash [2].

**Palabras clave:** Funciones generatrices, relaciones de recurrencia, fórmula Rodrigues, Relación con los polinomios de Legendre y expansiones de polinomios.

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