A DIFFERENTIAL EQUATION FOR POLYNOMIALS RELATED TO THE JACOBIAN CONJECTURE

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Abstract

We analyze a possible minimal counterexample to the Jacobian Conjecture P, Q *with* $\gcd(\deg(P), \deg(Q)) = 16$ *and show that its existence depends only on the existence of solutions for a certain Abel differential equation of the second kind.*

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1 Introduction

In a recent article [1], we managed to describe the shape of possible minimal counterexample to JC (the Jacobian conjecture as stated in [3]) given by a pair of polynomials (P,Q) with $\gcd(\deg(P),\deg(Q)) = B$, where

$$B := \begin{cases} \infty & \text{if JC is true,} \\ \min\bigl(\gcd(\deg(P),\deg(Q))\bigr) & \text{where } (P,Q) \text{ is a counterexample} \\ & \text{to JC, if JC is false.} \end{cases}$$

We arrived at the following theorem (See [1, Theorem 8.10]):

Theorem 1.1 *If $B = 16$, then there exist μ_0, μ_1, μ_2, $\mu_3 \in K$ with $\mu_0 \neq 0$ and $P, Q \in L := K[x,y]$ such that*

$$\ell_{1,-1}(P) = x^3y + \mu_3x^2, \quad \ell_{1,-1}(Q) = x^2y + \mu_3x$$

and

$$[P,Q] = x^4y + \mu_0 + \mu_1x + \mu_2x^2 + \mu_3x^3. \tag{1.1}$$

Moreover, there exists $j \in \mathbb{N}$ such that $\{(j,1)\} = \mathrm{Dir}(P) = \mathrm{Dir}(Q)$,

$$\mathrm{st}_{j,1}(P) = (3,1), \ \ \mathrm{st}_{j,1}(Q) = (2,1), \ \mathrm{en}_{j,1}(P) = (0,m), \ \mathrm{en}_{j,1}(Q) = (0,n),$$

where $m = 3j+1$ and $n = 2j+1$.

By [2, Theorem 2.23] we know that $B \geq 16$. Hence, if we can prove that such a pair cannot exist, necessarily $B > 16$.
In Section 2 we will show how the existence of such a pair (P,Q) would allow the construction of a counterexample to the Jacobian Conjecture. We use the notations of [1].
In Section 3 we write, according to Theorem 1.1,

$$P = x^3y + x^2p_2(y) + xp_1(y) + p_0(y) \quad \text{and} \quad Q = x^2y + xq_1(y) + q_0(y).$$

Then the condition (1.1) translates into a system of four first order differential equations for the polynomials p_0, p_1, q_0, q_1, q_2. We reduce this system to a single equation for two polynomials and we prove the following theorem:

Theorem 1.2 $B = 16$ *if and only if there exist* $A, q_1 \in K[y]$ *and* $\mu_0, \mu_1, \mu_2, \mu_3 \in K$ *with* $\mu_0 \neq 0$,

$$A(0) = -\frac{1}{4}\mu_3^2, \quad A'(0) = \mu_2 \quad \textit{and} \quad \mu_3 A''(0) = -6\mu_1 - 2\mu_3 q_1''(0), \tag{1.2}$$

such that

$$6\left(A - \frac{q_1^2}{4} + \frac{\mu_3}{4}q_1 - \frac{\mu_2}{6}y\right)^2 = 4yAA' + 6\left(\frac{\mu_3}{4}q_1 - \frac{\mu_2}{6}y\right)^2 - \mu_2 y q_1^2 + 3\mu_1 y^2 q_1 - 6\mu_0 y^3. \tag{1.3}$$

We were not able to obtain a solution of (1.3) satisfying (1.2) with $\mu_0 \neq 0$ (which would yield a counterexample to the JC), nor could we discard the existence of such a solution (which would prove $B > 16$). We analyze some particular cases of (1.3), for example we show that for $\mu_3 = \mu_2 = \mu_1 = \mu_0 = 0$ the only possible solutions are (ρ, σ)-homogeneous for $(\rho, \sigma) = (j, 1)$, where $j + 1 = \deg(q_1)$. We also recognize (1.3) as an Abel differential equation of second kind, for which no general solution is known. Using a standard trick we write this equation in a shorter form in (3.7) and in (3.8).

2 Construction of an counterexample

We reverse the order of the construction leading to Theorem 8.10 of [1]. Starting from a pair (P, Q) as in Theorem 1.1, we apply different automorphisms of L and $L^{(1)}$ and obtain a counterexample $(\tilde{P}, \tilde{Q})$ with $\gcd(\deg(\tilde{P}), \deg(\tilde{Q})) = 16$.

Recall from [1] the automorphisms $\psi_1 \in \mathrm{Aut}(L)$ and $\psi_3 \in \mathrm{Aut}(L^{(1)})$ given by

$$\psi_1(x) := y, \qquad \psi_3(x) := x^{-1},$$
$$\psi_1(y) := -x, \qquad \psi_3(y) := x^3 y.$$

For $(\rho,\sigma) \in \overline{\mathfrak{V}}$ and $k \in \{1,3\}$, we define $(\rho_k,\sigma_k) := \overline{\psi}_k(\rho,\sigma)$ by

$$\overline{\psi}_1(\rho,\sigma) := (\sigma,\rho) \quad \text{and} \quad \overline{\psi}_3(\rho,\sigma) := \begin{cases} (-\rho, 3\rho+\sigma) & \text{if } (\rho,\sigma) \le (-1,2), \\ (\rho, -3\rho-\sigma) & \text{if } (\rho,\sigma) > (-1,2). \end{cases}$$

We have following lemma (See [1, Lemma 6.6]):

Lemma 2.1 *Let $P \in L^{(1)}$. The maps ψ_1 and ψ_3 satisfy the following properties:*

1. *For all $i,j \in \mathbb{N}_0$ we have $v_{\rho_1,\sigma_1}(\psi_1(x^i y^j)) = v_{\rho,\sigma}(x^i y^j)$, and if $P \in L$, then*

   $$\ell_{\rho_1,\sigma_1}(\psi_1(P)) = \psi_1\left(\ell_{\rho,\sigma}(P)\right) \text{ and } \ell\ell_{\rho_1,\sigma_1}(\psi_1(P)) = \psi_1\left(\ell\ell_{\rho,\sigma}(P)\right).$$

2. *If $(\rho,\sigma) \le (-1,2)$, then we have $v_{\rho_3,\sigma_3}(\psi_3(x^i y^j)) = v_{\rho,\sigma}(x^i y^j)$ for all $i \in \mathbb{N}_0$ and $j \in \mathbb{Z}$,*

   $$\ell_{\rho_3,\sigma_3}(\psi_3(P)) = \psi_3\left(\ell_{\rho,\sigma}(P)\right) \text{ and } \ell\ell_{\rho_3,\sigma_3}(\psi_3(P)) = \psi_3\left(\ell\ell_{\rho,\sigma}(P)\right).$$

3. *If $(\rho,\sigma) > (-1,2)$, then $v_{\rho_3,\sigma_3}(\psi_3(x^i y^j)) = -v_{\rho,\sigma}(x^i y^j)$ for all $i \in \mathbb{N}_0$ and $j \in \mathbb{Z}$,*

   $$\ell_{\rho_3,\sigma_3}(\psi_3(P)) = \psi_3\left(\ell\ell_{\rho,\sigma}(P)\right) \text{ and } \ell\ell_{\rho_3,\sigma_3}(\psi_3(P)) = \psi_3\left(\ell_{\rho,\sigma}(P)\right).$$

Moreover clearly $\mathrm{Jac}(\psi_1) = [\psi_1(x),\psi_1(y)] = 1$ and $\mathrm{Jac}(\psi_3) = -x$. Let (P,Q) be as in Theorem 1.1.

FIRST STEP:
Set $P_1 := \psi_3(P)$ and $Q_1 := \psi_3(Q)$ and $(\tilde{\rho},\tilde{\sigma}) := (-j, 3j+1)$. Using Lemma 2.1 one checks that $\mathrm{Pred}_{P_1}(\tilde{\rho},\tilde{\sigma}) = \mathrm{Pred}_{Q_1}(\tilde{\rho},\tilde{\sigma}) = (1,-1)$,

$$\mathrm{en}_{\tilde{\rho},\tilde{\sigma}}(P_1) = (0,1), \quad \mathrm{en}_{\tilde{\rho},\tilde{\sigma}}(Q_1) = (1,1), \quad w(\ell\ell_{-1,3}(P_1)) = m(3,1),$$

and

$$w(\ell\ell_{-1,3}(Q_1)) = n(3,1),\ \ell_{-1,2}(P_1) = y+\mu_3 x^{-2},\ \ell_{-1,2}(Q_1) = xy+\mu_3 x^{-1},$$

where $m := 3j+1$ and $n := 2j+1$. Moreover, using that

$$[\varphi(P),\varphi(Q)] = \varphi([P,Q])[\varphi(x),\varphi(y)],$$

for all morphisms φ, we obtain

$$[P_1,Q_1] = -(y+\mu_0 x+\mu_1+\mu_2 x^{-1}+\mu_3 x^{-2}).$$

SECOND STEP
Set $P_2 := \varphi_0(P_1)$ and $Q_2 := \varphi_0(Q_1)$, where $\varphi_0(y) := y-(\mu_0 x+\mu_1+\mu_2 x^{-1}+\mu_3 x^{-2})$ and $\varphi_0(x) := x$ (note that $\mathrm{Jac}(\varphi_0) = 1$). Then $P_2, Q_2 \in L$ and

$$[P_2,Q_2] = -y,\ \mathrm{Dir}(P_2) = \mathrm{Dir}(Q_2) = \{(\tilde{\rho},\tilde{\sigma}),(1,1)\},\ \mathrm{en}_{\tilde{\rho},\tilde{\sigma}}(P_2) = (0,1),$$

and

$$\mathrm{en}_{\tilde{\rho},\tilde{\sigma}}(Q_2) = (1,1),\ \ \ell_{1,1}(P_2) = \lambda_P R_2^m \quad \text{and} \quad \ell_{1,1}(Q_2) = \lambda_Q R_2^n,$$

for $R_2 = x^3(y-\mu_0 x)$.

THIRD STEP
Since $P_2, Q_2 \in L$, we can apply ψ_1. We set $P_3 := \psi_1(P_2)$, $Q_3 := \psi_1(Q_2)$ and $(\overline{\rho},\overline{\sigma}) := (3j+1,-j)$. Then

$$[P_3,Q_3] = -x,\ \ \mathrm{Dir}(P_3) = \mathrm{Dir}(Q_3) = \{(\overline{\rho},\overline{\sigma}),(1,1)\},\ \ \mathrm{en}_{\overline{\rho},\overline{\sigma}}(P_3) = (1,0),$$

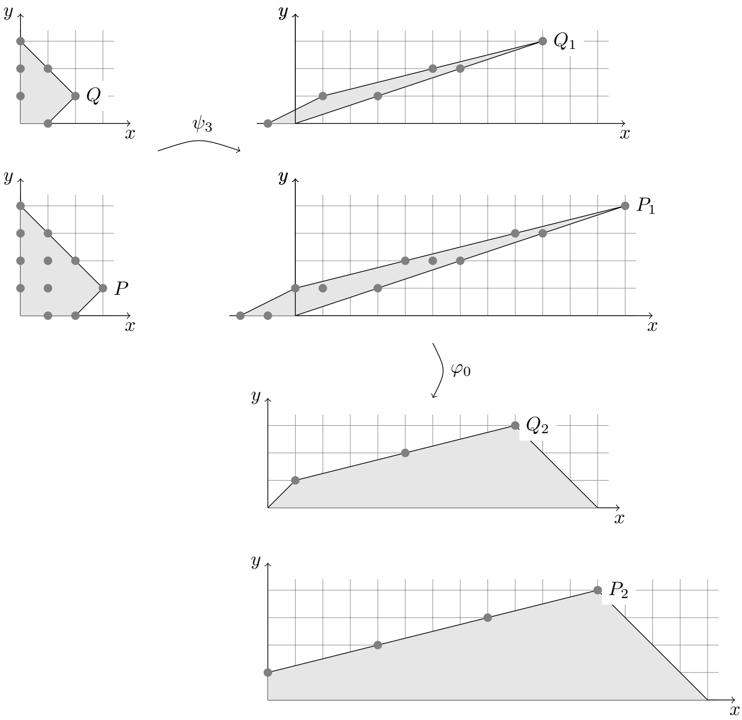

Figure 1: Illustration of the first two steps, for $j = 1$.

and

$$\mathrm{en}_{\overline{\rho},\overline{\sigma}}(Q_3) = (1,1),\ \ell_{1,1}(P_3) = \tilde{\lambda}_P R_3^m \quad \text{and} \quad \ell_{1,1}(Q_3) = \tilde{\lambda}_Q R_3^n,$$

for $R_3 = y^3(y + \frac{1}{\mu_0}x)$.

FOURTH STEP(Figure 2)

We set $P_4 := \psi_3(P_3)$, $Q_4 := \psi_3(Q_3)$ and $(\hat{\rho},\hat{\sigma}) := (-3j-1, 8j+3)$. Then

$$\mathrm{Dir}(P_4) = \mathrm{Dir}(Q_4) = \{(\hat{\rho},\hat{\sigma}), (-1,4)\},\ \mathrm{en}_{\hat{\rho},\hat{\sigma}}(P_4) = (-1,0).$$

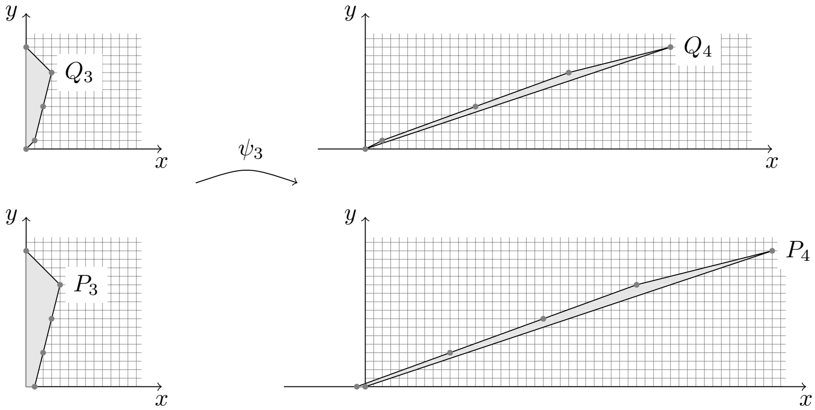

Figure 2: Illustration of the fourth step.

Moreover $[P_4, Q_4] = 1$ and

$$\mathrm{en}_{\hat{\rho},\hat{\sigma}}(Q_4) = (2,1), \quad \ell_{-1,4}(P_4) = \tilde{\lambda}_P R_4^m \quad \text{and} \quad \ell_{-1,4}(Q_4) = \tilde{\lambda}_Q R_4^n,$$

for $R_4 = y^3 x^{12}(y + \frac{1}{\mu_0} x^{-4})$.

FIFTH STEP

Set $P_5 := \varphi_1(P_4)$ and $Q_5 := \varphi_1(Q_4)$, where $\varphi_1(y) := y - \frac{1}{\mu_0} x^{-4}$ and $\varphi_1(x) := x$ (note that $\mathrm{Jac}(\varphi_1) = 1$). Then

$$\ell_{-1,4}(P_5) = \tilde{\lambda}_P R_5^m \quad \text{and} \quad \ell_{-1,4}(Q_5) = \tilde{\lambda}_Q R_5^n,$$

for $R_5 = y x^{12}(y - \frac{1}{\mu_0} x^{-4})^3$.

SIXTH STEP(Figure 3)

If $P_5, Q_5 \in L$, then we have a counterexample to JC, since $[P_5, Q_5] = 1$, $\deg(P) = 16m$ and $\deg(Q) = 16n$ with $m \nmid n$ and $n \nmid m$.

Else set $(\rho_1, \sigma_1) := \mathrm{Succ}_{P_5}(-1, 4)$. Then $[\ell_{\rho_1,\sigma_1}(P_5), \ell_{\rho_1,\sigma_1}(Q_5)] = 0$ and so

$$\ell_{\rho_1,\sigma_1}(P_5) = \hat{\lambda}_P R_6^m \quad \text{and} \quad \ell_{-1,4}(Q_5) = \hat{\lambda}_Q R_6^n,$$

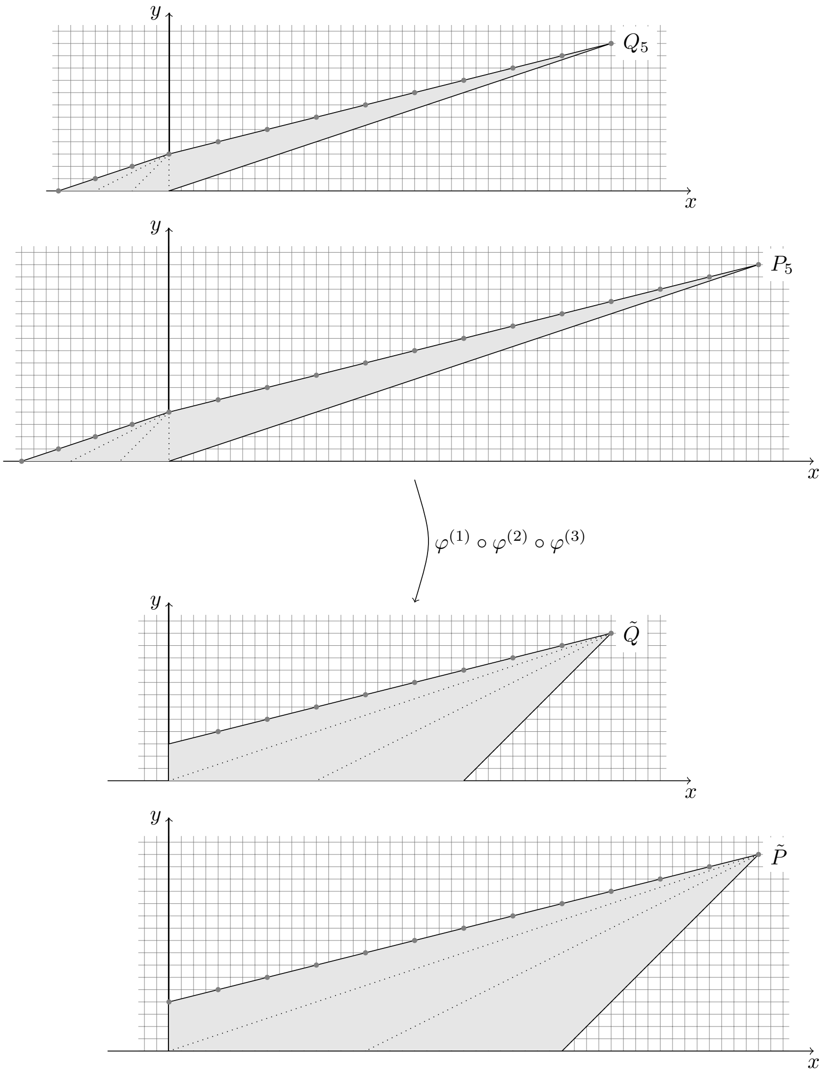

Figure 3: Illustration of the sixth step.

for some $R_6 = y + \lambda_k x^{-k}$ with $\lambda_k \in K^\times$ and $k \in \{1,2,3\}$. Note that $(\rho_1, \sigma_1) = (-1, k)$. If necessary, we apply successively $\varphi^{(k)}$ given by $\varphi^{(k)}(y) := y - \lambda_k x^{-k}$ and $\varphi^{(k)}(x) := x$, to obtain finally the desired counterexample $(\tilde{P}, \tilde{Q})$ given by

$$(\tilde{P}, \tilde{Q}) := (\varphi^{(1)}(\varphi^{(2)}(\varphi^{(3)}(P_5))), \varphi^{(1)}(\varphi^{(2)}(\varphi^{(3)}(Q_5)))) \in L.$$

3 Differential equations for polynomials

According to Theorem 1.1 we write

$$P = x^3y + x^2p_2(y) + xp_1(y) + p_0(y) \quad \text{and} \quad Q = x^2y + xq_1(y) + q_0(y).$$

Then the equality (1.1) yields

$$\begin{aligned}
x^4y &= [x^3y, x^2y] \\
\mu_3x^3 &= [x^3y, xq_1(y)] + [x^2p_2(y), x^2y] \\
\mu_2x^2 &= [x^3y, q_0(y)] + [x^2p_2(y), xq_1(y)] + [xp_1(y), x^2y] \\
\mu_1x &= [x^2p_2(y), q_0(y)] + [xp_1(y), xq_1(y)] + [p_0(y), x^2y] \\
\mu_0 &= [xp_1(y), q_0(y)] + [p_0(y), xq_1(y)].
\end{aligned}$$

The first equality is trivially true. Noting that

$$[x^kp_k(y), x^jq_j(y)] = x^{k+j-1}(kp_k(y)q_j'(y) - jp_k'(y)q_j(y)),$$

we obtain the system of four differential equations for the five polynomials p_0, p_1, p_2, q_0, q_1:

$$\begin{aligned}
\mu_3 &= 3yq_1' - q_1 + 2p_2 - 2yp_2' \\
\mu_2 &= 3yq_0' + 2p_2q_1' - p_2'q_1 + p_1 - 2yp_1' \\
\mu_1 &= 2p_2q_0' + p_1q_1' - p_1'q_1 - 2yp_0' \\
\mu_0 &= p_1q_0' - p_0'q_1.
\end{aligned}$$

Note that $\ell_{1,-1}(P) = x^3y + \mu_3x^2$ and $\ell_{1,-1}(Q) = x^2y + \mu_3x$ imply $q_1(0) = \mu_3$ and $p_2(0) = \mu_3$. Moreover, if we write $P = \sum_{i,j} a_{i,j}x^iy^j$, then we can assume $a_{2,1} = p_2'(0) = 0$, replacing P by $P - a_{2,1}Q$. Writing $Q = \sum_{i,j} b_{i,j}x^iy^j$, $[P,Q] = \sum_{i,j} c_{i,j}x^iy^j$ and noting that

$$c_{i,j} = \sum_{(k,l)+(s,t)=(i,j)+(1,1)} (kt - ls)a_{k,l}b_{s,t}, \tag{3.1}$$

one verifies that

$$0 = c_{3,1} = 2a_{3,1}b_{1,1} = 2b_{1,1},$$

using $b_{2,0} = b_{2,2} = a_{3,2} = a_{3,0} = 0$ and $b_{3,k} = a_{4,k} = 0$ for all k. It follows that $q_1'(0) = b_{1,1} = 0$ and so we can and will assume

$$q_1(0) = \mu_3, \quad q_1'(0) = 0, \quad p_2(0) = \mu_3 \quad \text{and} \quad p_2'(0) = 0.$$

This allows to solve the first equation in full generality. In fact, write $q_1 = \mu_3 + y^2F'$ and $p_2 = \mu_3 + yG$ for some $F, G \in K[y]$. From the first equation we obtain

$$\mu_3 = 3y(2yF' + y^2F'') - (\mu_3 + y^2F') + 2(\mu_3 + yG) - 2y(G + yG'),$$

from which we deduce the equality

$$2G' = 5F' + 3yF'' = (2F + 3yF')'$$

and so $G = F + (3/2)yF' + const$. Since $G(0) = 0$, we can assume $F(0) = 0$ and $G = F + (3/2)yF'$. Hence the general solution to the first equation is $q_1 = \mu_3 + y^2F'$ and $p_2 = \mu_3 + yF + (3/2)y^2F'$, for any choice of $F \in yK[y]$.
Using the second equation we can express q_0' as a function of F and p_1:

$$\begin{aligned} q_0' = \big(&-2p_1 + 2\mu_2 + 2\mu_3F + 4yp_1' - 6y^2FF' - \mu_3y^2F'' - 4y^3(F')^2 \\ &-4y^3FF'' - 3y^4F'F''\big)/6y \end{aligned} \tag{3.2}$$

The third equation yields p_0' as a function of F, p_1 and q_0':

$$p_0' = \frac{yp_1(2F' + yF'') - \mu_1 - p_1'(\mu_3 + y^2F') + (2\mu_3 + y(2F + 3yF'))q_0'}{2y} \tag{3.3}$$

Inserting the values into the fourth equation we obtain a (very big) differential equation for p_1 and F:

$$\begin{aligned}6\mu_0 y^2 = \quad & y p_1\Big(2(p_1-\mu_2-\mu_3F)-4yp_1'+y^2(6FF'+\mu_3F'')\\ &+4y^3((F')^2+FF'')+3y^4F'F''\Big)-\Bigg(3y^2p_1(2F'+yF'')\\ &-3\mu_1 y-3yp_1'(\mu_3+y^2F')-\tfrac{1}{2}(2\mu_3+y(2F+3yF'))\cdot(2p_1\\ &-2\mu_2-2\mu_3F-4yp_1'+6y^2FF'+\mu_3y^2F''+4y^3(F')^2\\ &+4y^3FF''+3y^4F'F'')\Bigg)(\mu_3+y^2F')\end{aligned} \tag{3.4}$$

Now we set

$$A := yp_1-q_1p_2+\frac{3}{4}q_1^2 = -\frac{1}{4}\mu_3^2+yp_1-\mu_3yF-\mu_3y^2F'-y^3FF'-\frac{3}{4}y^4(F')^2$$

and we can express (3.4) as a differential equation for A and q_1:

$$\begin{aligned}6\left(A-\frac{q_1^2}{4}+\frac{\mu_3}{4}q_1-\frac{\mu_2}{6}y\right)^2 =& 4yAA'+6\left(\frac{\mu_3}{4}q_1-\frac{\mu_2}{6}y\right)^2\\ &-\mu_2yq_1^2+3\mu_1y^2q_1-6\mu_0y^3\end{aligned} \tag{3.5}$$

Moreover we have

$$A(0) = -\frac{1}{4}\mu_3^2, \quad A'(0)=\mu_2 \quad \text{and} \quad u_3A''(0) = -6\mu_1-2\mu_3q_1''(0). \tag{3.6}$$

In fact, from the definition of A we have that $A(0) = -q_1(0)p_2(0) + \frac{3}{4}q_1(0)^2 = -\frac{1}{4}\mu_3^2$. The other two conditions follow from the requirement that $q_0'(y)$ and $p_0'(y)$ defined by (3.2) and (3.3) are polynomials.

This proves Theorem 1.2 and is a great simplification with respect to (3.4), not only in the number of terms involved, but in the type of differential equation. In fact, (3.4) is a quadratic first order differential equation for A, called an Abel differential equation of second kind. For q_1 it is a cuartic equation with no derivative of q_1 involved. However we were not able to obtain a solution of (3.5) with $\mu_0 \neq 0$ and such that (3.6) is satisfied (which would yield a counterexample to the JC), nor could we discard the existence of such a solution (which would prove $B > 16$). In the sequel, we will analyze some aspects of this differential equation.

3.1 Solutions without (3.6).

If we don't require (3.6), then there exist solutions of (3.5) with $\mu_0 \neq 0$. Take for example $A = 1 - y^3 - y^6/4$ and $q_1(y) = y^3 + 2$. Then (3.5) is satisfied for $\mu_0 = 1$, $\mu_1 = 0 = \mu_2$ and $\mu_3 = 2$. If we try to construct a counterexample, we obtain $p_1(y) = y^5 + 2y^2 + \frac{2}{y} \notin K[y]$. In fact this solution yields

$$P = x^3 y + 2x^2(y^3+1) + x\left(y^5 + 2y^2 + \frac{2}{y}\right) + \frac{y^7}{7} + \frac{y^4}{2} + \frac{1}{y^2}$$

and

$$Q = x^2 y + x(y^3 + 2) + \frac{y^5}{5} + y^2 + \frac{2}{y}.$$

Note that $P, Q \in K[x, y, y^{-1}]$ and $[P, Q] = x^4 y + \mu_0 + \mu_1 x + \mu_2 x^2 + \mu_3 x^3$, with $\mu_0 = 1 \neq 0$.

3.2 The case $\mu_3 = \mu_2 = \mu_1 = \mu_0 = 0$: Homogeneous solutions.

Consider the case $\mu_3 = \mu_2 = \mu_1 = \mu_0 = 0$. Then (3.5) reads

$$6\left(A - \frac{q_1^2}{4}\right)^2 = 4yAA',$$

and clearly, any irreducible factor of A must be y, since any other linear factor of A would have multiplicity $2t$ on the left hand side and $2k-1$ on the right hand side. Then we can assume $A = y^k$ for some k and necessarily $q_1^2 = 4y^k\left(1 \pm \sqrt{\frac{2k}{3}}\right)$, hence $k = 2(j+1)$ and $q_1 = 2Ry^{j+1}$, for $R := \pm 2\sqrt{1 \pm \sqrt{\frac{4j+2}{3}}}$. Then it is straightforward to verify that $p_2 = \left(\frac{3}{2} + \frac{1}{j}\right) Ry^{j+1}$ and $p_1 = y^{2j+1}\left(1 - \left(\frac{1}{j} + \frac{3}{4}\right) R^2\right)$. We also obtain $q_0 = \lambda y^{2j+1}$ and $p_0 = \lambda_1 y^{3j+1}$ for some λ, λ_1. Hence P and Q are (ρ, σ)-homogeneous for $(\rho, \sigma) = (j, 1)$.

3.3 Standard methods for solving Abel differential equations.

For Abel differential equations no general solution is known. However, some methods are available: The standard method for simplifying an Abel differential equation of the second kind suggests the substitution $A = y^{3/2}T$ in (3.5). This yields the equation

$$TT' = F_1(y)T + F_0(y) \tag{3.7}$$

with

$$F_1(y) = -\frac{1}{4y^{5/2}}(3q_1^2 - 3\mu_3 q_1 + 2\mu_2 y)$$

and

$$F_0(y) = \frac{3}{32y^4}(q_1^4 - 2\mu_3 q_1^3 + 4\mu_2 y q_1^2 - 8\mu_1 y^2 q_1 + 16\mu_0 y^3)$$

We could't bring the equation (3.7) into any of the 80 solvable cases listed in [4, 1.3.3], nor could we discard the existence of solutions.

Following the book [4] we set $U = \frac{1}{T}$ and then (3.7) reads

$$U' + F_1(y)U^2 + F_0(y)U^3 = 0, \tag{3.8}$$

an Abel differential equation of the first kind. Again, we couldn't find a solvable case in [4] that corresponds to (3.8) and it is also impossible to choose $\mu_0 \neq 0$, μ_1, μ_2, μ_3, q_1 and α such that

$$\left(\frac{F_0}{F_1}\right)' = \alpha F_1,$$

which is one of the known cases that allow further simplification of equation (3.8).

3.4 The case $\mu_3 = 0 = \mu_1$.

Let us analyze the equation (3.7) in one particular case. Note that by (3.1) we have

$$\mu_1 = c_{1,0} = 2a_{2,0}b_{0,1} - a_{1,1}b_{1,1} = \mu_3(2b_{0,1} - a_{1,1}).$$

Consequently, if $\mu_3 = 0$, then $\mu_1 = 0$. We will consider the case $\mu_3 = 0 = \mu_1$. In this case

$$F_1(y) = -\frac{1}{4y^{5/2}}(3q_1^2 - 2\mu_2 y)$$

and

$$F_0(y) = \frac{1}{32y^4}(3q_1^4 + 4\mu_2 y q_1^2 + 48\mu_0 y^3).$$

Again, we were unable to transform (3.7) into one of the solvable cases of [4].

We also can try to solve the case $\mu_1 = 0$ and $\mu_3 = 0$ directly in (3.5). In that case we can set $S := \frac{q_1^2}{4} + \frac{\mu_2 y}{6}$ and then (3.5) reads

$$3(A - S)^2 = 2yAA' - 2\mu_2 yS + \frac{5}{12}\mu_2^2 y^2 - 3\mu_0 y^3.$$

We couldn't find solutions with $\mu_0 \neq 0$ such that $S - \frac{\mu_2 y}{6}$ is a square.

3.5 Low degree cases.

Finally we solve (3.5) with the initial conditions (3.6) for some low degree cases. One can show that $\deg(A) = 2\deg(q_1)$, and we were able to solve the cases $\deg(q_1) = 2, 3, 4$, assuming q_1 monic and setting $\mu_0, \mu_1, \mu_2, \mu_3$ and the coefficients of q_1 and A as variables. For $\deg(q_1) = 3$ we obtain the solution $\mu_2 = \mu_1 = \mu_0 = 0$ and $A = -y^6/4 - \mu_3 y^3/2 - \mu_3^2/4$ which gives

$$P = x^3 y + x^2(2y^3 + \mu_3) + x\left(y^5 + \mu_3 y^2\right) + \frac{y^7}{7} + \frac{\mu_3 y^4}{4}$$

and

$$Q = x^2y + x(y^3 + \mu_3) + \frac{y^5}{5} + \frac{\mu_3 y^2}{2}.$$

Note that $P, Q \in K[x, y]$ and $[P, Q] = x^4y + \mu_3x^3$. This example is closely related to the example obtained in 3.1, in fact if we apply the procedure of section 1, with $\mu_0 = 1$, $\mu_1 = 0 = \mu_2$ and $\mu_3 = 2$ as in 3.1 then we can construct a pair $P, Q \in K[x, y]$ with $\deg(P) = 112$, $\deg(Q) = 80$ and $[P, Q] = 2x^3 + x^4y$.

The only other solutions were the homogeneous solutions with $\mu_3 = \mu_2 = \mu_1 = \mu_0 = 0$. For $deg(q_1) = 5$, after an hour the PC hadn't solved the resulting system. We also were able to show that in the case $\mu_1 = 0 = \mu_2$ (and q_1 with arbitrary degree), any solution of (3.5) satisfying (3.6) must have $\mu_0 = 0$.

Based on this partial results, we state the following conjecture:

CONJECTURE: The only solutions of (3.5) are the solutions with $\mu_2 = \mu_1 = 0$.

If the conjecture is true, then the only solutions of (3.5) satisfying (3.6) are the solutions with $\mu_2 = \mu_1 = \mu_0 = 0$, which implies $B > 16$.

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Resumen

Analizamos un posible contraejemplo P, Q a la conjetura del jacobiano con $\gcd(\deg(P), \deg(Q)) = 16$ y mostramos que su existencia depende exclusivamente de la existencia de soluciones de una cierta ecuación diferencial de Abel de segundo tipo.

Palabras clave: Jacobiano, ecuación diferencial de Abel

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