HYPERSATO STRUCTURES

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Abstract

We define hypersato structures: these structures admit three inequivalent Sasakian structures such that each of these structures shares a common Reeb vector field ξ and a common contact form η with the others two. It is interesting to notice that hypersato manifolds can be viewed as $U(1)$ principal orbibundles with base space a $4n$-dimensional hyperkähler orbifold. We also discuss some results on the moduli problem of these structures.

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1 Introduction

In [10], inspired by a paper of Satô [22], we defined *hypersato structures*: a manifold is said to be hypersato if it admits three inequivalent Sasakian structures such that each of these structures shares a common Reeb vector field ξ and a common contact form η with the others two. These are a variation on the definition of 3-Sasakian structures [6] which admit three different Sasakian structures $(\Phi_i, \xi_i, \eta_i)_{i=1,2,3}$, and only occur in dimensions manifolds of dimension $4n+3$. Hypersato structures can be found in certain manifolds of dimension $4n+1$. A manifold endowed with hypersato structure is provided with three $(1,1)$ tensors which leads to the existence of three complex structures on its associated transverse structure. We will expand some ideas from [10] and also take advantage of the fact that all these manifolds are spin and admit null Sasaki metrics (and hence η-Einstein by a result of El Kacimi Alaoui, see [13]) to apply previous results given in [11]. It is interesting to notice that hypersato manifolds can be viewed as $U(1)$ principal orbibundles with base space a $4n$-dimensional hyperkähler orbifold. We also discuss some results on the moduli problem of these structures.

2 3-Structures of Second Type

Let us briefly review some aspects of Sasakian geometry, the standard reference here is [9].

Consider a $(2n+1)$-dimensional manifold M, one says that M is a contact manifold if there exists a nowhere vanishing 1-form η, called a **contact form**, on M such that

$$\eta \wedge (d\eta)^n \neq 0.$$

It is not difficult to show that any contact manifold (M, η) admits a unique vector field ξ, called the **Reeb vector field**, satisfying the two conditions

$$\xi \rfloor \eta = 1, \quad \xi \rfloor d\eta = 0.$$

If additionally (M, η, ξ) admits a $(1,1)$ tensor Φ such that the triple (ξ, η, Φ) satisfies

$$\eta(\xi) = 1 \,\text{and}\; \Phi^2 = -\mathbb{I} + \xi \otimes \eta,$$

where $\mathbb{I}$ is the identity endomorphism on TM, one says that M admits an **almost contact structure,** and a smooth manifold with such a structure is called an **almost contact manifold.** A Riemannian metric g on M is said to be **compatible** with the almost contact structure if for any vector fields X, Y on M we have

$$g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.1}$$

An almost contact structure with a compatible metric is called an **almost contact metric structure**. A contact metric structure (ξ, η, Φ, g) is called **K-contact** if ξ is a Killing vector field of g. This metric is called **Sasakian** if the metric cone $(C(M) = M \times \mathbb{R}^+, dr^2 + r^2 g)$ is Kähler. Here the Kähler form is given by $d(r^2\eta)$ and the complex structure I in $C(M)$ is given by the rule

$$IY = \Phi Y + \eta(Y) r \frac{\partial}{\partial r} \text{and } I \frac{\partial}{\partial r} = -\xi.$$

The Reeb vector field ξ defines a one dimensional Riemannian foliation: the **characteristic foliation** $\mathcal{F}_\xi$ defined on M whose leaves are geodesics with respect to the Sasakian metric g, it is not difficult to verify that this metric is bundle-like.

Definition 2.1. The characteristic foliation $\mathcal{F}_\xi$ is said to be *quasi-regular* if there is a positive integer k such that each point has a foliated coordinate chart (U, x) such that each leaf of $\mathcal{F}_\xi$ passes through U at most k times. If $k = 1$ then the foliation is called *regular.* If $\mathcal{F}_\xi$ is not quasi-regular, it is said to be *irregular.*

Let (M, ξ, η, Φ, g) be a Sasakian manifold, and consider the subbundle $\mathcal{D} = \ker \eta$. There is an orthogonal splitting of the tangent bundle as

$$TM = \mathcal{D} \oplus L_\xi, \tag{2.2}$$

where L_ξ is the trivial line bundle generated by the Reeb vector field ξ. The subbundle $\mathcal{D}$ is called **contact subbundle** is just the normal bundle to the characteristic foliation $\mathcal{F}_\xi$. It is naturally endowed with both a complex structure $J = \Phi|\mathcal{D}$ and a symplectic structure $d\eta$. Hence, $(\mathcal{D}, J, d\eta)$ gives M a *transverse Kähler* structure with Kähler form $d\eta$ and metric $g_{\mathcal{D}}$ defined by

$$g_{\mathcal{D}}(X, Y) = d\eta(X, JY) \tag{2.3}$$

which is related to the Sasakian metric g given by

$$g = g_{\mathcal{D}} \oplus \eta \otimes \eta. \tag{2.4}$$

For compact quasi-regular Sasakian manifolds the space of leaves is a compact Riemannian orbifold $\mathcal{Z}$. But since the transverse geometry is Kähler, the orbifold must be Kähler as well. Moreover, in the quasi-regular case, it follows that M is the total space of a V-bundle over $\mathcal{Z}$, and the curvature of the connection form η is precisely the pullback of the Kähler form on $\mathcal{Z}$. Thus, $\mathcal{Z}$ satisfies an orbifold integrability condition. This integrability condition builds up a bridge between Sasakian geometry on compact manifolds to projective algebraic geometry. For instance, at the level of cohomology groups this relationship is quite explicit. To see this, first we have to review *basic cohomology.*

A smooth p-form α on M is called *basic* if

$$\xi \rfloor \alpha = 0, \quad \mathcal{L}_\xi \alpha = 0,$$

and we let Λ^p_B denote the sheaf of germs of basic p-forms on M, Ω^p_B will denote the set of global sections of Λ^p_B on M. The sheaf of Λ^p_B is a module under the ring, Λ^0_B, of germs of smooth basic functions on M. We let

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$C^\infty_B(M)=\Omega^0_B$, *i.e.*, the ring of smooth basic functions on M. Since the exterior differentiation preserves basic forms we get a de Rham complex

$$\cdots \longrightarrow \Omega^p_B \xrightarrow{d} \Omega^{p+1}_B \longrightarrow \cdots$$

whose cohomology $H^*_B(\mathcal{F}_\xi)$ is called the *basic cohomology* of $(M,\mathcal{F}_\xi)$.

The basic cohomology ring $H^*_B(\mathcal{F}_\xi)$ is an invariant of the foliation $\mathcal{F}_\xi$ and hence, of the Sasakian structure on M. When M is compact, the relationship between the de Rham cohomology and the basic cohomology is given by the following generalization of the Gysin sequence

$$\cdots \to H^p_B(\mathcal{F}_\xi) \xrightarrow{\iota_*} H^p(M,\mathbb{R}) \xrightarrow{j_p} H^{p-1}_B(\mathcal{F}_\xi) \xrightarrow{\delta} H^{p+1}_B(\mathcal{F}_\xi) \to \cdots$$

where ι_* is the natural inclusion and δ the connecting homomorphism given by $\delta[\alpha]=[d\eta\wedge\alpha]$ and j_p is defined by composing the map induced by contraction with ξ and the isomorphism $H^r(\Omega(M)^{\mathfrak{T}})\approx H^r(M,\mathbb{R})$. Here $\mathfrak{T}$ denotes the closure of the leaves of ξ, which is a torus, and $H^r(\Omega(M))^{\mathfrak{T}}$ denotes the $\mathfrak{T}$-invariant cohomology defined from the $\mathfrak{T}$-invariant r-forms in $\Omega(M)^{\mathfrak{T}}$. Since the transverse geometry is Kähler one can define *basic Dolbeault cohomology groups* $H^{p,q}_B(\mathcal{F}_\xi)$ that give rise to a transverse Hodge decomposition. These groups are fundamental invariants of the Sasakian structure which share many of the properties of the ordinary Dolbeault cohomology of a Kähler structure. The *basic Betti numbers* and the *basic Hodge numbers* are defined as follows: $b^B_r(\mathcal{F}_\xi)=\dim H^r_B(\mathcal{F}_\xi)$ and $h^{p,q}_B(\mathcal{F}_\xi)=\dim H^{p,q}_B(\mathcal{F}_\xi)$. Of course they satisfy $b^B_r(\mathcal{F}_\xi)=\sum_{p+q=r} h^{p,q}_B(\mathcal{F}_\xi)$.

If the Sasakian structure is quasi-regular the one obtains (a direct consequence of work of Girbau, Haefiger and Sundaraman [15]) the following isomorphism identifying *orbifold cohomology* and basic cohomology (*c.f* [9])

$$H^*_{orb}(M/\mathcal{F}_\xi,\mathbb{R})=H^*(M/\mathcal{F}_\xi,\mathbb{R})\approx H^*_B(\mathcal{F}_\xi).$$

Now let us complexity the contact subbundle $\mathcal{D}$ so it has a first Chern class $c_1(\mathcal{D})\in H^2(M,\mathbb{Z})$. Consider the long exact sequence given above and the map $H^2(M,\mathbb{Z}) \longrightarrow H^2(M,\mathbb{R})$ whose kernel is the torsion part of $H^2(M,\mathbb{Z})$. One can show there is a sequence

$$\begin{array}{ccccc} & & & H^2(M,\mathbb{Z}) & \\ & & & \downarrow & \\ 0 \to \mathbb{R} \xrightarrow{\delta} H^2_B(\mathcal{F}_\xi) & \xrightarrow{\iota_*} & & H^2(M,\mathbb{R}) & \longrightarrow H^1_B(\mathcal{F}_\xi) \approx H^1(M,\mathbb{R}) \longrightarrow \cdots . \end{array} \tag{2.5}$$

The map δ is given by $\delta(c) = c[d\eta]$ where $c \in \mathbb{R}$. Now on a Sasakian manifold the vector bundle $\mathcal{D}^{1,0}$ is holomorphic with respect to the CR structure so we can compute the free part of $c_1(\mathcal{D}) = c_1(\mathcal{D}^{1,0})$ from the Kähler transverse geometry in the usual way. That is $c_1(\mathcal{D})$ can be represented by a basic real closed $(1,1)$-form ρ_B. The class $c_1^B = [\rho_B] \in H^2_B(\mathcal{F}_\xi)$ is independent of the transverse metric and the basic connection used to compute it, and depends ony on the foliated manifold $(M,\mathcal{F}_\xi)$ with its CR-structure. We refer to this class $c_1(\mathcal{F}_\xi) \in H^2_B(\mathcal{F}_\xi)$ as the **basic first Chern class** of the foliation $\mathcal{F}_\xi$.

A Sasakian structure (ξ,η,Φ,g) is said to be **positive (negative)** if $c_1(\mathcal{F}_\xi)$ is represented by a positive (negative) definite $(1,1)$ form. If either of these two conditions is satisfied (ξ,η,Φ,g) is said to be **definite**, and otherwise (ξ,η,Φ,g) is called **indefinite**. (ξ,η,Φ,g) is said to be **null** if $c_1(\mathcal{F}_\xi) = 0$.

The motivation for the following definition, given in [22], was to discuss a structure *similar* to 3-Sasakian structures (see Definition 3.3 below) in manifolds of dimension $4n+1$.

Definition 2.2. Let M be a differentiable manifold of dimension n, which admits two contact structures (Φ_1,ξ,η) and (Φ_2,ξ,η) such that

$$\Phi_1\Phi_2 + \Phi_2\Phi_1 = 0.$$

Then we say that M has a **contact 3-structure of second type.**

If we consider a third endomorphism $\Phi_3 = \Phi_1\Phi_2$, a straightforward calculation shows that the triple (Φ_3,ξ,η) is a contact structure as well.

The following equalities, that relate these three different structures are valid:

$$\Phi_1^2 = \Phi_2^2 = \Phi_3^2 = -I + \eta \otimes \xi$$

$$\Phi_3 = \Phi_1\Phi_2 = -\Phi_2\Phi_1,\ \Phi_1 = \Phi_2\Phi_3 = -\Phi_3\Phi_1,\ \Phi_2 = \Phi_3\Phi_1 = -\Phi_1\Phi_3 \tag{2.6}$$

$$\Phi_1\xi = \Phi_2\xi = \Phi_3\xi = 0,\ \eta\Phi_1 = \eta\Phi_2 = \eta\Phi_3 = 0$$

Remark 2.1. Unlike 3-Sasakian structures, we require from these structures to have, for the three distinct endomorphisms, just one Reeb vector field and one contact form. Of course, this detail would lead us to substantial differences between these two structures.

As usual we have the splitting $TM = \mathcal{D} \oplus \mathcal{L}_\xi$ and is clear from the relations given in (2.6) that the contact bundle admits a quaternionic structure at every point, *i.e.*, such that $\mathcal{D}_p = \{X \in T_pM \mid \eta(X) = 0\}$ has dimension $4n+1$ at every point. Thus, we have the following proposition.

Proposition 2.3. *Let M be differential manifold that admits a 3-structure of second type, then the dimension of M is $4n+1$.*

■

Remark 2.2. It is a well-known result [23] that a differentiable manifold with a quaternionic structure in it, does not admit a fourth structure. Hence, it is impossible to have a fourth contact structure (Φ_4, ξ, η).

3 Hypersato Structures

In this section we add metric to the contact 3-structures of second type. It is known [22] that these structures admit positive definite metrics g compatible with the three structures under discussion, that is, metrics satisfying

$$\begin{aligned}
\eta(X) &= g(X, \xi)\\
g(\Phi_1X, \Phi_1Y) &= g(\Phi_2X, \Phi_2Y) = g(\Phi_3X, \Phi_3Y) = g(X, Y) - \eta(X)\eta(Y)
\end{aligned}$$

for any vector field X, Y on M. Metrics with this quality are called *associated metrics to the structure.* However this metric is not coming, necessarily, from the contact structure, that is, metrics g_i of the form $g_i(X,Y) = d\eta(\Phi_i X, Y)$ that will produce *contact metric structures* which are the structures of interest to us. In contact 3-structures of second type it will be possible to define three metrics with this feature (similar to 3-Sasakian structures). We have the following definition.

Definition 3.1. A M^{4n+1} manifold that has a 3-structure of second type in it, is called **hypersato** if each of these structures (Φ_i, ξ, η) is Sasakian. The corresponding metrics $\{g\}_{i=1,2,3}$ associated to the hypersato structures $\mathcal{S}_i$ are given by the obvious

$$g_i = d\eta \circ (\Phi_i \otimes I) + \eta \otimes \eta. \tag{3.1}$$

Remark 3.1. Recall that, implicitly, this definition requires the compatibility of the three endomorphisms Φ_i with the symplectic form $d\eta$, that is,

$$d\eta(\Phi_i X, \Phi_i Y) = d\eta(X,Y) \text{ for all } X, Y \qquad d\eta(\Phi_i X, X) > 0 \text{ for all } X \neq 0. \tag{3.2}$$

The first condition will allow us to have a contact metric compatible with the endomorphisms. The second condition will give us a strictly (strongly) pseudo-convex Levi form L_η on the corresponding CR-structure.

If we attempt to use one these metrics, say $g_1 = d\eta \circ (\Phi_1 \otimes I) + \eta \otimes \eta$ as part of the one of the other Sasakian structure, say $\mathcal{S}_2 = (\Phi_2, \xi, \eta)$ we will obtain the same Sasakian structure, that is, $\mathcal{S}_1 = \mathcal{S}_2$. This follows from a result of Tanno [9]. Below we reformulate this result as follows

Theorem 3.2. *Let $\mathcal{S}_1 = (\xi_1, \eta_1, \Phi_1, g)$ and $\mathcal{S}_2 = (\xi_2, \eta_2, \Phi_2, g)$ be two Sasakian structures on a hypersato manifold M^{4n+1} sharing the same metric, then either $\mathcal{S}_1 = \mathcal{S}_2$ or one is conjugate of the other, i.e., $\mathcal{S}_2 = (-\xi_1, -\eta_1, -\Phi_1, g)$.*

■

Recall that the group of the tangent bundle of a 3-Sasakian manifold is reducible to the group $Sp(n)\times I_3$, where I_3 is the three by three identity matrix, see [6] for details. The following is an expected result.

Proposition 3.3. *The structure group on any manifold M^{4n+1} admitting hypersato structure is reducible to $Sp(n)\times 1$.*

Proof. Consider $\{U_\alpha\}$ an open covering of M. Let X_i be a unit vector field over U_α orthonormal to ξ with respect to a compatible metric g. Then we obtain $4n+1$ orthonormal vector fields $\xi, X_i, \Phi_1 X_i, \Phi_2 X_i$ and $\Phi_3 X_i$ on U_α with $i = 1\dots n$. Repeating the argument for every U_α we obtain an adapted frame $\mathcal{B}$

$$\xi, X_i, \Phi_1 X_i, \Phi_2 X_i, \Phi_3 X_i,\ i = 1\dots n.$$

We can write the structure tensors g, Φ_1, Φ_2 and ξ in terms of this frame:

$$g = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}, \qquad \Phi_1 = \left(\begin{array}{cccc|c} 0 & I_n & 0 & 0 & \\ -I_n & 0 & 0 & 0 & \\ 0 & 0 & 0 & I_n & 0 \\ 0 & 0 & -I_n & 0 & \\ \hline & & 0 & & 0 \end{array}\right)$$

$$\Phi_2 = \left(\begin{array}{cccc|c} 0 & 0 & I_n & 0 & \\ 0 & 0 & 0 & -I_n & \\ -I_n & 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 & \\ \hline & & 0 & & 0 \end{array}\right), \qquad \xi = (0, 0, \dots, 0, 1)^t$$

where I_n denotes the $n\times n$ identity matrix. If we consider another adapted frame $\mathcal{W}$, there is an orthogonal matrix C of the form $\begin{pmatrix} A_{4n} & 0 \\ 0 & 1 \end{pmatrix}$ such that $\mathcal{B} = C\mathcal{W}$. Since the tensors g, Φ_1 and Φ_2 have the same components as before, A_{4n} has to be of the form

$$A_{4n} = \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix}$$

where a, b, c and d are $n \times n$ matrices. Hence the group of the tangent bundle of M can be reduced to $Sp(n) \times 1$ ■

Corollary 3.4. *Every hypersato manifold M is a spin manifold.*

Proof. From the natural splitting $TM = \mathcal{D} \oplus L_\xi$ where L_ξ is the trivial (real) line bundle generated by ξ one obtains

$$w_2(M) = w_2(TM) = w_2(\mathcal{D}),$$

which is the mod 2 reduction of $c_1(\mathcal{D}) \in H_2(M, \mathbb{Z})$. But from Proposition 3.3 we have that $c_1(\mathcal{D}) = 0$. ■

Recall the exact sequence given in (5). Now, if $c_1(\mathcal{D}) = 0$ we have that the basic first Chern class $c_1(\mathcal{F}_\xi) \in H^2_B(\mathcal{F}_\xi)$ is of the form $a[d\eta]$ for $a \in \mathbb{R}$. In [13] it was shown that the transverse Monge-Ampère problem has solution for the null ($a = 0$) and negative case ($a < 0$), so this implies the existence of either null or negative Sasaki η-Einstein structures. For hypersato structures we have the following result

Theorem 3.5. *A hypersato structure $(M^{4n+1}, \xi, \eta, \Phi_i, g)$ is a null Sasaki η-Einstein structure.*

Proof. Since M^{4n+1} admits a hypersato structure, M admits a one dimensional Riemannian foliation $\mathcal{F}_\xi$. In terms of Haefliger cocycles, we have a foliated atlas (U_α, ϕ_α), local submersions $f_\alpha : U_\alpha \to \mathbb{R}^{4n} = \mathbb{C}^{2n}$ and continuous maps $\tau_{\alpha\beta} : f_\alpha(U_\alpha \cap \beta) \to f_\beta(U_\alpha \cap \beta)$ satisfying certain cocycle conditions, see [9] for details. Furthermore, in [9] it is shown that a Riemannian foliation $(M, \mathcal{F}_\xi)$ has metrics h_α in $f_\alpha(U_\alpha) = \widetilde{U}_\alpha$ with $h_\alpha = \tau^*_{\alpha\beta} h_\beta$ (here the h_α's are the pull-backs on $\mathbb{R}^{4n}$ by $f'_\alpha s$). So it is enough to prove this result locally.

Since all hypersato structures satisfy $c_1(\mathcal{D}) = 0$ we have that $c_1(\mathcal{F}_\xi) = a[d\eta]$ is definite or null. Let us assume that $c_1(\mathcal{F}_\xi)$ is definite, then $\Lambda^{2n,0}\widetilde{U}_\alpha$ is non-trivial (here $\Lambda^{2n,0}\widetilde{U}_\alpha$ denotes $\Lambda^{2n,0}\mathcal{D}^*{}_\mathbb{C}$ restricted to $\widetilde{U}_\alpha$). The Levi-Civita connection Δ of h_α induces a connection $\widetilde{\Delta}$ on $\Lambda^{2n,0}\widetilde{U}_\alpha$. Since h_α is, almost by definition, compatible with $\Phi_i|_\mathcal{D}$ for $i = 1, 2, 3$, h_α is invariant under the action of $Sp(n)$, that is, $\mathrm{Hol}^0(\Delta) \subseteq Sp(n)$. Recall that the action of any $\Gamma \in Sp(n) \subset SU(n)$ on an open set $U \subset \mathbb{C}^{2n}$ induces multiplication by the determinant $\det(\Gamma)$ on $\Lambda^{2n,0}\widetilde{U}_\alpha$. It follows that

$$\mathrm{Hol}^0(\widetilde{\Delta}) = \det \mathrm{Hol}^0(\Delta) = \{1\},$$

hence $\widetilde{\Delta}$ is flat, that is, the curvature of $\widetilde{\Delta}$ is zero, but this curvature is exactly the Ricci form of h_α. Thus, $c_1(\mathcal{F}_\xi) = 0$. ■

We apply Theorem 8.1.14 in [9] and we immediately obtain

Corollary 3.6. *Compact hypersato manifolds* $(M^{4n+1}, \xi, \eta, \Phi_i, g)$ *are quasi-regular.*

■

In the sequel we will need the definition of hyperkähler structures. In the literature there are several definitions [19], [17] (most of them equivalent). Here we give a definition that will be suitable to our purposes.

Definition 3.7. Let Z be a smooth manifold equipped with three complex structures $\{I_j\}_{j=1,2,3}$ that satisfy the quaternionic identities

$$I_1 I_2 = -I_2 I_1 = I_3 \tag{3.3}$$

Z is said to be **hypercomplex** or to admit *hypercomplex structure.* A Riemannian metric g on a hypercomplex manifold (Z, I_1, I_2, I_3) is called **hyperhermitian** if it is compatible with respect to every complex structure J induced by I_1, I_2, I_3. In addition, if the hyperhermitian metric is Kähler for all complex structures in Z, then one says that this manifold is **hyperkähler**. Similarly, we can define hyperkähler structures for orbifolds.

Remark 3.2. The definition of hyperkähler manifolds (M^{4n}, g) given above, is equivalent to the inclusion of the holonomy group of the metric g in the group $\mathrm{Sp}(n)$ (see [19]), that is (M, g) admits hyperkähler structure if $\mathrm{Hol}(g) \subset \mathrm{Sp}(n)$. In the sequel, we will indisctintly use both definitions.

Definition 3.8. Let M be a Riemannian manifold. One says that (M, g) is 3-**Sasakian** if its corresponding metric cone $(C(M), \overline{g}) = (M \times \mathbb{R}^+, dr^2 + r^2 g)$ is hyperkähler.

It is known (see [6]) that on a compact 3-Sasakian manifold M has dimension $4n + 3$ and we have obtain three inequivalent Reeb vector fields which generate a 3-dimensions foliation $\mathcal{F}_3$. Hence the space of leaves $M/\mathcal{F}_3$ has the structure of a *quaternionic Kähler orbifold* of dimension $4n$ such that the natural projection $\pi : M \to M/\mathcal{F}_3$ is a principal orbibundle with group $SU(2) = Sp(1)$ or $SO(3)$ and a Riemannian orbifold submersion.

The situation for hypersato structures is different since we have at our disposal only one Reeb vector fiel and hence, only an S^1 action on the manifold. However from the definition one concludes that the transverse structure of hypersato manifold ends up having three different complex structures, hence three different Kähler 2-forms $\omega_{_i}$. Moreover, from Theorem 3.5 we obtain as transverse space a hyperkähler orbifold.

On the other hand, if one starts with a projective hyperkähler orbifold $\mathcal{Z}$ with $[p^*\omega_1] \in H^2_{orb}(X, \mathbb{Z})$, then we have an S^1 V-bundle defined by $[\omega_1]$ where M is the total space with Sasakian structure (ξ, η, Φ_1) such that $d\eta = \pi^*\omega_1$. Here $d\eta$ is the curvature of the connection form η. From the finer bundle determined by $[\omega_1]$ it is easy to guarantee that the lift of one complex structure I_1 on $\mathcal{Z}$ (according to the choice of index above) works for Φ_1. However for the remaining two I_2 and I_3 we have to be more careful. Define $\Phi_{_i}$ on the total space M as follows:

$$\Phi_{_i}(\widetilde{X}) = \widetilde{I_i(X)},$$

where $X \in \mathcal{Z}$ and $\widetilde{X}$ denotes the horizontal lifting of X. Clearly, defined in this way, $\Phi_{_i}$ is the endomorphism that we were looking for, just extend it to all TM, in the usual way, adding the condition $\Phi_{_i} \circ \xi = 0$. In doing so, we have $\Phi^2_{_i} = -\mathbb{I} + \xi \otimes \eta$. Another technical problem that arises is how to ensure that there exists, in fact, a projection. Notice that, so far, the action of $\Phi_{_i}$ on $X \in TM$ could take this vector field to the vertical part L_ξ. That case would be omitted if we assume that $\Phi_i(X)$ is invariant under the flow of ξ, that is, if the Lie derivative $\pounds_\xi \Phi_i = 0$, this condition together with the integrability of the CR-structure establish the normality on the total space with respect to each of the three $\Phi_i{}' s$, that is, the corresponding almost complex structures J_i on the cone $C(M)$ are integrable.

Something that remains to be shown is that each Φ_i is compatible in the sense of equation (2.1). This is a consequence of the invariance of Φ_i under the flow of ξ and the definition $\Phi_i X$: any horizontal vector field of the form $\Phi_i(\widetilde{X}) \in M$ is the lift of a vector field in $\mathcal{Z}$.

$$\begin{aligned} d\eta(\Phi_i X, \Phi_i Y) &= d\eta(\Phi_i \widetilde{X}, \Phi_i \widetilde{Y}) \\ &= \pi^*\omega_1(\widetilde{I_i X}, \widetilde{I_i Y}) \\ &= \pi^*\omega_1(X, Y) \\ &= d\eta(X, Y). \end{aligned}$$

We put all this discussion in form of propositions.

Proposition 3.9. *Let $(M^{4n+1}, \xi, \eta, \Phi_i)$ be a compact hypersato manifold. Then M is the total space of a principal circle V-bundle over a hyperkähler orbifold $\mathcal{Z}$.*

■

Proposition 3.10. *Let $(\mathcal{Z}^{4n}, \omega_i, I_i)$ be a hyperkähler orbifold with at least one integral class $[p^*\omega_{i_0}] \in H^2_{orb}(\mathcal{Z}, \mathbb{Z})$. Let M denote the total space of the circle V-bundle defined by the class of the fixed form $[\omega_{i_0}]$. Then the manifold admits a hypersato structure (ξ, η, Φ_i) such that $[d\eta] = \pi^*_{i_0}[\omega_{i_0}]$, where π_{i_0} is the natural projection map defined by $[\omega_{i_0}]$, and $\Phi_i(\widetilde{X}) = \widetilde{I_i(X)}$ with $\pounds_\xi \Phi_i = 0$, for $i \neq i_0$*

■

Now let us try to find some manifolds that admit this structure. In dimension 5 it is helpful to bear in mind what possibilities we have on the transverse space. Compact hypercomplex four-manifolds were classified by Boyer in [5] where it was shown that a compact hypercomplex four-manifold is either a torus, a K3 surface or a special type of Hopf surface. One can endow tori and K3 surfaces with hyperkähler metrics (see [4] for details). On the other hand, any Hopf surface has first Betti number one, so Hopf surfaces are the only one compact hypercomplex four-manifolds where the hyperhermitian metrics are never hyperkähler. From Theorem A in [11] we have the immediate

Theorem 3.11. *Let $M^5 = \#k(S^2 \times S^3)$ with $3 \leqslant k \leqslant 21$. Then M^5 admits hypersato structures.*

■

Remark 3.3. The close relationship that one observes between null Sasakian and hypersato structures holds for dimension 5 in a natural way, this follows from the fact that hyperkähler manifolds (orbifolds) and Calabi-Yau manifolds (orbifolds) coincide in dimension 4. In higher dimensions, at least for simply connected manifolds, hyperkähler manifolds have been defined in order to have as many similarities as possible with K3 surfaces, these higher dimensional analogues are called *complex symplectic manifolds* if priority is given to the algebraic geometry of the underlying complex manifold, see [17] for an excellent reference. Now we exhibit more examples, for this we need to apply some techniques developed by Boyer, Galicki and Ornea in [8].

4 Hypersato Structures and The Join Construction

In the next lines we recall the *join construction* for regular structures, for details about this construction on the quasi-regular case we refer to [8]. We denote by $\mathcal{SM}$ the set of regular Sasakian manifolds. For each pair of positive integers (k_1, k_2) with $\gcd(k_1, k_2) = 1$ we have the graded multiplication (here, the set $\mathcal{SM}$ is graded by dimension, that is, $\mathcal{SM} = \oplus_{n=0}^{\infty}\mathcal{SM}_{2n+1}$):

$$\star k_1, k_2 : \mathcal{SM}_{2n_1+1} \times \mathcal{SM}_{2n_2+1} \to \mathcal{SM}_{2(n_1+n_2)+1}. \tag{4.1}$$

defined as follows: consider $\mathcal{M}_1, \mathcal{M}_2 \in \mathcal{SM}$. There is the natural free action of T^2, induced by the free action of the Reeb vector field on both $\mathcal{M}_1$, $\mathcal{M}_2$, on $\mathcal{M}_1 \times \mathcal{M}_2$ and the quotient manifold is the product of the corresponding Kähler manifolds $Z_1 \times Z_2$. If $[\omega_i] \in H^2(Z_i, \mathbb{Z})$ then $[k_1\omega_1 + k_2\omega_2] \in H^2(Z_1 \times Z_2, \mathbb{Z})$ defines a S^1-bundle over the manifold $Z_1 \times Z_2$ whose total space is the manifold $\mathcal{M}_1 \star_{k_1,k_2} \mathcal{M}_2$ and refer to it as the (k_1, k_2)-join of $\mathcal{M}_1$ and $\mathcal{M}_2$. This Sasakian structure is unique up to gauge transformation. This defines the map (4.1).

As stated above, this construction can be generalized to the quasi-regular case (however, we will not go further than the regular case). We have the following result [8].

Proposition 4.1. *The (k_1, k_2)-join of two compact null Sasaki manifolds is null Sasaki manifold.*

■

Let us denote by $\mathcal{HS}$ the set of compact hypersato orbifolds. This set is topologized with the $C^{m,\alpha}$ topology, and $\mathcal{HS}$ is graded by $\mathcal{HS} = \oplus_{n=1}^{\infty}\mathcal{HS}_{4n+1}$.

Proposition 4.2. *The join of two hypersato orbifolds is hypersato. Moreover, the join operation $\star$ is continuous on both factors. This operation defines an operation on hypersato structures:*

$$\mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathcal{HS} \times \mathcal{HS} \longrightarrow \mathcal{HS}$$

$$[(k_1, k_2), (\mathcal{M}_1, \mathcal{M}_2)] \mapsto \mathcal{M}_1 \star_{k_1,k_2} \mathcal{M}_2.$$

Proof. The join of two orbifolds M_1 and M_2 that posses hypersato structures comes from the product $(Z_1 \times Z_2, g_1 \times g_2)$ of two hyperkähler orbifolds (Z_1, g_1) and (Z_2, g_2) each with $\mathrm{Hol}(g_i) \subset \mathrm{Sp}(n)$. Hence $\mathrm{Hol}(g_1 \times$

$g_2) = \mathrm{Hol}(g_1) \times \mathrm{Hol}(g_2) \subset \mathrm{Sp}(2n)$. Thus, $M_1 \star_{k_1,k_2} M_2$ has hypersato structure. ■

For non-simply connected hypersato structures, we consider compact Heisenberg manifolds M (which are the quotient of the Heisenberg group by a lattice subgroup of it). It is not difficult to see that M is a circle bundle over an abelian variety [8]. Folland has proved that these Heisenberg manifolds are in one to one correspondence, up to holomorphic equivalence, with polarized abelian varieties [14]. Based on this result, let us specialize the join construction to abelian varieties. The product of an abelian variety of dimension n and one of dimension m is in a natural way isomorphic to an abelian variety of dimension $m+n$. Suppose $\mathcal{A}$ is an abelian surface isomorphic to a product of two elliptic curves $\mathcal{A} = E_1 \times E_2$. As mentioned above, for each elliptic curve E_i, together with a Kähler class $[\omega_i] \in H^2(E_i, \mathbb{Z})$, there is a Heisenberg manifold $\mathfrak{N}_i^3$ of dimension 3 with null Sasakian structure. If we apply Proposition 4.2 we have that $\mathcal{A} = E_1 \times E_2$ with $[k_1\omega_1 + k_2\omega_2] \in H^2(\mathcal{A}, \mathbb{Z})$ defines a null Sasakian structure on $\mathfrak{N}_1^3 \star \mathfrak{N}_2^3$. On the other hand, for $[k_1\omega_1 + k_2\omega_2] \in H^2(\mathcal{A}, \mathbb{Z})$ there is a Heisenberg manifold $\mathfrak{N}^5$ admitting this null Sasakian structure. Thus we have the following

Corollary 4.3. *Let $\mathcal{A}$ be an abelian surface isomorphic to the product of two elliptic curves $\mathcal{A} = E_1 \times E_2$. Let $\mathfrak{N}_1^3$ and $\mathfrak{N}_2^5$ the corresponding null Sasakian structures induced by that integral classes $[\omega_1] \in H^2(E_1, \mathbb{Z})$ and $[\omega_2] \in H^2(E_2, \mathbb{Z})$. Then, for every positive integers k_1, k_2, $[k_1\omega_1 + k_2\omega_2] \in H^2(\mathcal{A}, \mathbb{Z})$ defines a circle bundle $S^1 \hookrightarrow \mathfrak{N}^5 \to \mathcal{A}^{2n}$. Moreover, $\mathfrak{N}^5 = \mathfrak{N}_1^3 \star \mathfrak{N}_2^3$ admits hypersato structure.*

■

When an abelian variety $\mathcal{A}$ is *reducible*, that is, it is the product of abelian varieties, provided that $\dim_{\mathbb{R}} \mathcal{A} \equiv 0 \bmod 4$, one can always generalize this idea, and one has that reducible hypersato manifolds of *Heisenberg type* are products $\mathfrak{N}_{n_1} \star \mathfrak{N}_{n_2} \star \ldots \star \mathfrak{N}_{n_k}$, with $n_1 + n_2 + \ldots + n_k =$

$\dim_{\mathbb{R}} \mathcal{A} + 1$. So we have the following diagram

$$\begin{array}{ccc} \mathfrak{N}^{2n_1+1} \star \mathfrak{N}^{2n_2+1} \star \ldots \star \mathfrak{N}^{2n_k+1} & \longrightarrow & \mathfrak{N}^{2(n_1+n_2+\ldots+n_k)+1} \\ \downarrow & & \downarrow \\ \mathcal{A}_{n_1} \times \mathcal{A}_{n_2} \times \ldots \times \mathcal{A}_{n_k} & \longrightarrow & \mathcal{A}_{n_1+n_2+\ldots n_k}. \end{array} \tag{4.2}$$

It may happen that for quasi-regular hypersato manifolds $M_1 \star_{k_1,k_2} M_2$ is not a manifold but only an orbifold. In [8] we find under what conditions $M_1 \star_{k_1,k_2} M_2$ is a manifold. Let υ_i denote the order of the quasi-regular Sasakian manifold M_i, that is, υ_i is the least common multiple of the orders of the leaf holonomy groups of M_i. We have the following proposition, that is a rephrasing of Proposition 2.1 in [8] in terms of hypersato structures.

Proposition 4.4. *Let M_1 and M_2 two quasi-regular hypersato manifolds. For each pair of relative prime positive integers k_1, k_2, the orbifold $M_1 \star_{k_1,k_2} M_2$ is a smooth quasi-regular hypersato manifold if and only if* $\gcd(\upsilon_1 k_2, \upsilon_2 k_1) = 1$. *In particular, If M_i are regular hypersato manifolds, then so is $M_1 \star_{k_1,k_2} M_2$.*

■

As mentioned in Section 3, in (real) dimension 4 the only compact manifolds admitting hyperkähler structure are tori and K3 surfaces. in [10] we show that the total space of projective K3 surfaces, via Seifert bundles, is given by 21 connected sums of $S^2 \times S^3$. For tori, we have the Heisenberg manifolds $\mathfrak{N}^{4n+1}$. Now if we consider the product of a non-singular projective K3 surface X and an abelian surface A_{4n} of real dimension $4n$, via the join construction we obtain:

$$S^1 \hookrightarrow \#21(S^2 \times S^3) \star_{k_1,k_2} \mathfrak{N}^{4n+1} \to X \times A_{4n}.$$

From the above discussion we have that $\#21(S^2 \times S^3) \star_{k_1,k_2} \mathfrak{N}^5$ is a hypersato manifold of real dimension $4n + 5$ for any k_1, k_2 such that $\gcd(k_1, k_2) = 1$. Moreover, using Theorem A in [11] we can extend this idea and obtain the following result.

Corollary 4.5. *$\#k(S^2 \times S^3) \star_{k_1,k_2} \mathfrak{N}^{4n+1}$ and $\#k(S^2 \times S^3) \star_{k_1,k_2} \#l(S^2 \times S^3)$ admit hypersato structures for $3 \leqslant k, l \leqslant 21$. Moreover, these two orbifolds are quasi- regular hypersato manifolds if and only if* $\gcd(\upsilon_1 k_2, \upsilon_2 k_1) = 1$.

■

Next, we state the following structure theorem for Ricci-flat manifolds due to Beauville [3]. Here we give a simplified version, enough for our purposes.

Theorem 4.6. *Any simply connected Calabi-Yau manifold is given as a product $\prod_i Y_i \times \prod_j Z_j$ where:*
a) Each Y_i is a projective Calabi-Yau manifold with $H^0(Y_i, \Omega^p_{Y_i}) = 0$ for $0 < p < \dim(V_i)$;
b) The manifolds Z_j are simply connected hyperkähler.

■

This theorem has certain analogue for regular null Sasaki manifolds. Let us suppose X is a compact simply connected Calabi-Yau manifold that is *reducible*, that is, the product of two projective manifolds $Y \times Z$, where Y and Z are given as in a) in b), respectively. We have the circle bundle $S^1 \hookrightarrow M_1 \to Y$ defined by certain class $[\omega_1] \in H^2(Y, \mathbb{Z})$ which endows M_1 with a null Sasaki structure. Similarly, $[\omega_2] \in H^2(Z, \mathbb{Z})$ defines the circle bundle $S^1 \hookrightarrow M_2 \to Z$, with M_2 admitting a hypersato structure. Its product defines the circle bundle $S^1 \hookrightarrow M_1 \star_{k_1,k_2} M_2 \to Y \times Z = X$. Moreover $M_1 \star_{k_1,k_2} M_2$ is a null Sasakian manifold for any k_1, k_2 such that $\gcd(k_1, k_2) = 1$. So we have

Lemma 4.7. *Reducible simply connected regular null Sasakian manifolds can be factored, in the sense given in the previous paragraph, in terms of (regular) null and hypersato structures.*

■

In the literature, it is common to find a stronger condition in the definition of hyperkähler manifolds. It is said that a manifold (M^{4n}, g) admits hyperkähler structure if the holonomy group of g *equals* $\mathrm{Sp}(n)$. If this is the case, a result, similar to the one given in Lemma 4.7, for simply connected hypersato structures is not possible. If a simply connected hyperkähler manifold (M, g) is the product of two hyperkähler manifolds $X = Y \times Z$, Kunneth formula will imply that $\dim H^0(M, \Omega^2_X) > 1$. However, simply connected hyperkähler manifolds –that is, manifolds with $\mathrm{Hol}(g) = \mathrm{Sp}(n)$– admit a holomorphic symplectic form ω, hence $H^0(M, \Omega^2_X) = \mathbb{C}\omega$. Hyperkähler manifolds with this stronger condition are called **irreducible**. Actually, we have the following proposition, due to Beauville [3], that will be used in the next section.

Proposition 4.8. *Let X be a compact Kähler manifold of dimension $2n$. The following conditions are equivalent:*

(i) X admits a Kählerian metric g such that $\mathrm{Hol}(g) \subset \mathrm{Sp}(n)$ *(respectively,* $\mathrm{Hol}(g) = \mathrm{Sp}(n)$*),*

(ii) X admits a symplectic structure (respectively, X is simply connected and, up to scalar, admits a unique symplectic structure).

■

5 Deformations for Hypersato Manifolds

In this section we will take advantage of some known results on deformation of hyperkähler manifolds. The main reference used here is [3]. As done in Section 4 in [11], we will deform the hypersato structure in a manifold $(M^{4n+1}, \xi, \eta, \Phi_i, g)$ deforming one of the complex structures of the hyperkähler manifold $M/\mathcal{F}_\xi$. This deformation will produce deformations of the hypersato structure (at least when the hyperkähler manifold is irreducible).

5.1 Deformations of Transverse Holomorphic Sasakian Structures

An analogue to the theory of local deformations of complex structures developed by Kodaira and Spencer (see [20]). This theory was developed by Duchamp and Kalka and by Gomez-Mont, (see [12] and [15] for details on this theory). We briefly review some basic notions of this theory.

A germ of a deformation of a transverse holomorphic foliation $\mathcal{F}$ on a manifold M with base space $(B, 0)$ is given by an open cover $\{U_\alpha\}$ of M and a family of local submersions $f_{\alpha,t} : U_\alpha \to \mathbb{C}^n$ parametrized by $(B, 0)$ that are holomorphic in $t \in B$ for each $x \in U_\alpha$. Consider the holomorphic family of biholomorphism $\rho^t_{\alpha\beta} : f_{\beta,t}(U_\alpha \cap U_\beta) \to f_{\alpha,t}(U_\alpha \cap U_\beta)$. Then we have the (expected) condition

$$f_{\alpha,t} = \rho^t_{\alpha\beta} \circ f_{\beta,t} \text{ on } U_\alpha \cap U_\beta. \tag{5.1}$$

Infinitesimal deformations are obtained, as in the complex case, by differentiating equation (5.4) with respect to t and evaluating at $t = 0$.

Let us denote by $\Theta_\mathcal{F}$ the sheaf that encodes information about the aforementioned infinitesimal deformations. (Formally, first, one defines the sheaf of germs of vector fields on M that are infinitesimal automorphisms of the transverse holomorphic foliation $\mathcal{F}$, then this sheaf is quotiented by the sheaf of smooth vector fields tangent to the leaves of $\mathcal{F}$ to obtain $\Theta_\mathcal{F}$.) Here, we also have a Kodaira-Spencer map $\rho : T_0B \to H^1(M, \Theta_\mathcal{F})$ that sends $\frac{\partial}{\partial t}$ to certain class in $H^1(M, \Theta_\mathcal{F})$ defined by a section $\theta_{\alpha,\beta}$ of the sheaf $\Theta_\mathcal{F}|U_\alpha \cap U\beta$. One can consider the full cohomology ring $H^*(M, \Theta_\mathcal{F})$, these were proven to be finite dimensional.

In [15] it is showed that there is a Kuranishi space of deformations given by the map $\Phi : U \to H^2(M, \Theta_\mathcal{F})$, for U open set in $H^1(M, \Theta_\mathcal{F})$, here, as before, the base of parametrizations is given by $\Phi^{-1}(0)$.

Remark 5.1. We have the following (see [15]):

1. If $H^2(M, \Theta_\mathcal{F}) = 0$, then the Kuranishi family of deformations of $\mathcal{F}$ is isomorphic to an open set in $H^1(M, \Theta_\mathcal{F})$.

In the case that M is a compact Sasakian manifold we have the evident

Proposition 5.1. *The characteristic foliation $\mathcal{F}_\xi$ of a Sasakian structure $\mathcal{S}$ is a transverse holomorphic foliation.*

■

For quasi-regular Sasakian structures we have the following result that follows from an argument on spectral sequences (see [9] for details).

Proposition 5.2. *Let $\mathcal{S}$ be a quasi-regular Sasakian structure on a compact manifold with charactersitic foliation $\mathcal{F}_\xi$ and with quotient projection $\pi : M \to X$. Then there is an exact sequence*

$$0 \to H^1(X, T_X) \to H^1(M, \Theta_{\mathcal{F}}) \to H^0(X, T_X) \to H^2(X, T_X). \qquad (5.2)$$

■

From this exact sequence we obtain the following corollary.

Corollary 5.3. *We have*

(i) If $H^1(X, T_X) = 0$ and $H^2(X, T_X) = 0$, then we have $H^0(X, T_X) \cong H^1(M, \Theta_{\mathcal{F}})$.

(ii) If $H^0(X, T_X) = 0$ then $H^1(X, T_X) \cong H^1(M, \Theta_{\mathcal{F}})$.

We are particularly interested in the second part of this corollary: rephrasing (ii): if there are no infinitesimal holomorphic automorphisms, then all the deformations of the transverse holomorphic structure come from the deformations of the complex structures on X. However, we have to be a little careful, unlike the kählerian case, where the locally infinitesimal deformation of kählerian structures remain kählerian, the Riemannian foliation does not remain Riemannian necessarily. This technical difficulty can be overcome if we consider local Killing vector fields for the transverse metric g_T, where we take $g_T = f_\alpha^* g$ for all α. Let us denote by $\Theta_{\mathcal{F},\mathfrak{L}}$ the sheaf of such vector fields. Being M Sasakian, there is an orbifold submersion $\pi : M \to X$ onto a compact Kähler orbifold X whose Kähler metric h satisfies $g_T = \pi^* h$. The subsheaf of T_X that leaves this metric invariant will be sometimes denoted by $T_{X,\mathcal{L}}$. This is the main reason why we will be interested solely in polarized moduli spaces. When finding moduli spaces we will consider a fixed ample line bundle, or equivalently, a fixed Kähler metric. The sequence (5.2) given in Propositon 5.2 remains unaltered.

Proposition 5.4. *Let $\mathcal{S}$ be a quasi-regular Sasakian structure on a compact manifold with charactersitic foliation $\mathcal{F}_\xi$ and with quotient projection $\pi : M \to X$. Then there is an exact sequence*

$$0 \to H^1(X, T_{X,\mathcal{L}}) \to H^1(M, \Theta_{\mathcal{F},\mathfrak{L}}) \to H^0(X, T_{X,\mathcal{L}}) \to H^2(X, T_{X,\mathcal{L}}). \tag{5.3}$$

■

Let $f : N \to \mathcal{Z}$ be a principal S^1- orbibundle with N a smooth manifold and $\mathcal{Z}$ a Kähler orbifold. Consider the following two sets: the Lie algebra $\mathfrak{aut}(J, g_T)$ of infinitesimal automorphisms of the transverse Kähler structure, with J the transverse complex structure and g_T a compatible Kähler metric, and the Lie algebra $\mathfrak{aut}(\xi, \eta, \Phi, g)$ of the automorphism group of the underlying Sasakian structure. It is known (see Chapter 8 of [9]) that, under polarization $[\omega] \in H^2_{orb}(\mathcal{Z}, \mathbb{Z})$, any infinitesimal automorphisms $\check{X} \in \mathfrak{aut}(J, g_T)$ lifts to an automorphism $X \in \mathfrak{aut}(\xi, \eta, \Phi, g)$ of the induced Sasakian structure on the total space N of the circle V-bundle, with orbifold first Chern class $[\omega]$, if and only if $\check{X}$ is Hamiltonian, that is, if $[\check{X} \lrcorner dn]_B \in H^1_{orb}(\mathcal{Z}, \mathbb{R}) = H^1_B(\mathcal{F}_\xi)$ vanishes. For the null case, we have the following lemma that is a particular case of more general lemma proved in [11].

Lemma 5.5. *Let $f : N \to \mathcal{Z}$ be a principal S^1- bundle with N a smooth simply connected hypersato manifold and $\mathcal{Z}$ a hyperkähler orbifold. Then $H^0(\mathcal{Z}, T_Z) = 0$, where T_Z denotes the sheaf of holomorphic vector fields on $\mathcal{Z}$.*

Proof. Since the action of ξ is quasi-regular, one can identify the Lie algebra $H^0(Z, T_Z)$ of infinitesimal automorphisms of the Kähler orbifold with $\mathfrak{aut}(J, g_T)$. For null Sasakian structures, it was proven in [9] that $\mathfrak{aut}(\xi, \eta, \Phi, g) = \{\xi\}$. It follows from the discussion in the previous paragraph that Z has no Hamiltonians. For any $\check{X} \in H^0(Z, T_Z)$, one has the equality $[\check{X} \lrcorner dn]_B = [\alpha]_B \in H^1_B(\mathcal{F}_\xi) = H^1_{orb}(Z, \mathbb{R}) = 0$, but under the absence of Hamiltonians, the only possibility for α is to be 0 which implies $\check{X} = 0$, due to non-degeneracy of the symplectic form $d\eta$. ■

Recall that a *small deformation* is a 1-parameter family of deformations $\mathcal{X} = (X, f, \Delta)$ of a compact complex manifold X_0 with $\Delta \in \mathbb{C}$ and $t \in \Delta$ small. There are some results on stability for complex analytic manifolds due to Kodaira and Spencer [20] that ensures that any *small deformation* of a Kähler manifold remains Kähler. Since the Hodge number $h^{2,0}$ is constant in families of compact Kähler manifolds [20], any small deformation of a irreducible hyperkähler manifold admits a unique non-trivial two form which is everywhere non-degenerate. Moreover, using Theorem 3.6, Beauville proves in [3]

Proposition 5.6. *Let $f : X \to B$ a smooth proper morphism of complex manifolds. Let $0 \in B$, and X_0, the fiber of f at 0, a hyperkähler manifold. Then there is a neighborhood U of 0 in B such that X_s is hyperkähler for $s \in U$. Moreover, any Kähler deformation of an irreducible hyperkähler manifold is irreducible hyperkähler.*

■

If we consider deformations of a complex structure on polarized projective hyperkähler manifold that are irreducible we will obtain Kähler deformations that are projective, hence, by Proposition 5.6, projective hyperkähler deformations that are irreducible. Since every hyperkähler manifold has trivial canonical bundle, X admits a smooth versal deformation (see [24]). The irreducibility of X implies $H^0(X, \Omega_X) = 0$. From the triviality of the canonical bundle K_X one obtains the isomorphism $\Omega_X = T_X$ and thus $H^0(X, T_X) = 0$. So the Kuranishi family is universal. Moreover, from Corollary 5.3 (ii) all the deformations of the transverse holomorphic structure that are Sasakian come from the deformations of one (of the three) complex structure on X. Since each Kähler deformation gives rise to a hyperkähler manifold, we obtain deformations of the hypersato structure on the corresponding manifold. In particular, from Theorem B in [11] we have

Lemma 5.7. *The space of deformations of hypersato structures on $\#21(S^2 \times S^3)$ has real dimension 38.*

■

There are two standard series of examples of irreducible hyperkähler manifolds for each dimension (for details, see [3]):

The Hilbert scheme of points on a K3 surface. Consider a K3 surface X. The Hilbert scheme $X^{[r]}$ is the moduli space of all zero-dimensional subspaces $Z \subset X$ of length $l(\mathcal{O}_Z) = r$. If X is projective, then $X^{[r]}$ is projective [18]. $X^{[r]}$ is an irreducible hyperkähler manifold of dimension $2r$.

The generalized Kummer Variety. Let A be a two-dimensional complex torus. Then $A^{[r+1]}$ is symplectic but not simply-connected. It admits a smooth surjective map $P : A^{[r+1]} \to A$, which is the composition of the map $A^{[r+1]} \to A^{(r+1)}$ (here $A^{(r+1)} := A^{r+1}/\mathfrak{S}_{r+1}$ denotes the symmetric product) and the sum map $A^{(r+1)} \to A$. The fiber $\mathrm{K}_r = P^{-1}(0)$ is an irreducible hyperkähler manifold of dimension $2r$. Notice that K_1 is the Kummer surface associated to A. In [3] we have the following lemma.

Lemma 5.8. *For $r > 1$, the Hilbert scheme of a K3 surface $S^{[r]}$ satisfies*

$$H^1(S^{[r]}, T_{S^{[r]}}) \cong H^1(S, T_s) \oplus \mathbb{C} \cong \mathbb{C}^{21}.$$

Similarly, for $r > 1$, the generalized Kummer variety K_r has

$$H^1(\mathrm{K}_r, T_{\mathrm{K}_r}) \cong H^1(A, T_A) \oplus \mathbb{C} \cong \mathbb{C}^5.$$

■

Analogous argument to the one given in the proof of Lemma 5.7 yields to the following lemma.

Lemma 5.9. *Let $S^{[r]}$ and K_r be projective with $r > 1$. Then the space of deformations of hypersato structures on a manifold $S^1 \hookrightarrow M^{4r+1} \to S^{[r]}$ has real dimension 40. The space of deformations of hypersato structures on a manifold $S^1 \hookrightarrow M^{4r+1} \to K_r$ has real dimension 8.*

■

Remark 5.2. Lemma 5.9 not only gives us information about the space of deformation at the dimensional level. For instance $H^1(S^{[r]}, T_{S^{[r]}}) \cong H^1(S, T_s) \oplus \mathbb{C}$ is saying that the deformations of the Hilbert scheme $S^{[r]}$ coming from deforming the K3 surface form a hypersurface in the space of deformation of the $S^{[r]}$. Similar situation occurs for the generalized Kummer variety. It will be interesting to obtain explicit versions of these two series of examples of hyperkähler manifolds at the level of hypersato structures. Are these related to 21 connected sums of $S^2 \times S^3$ (for the Hilbert schemes) and the Heisenberg manifolds (for generalized Kummer varieties)? Can we find and explicit description of the moduli of hypersato structures as achieved for the null case in [11]?

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Resumen

Definimos *variedades hipersato:* estas variedades admiten tres estructuras del tipo Sasaki inequivalentes de tal manera que estas tres estructuras poseen un campo vectorial del tipo Reeb ξ y una forma de contacto η en común. Variedades que admiten estructura hipersato pueden considerarse como espacios totales de un fibrado principal $U(1)$ del tipo orbifold, donde el espacio base admite una métrica singular hiperkähler. Discutimos también algunos resultados acerca del espacio moduli de variedades admitiendo estas estructuras.

Palabras Clave: Geometría riemanniana, geometría Sasaki, métricas hiperkähler.

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